# UNIQUE UNDEFINABLE ELEMENTS

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Abstract. [Fu68] presents structures, in a finite relational type, with unique undefinable elements and recursive theories. We present somewhat simpler examples, with somewhat stronger properties. We also present structures, in a finite relational type, whose unique undefinable element is weak second order undefinable. We convert all examples of a similar nature to corresponding examples in the form of bipartite graphs and atomic inclusions. We show that "there is a structure, in a finite relational type, with a unique second order undefinable element" is not provable in ZFC (assuming ZFC is consistent). We explore some properties of structures with unique undefinable elements.



## 1. Introduction.

In [Fu68], there is an affirmative answer to the question: is there a structure in a finite relational type with a unique undefinable element? Here the relational type is the set of constant, relation, and function symbols interpreted by the structure. Definability and undefinability always refer to the usual first order predicate calculus with equality (unless stated otherwise). The theory of each example in [Fr68] (i.e., the set of all first order sentences true in the structure) is shown in [Fr68] to be recursive.

According to [Fu68], the question was posed by J. Mycielski and C. Ryll-Nardzewski at the University of Wroclaw. Without being aware of this history, we recently formulated the question in connection with rudimentary aspects of theology - as we discuss below.

The requirement of a finite relational type is entirely appropriate, and needed to avoid trivialities. E.g., (N,1,2,3,...), which is the set N of all nonnegative integers with constants for 1,2,3,... . Clearly 0 is the unique undefinable element, since every definition can only mention finitely many of the constants 1,2,3,... . For this reason, it is also clear that in (N,1,2,3,...), 0 is second order undefinable.

The "unique undefinable element" condition on a structure occurred to us as a result of interactions at a conference [SR12], where a number of theologians were present with mathematical and scientific interests.

God has been proposed to be an entity not subject to any of the limitations that all other entities are subject to. Definability has been proposed as a limiting feature on entities. The idea that God is the unique undefinable entity naturally arises.

Under this conception, it is natural to strengthen the undefinability condition on the unique undefinable element, and also strengthen the definability condition on the remaining elements. These refinements can be viewed as corresponding to strengthening the contrast between God and all other entities. We follow this approach here.

The [Fu68] examples are presented in the form $M = (N \cup \{A_0,A_1,...\},\varepsilon,S_r)$, where $A_0,A_1,... \subseteq N$, $\varepsilon$ is the binary membership relation between N and $\{A_0,A_1,...\}$, and $S_r$ is the binary successor relation on N. Here $A_1,A_2,...$ are periodic, and $A_0$ is not periodic. It is immediate that the A's are definable in M. Quantifier elimination is used to show that $A_0$ is not definable in M. The qe depends on the A's being carefully chosen. The qe also shows that the theory of M is recursive.

In section 2, we present new examples of the form $(N \cup \{A_0,A_1,...\},\varepsilon,0,S_f)$, where $A_0,A_1,... \subseteq N$, $\varepsilon$ is membership, and $S_f(x) = x+1$ if $x \in N$; 0 otherwise, such that $A_0$ is the unique undefinable element, where the theory (set of first order sentences that hold) is recursive.

The construction here is somewhat simpler than the construction in [Fu68]. Here we use the subscripts f,r on S to distinguish between the successor function and the successor relation. As expected, the $S_r$ in [Fu68] can be straightforwardly replaced by $S_f$ or $0,S_f$.

In the section 2 examples with $0,S_f$, the definable elements are defined by a conjunction of atomic formulas. In particular, the unique $\Sigma_0$ undefinable element is undefinable.

Note the particularly strong contrast between the unique undefinable element and the other elements, illustrating a principal motivating theme as discussed in the fifth and sixth paragraphs of this Introduction.

In the [Fu68] examples, the definable elements are defined by single universally quantified formulas, and no better - assuming that we replace $S_r$ by $0,S_f$. Thus in the [Fu68] examples with $0,S_f$, the unique $\Pi_1$ undefinable element is undefinable. This represents a somewhat weaker contrast between the unique undefinable element and the other elements.

In section 2, we also show how to simplify the form further, to $(N \cup \{A_0,A_1,...\},\varepsilon,0)$, and to $(N \cup \{A_0,A_1,...\},\varepsilon)$. In the form with 0, the unique $\Sigma_1$* undefinable element is undefinable. This represents the strongest definability possible for the elements other than the unique undefinable element, assuming that we remain in a finite relational type without function symbols (at least in terms of quantifier complexity). As we mentioned earlier, in an infinite relational type, we have $(N,1,2,...)$, where 0 is second order undefinable and 1,2,... are constants.

Adjustments of the [Fu68] examples without function symbols appear to achieve only that the unique $\Sigma_2$* undefinable element is undefinable. The extra quantifier arises because of the use of periodic sets in [Fu68].

Weak second order logic extends first order logic by adding quantifiers over all finite relations of any fixed arity. Second order logic is much stronger still, and instead uses quantifiers over all relations of any fixed arity.

In section 5, we show that, in any finite relational type, for any $n \geq 0$, any unique $\Sigma_n$* undefinable element must be weak second order definable. This puts a limitation on the contrast between the unique undefinable element and the other elements.

It follows immediately that in the [Fu68] and section 2 examples, the unique undefinable element is weak second order definable (although of course this can easily be seen directly).

In section 5 recursion theoretic forcing is used to present structures in a finite relational type whose unique undefinable element is weak second order undefinable. This strengthens the contrast between the unique undefinable element and the other elements.

These examples take the form $(N \cup \{A_0,A_1,...\},\varepsilon,+,\bullet)$. In fact, we show that such examples can take the form $(N \cup \{A_0,A_1,...\},\varepsilon,+,\bullet,T)$, for any given $T \subseteq N^2$. Thus these examples can be arbitrarily rich. In sections 3,4 we set up the recursion theoretic forcing constructions used in section 5. We no longer have recursive theories. See Question 1.3 below.

In section 6, we put the examples from section 5 into the form $(N \cup \{A_0,A_1,...\},\varepsilon)$. For the section 5 examples, we maintain the "arbitrary richness". We also put the examples from sections 2,5 into the form of a bipartite graph and an atomic inclusion. These are both familiar kinds of mathematical structures. See Theorems 6.6, 6.7.

In section 7, we address the question: is there a structure in a finite relational type with a unique second order undefinable element? We show that an affirmative answer cannot be given within ZFC.

In section 8, we explore some basic properties of structures with a unique undefinable element.

We now sketch proofs of two easy results that set the stage for the developments, and raise some questions.

THEOREM 1.1. Let M be a structure in a countable relational type. If M has a unique undefinable element, then M is countably infinite. Similarly for weak second order undefinable and second order undefinable.

Proof: Let M be as given, with domain D. Assume M has a unique definable element. By cardinality considerations, obviously D is countable. To show that D is infinite, suppose $D = \{d_1,...,d_k,d\}$, where $d_1,...,d_k$ are definable. Let $\varphi_1(v),...,\varphi_k(v)$ define $d_1,...,d_k$, respectively, in M. Then $\neg\varphi_1(v) \wedge ... \wedge \neg\varphi_k(v)$ defines d. The remaining two claims are proved the same way. QED

THEOREM 1.2. Let M be a structure of finite relational type. The unique undefinable element, if any, is second order definable. The unique weak second order undefinable element, if any, is second order definable.

Proof: Let M be a structure of finite relational type, where u is the unique undefinable element. By Theorem 1.1, M is infinite. Using second order logic, we can quantify over relations that provide finite sequence coding, and define a first order satisfaction relation/ In this way, we give a second order definition over M of "x is an element of M that is first order definable over M". Hence u is second order defined over M by "u is not first order definable". The second claim follows since weak second order logic can also be treated similarly within second order logic. QED

QUESTION 1.3. Is there a structure in a finite relational type with a unique weak second order undefinable element and recursive theory? Is there a structure in a finite relational type whose unique undefinable element is weak second order undefinable, with recursive theory?

QUESTION 1.4. Is there a linear ordering with a unique undefinable element? Is there a linear ordering with a unique undefinable element and recursive theory? What about Question 1.3 for linear orderings?

QUESTION 1.5. What about Questions 1.3, 1.4 for compact sets of rationals under < (or equivalently, complete linear orderings)?

QUESTION 1.6. Is there a structure in a finite relational type with a unique second order undefinable element?

In section 7, we show that an affirmative answer to Question 1.6 cannot be obtained within ZFC.

In [Hj10], a structure of cardinality $\omega_1$, in a finite relational type, is given which has a unique $L_{\omega\_1\omega}$ undefinable element, answering a question in [Mi91]. As remarked in [Hj10], any such structure must be uncountable.

We adhere to the following conventions concerning definability.

a. Definability always refers to the first order predicate calculus with equality. Weak second order and second order definability are always explicitly mentioned.

b. We use "definable in" for elements of a structure, and "definable over" for relations and functions on the elements of a structure. In the former case, we of course do not allow parameters, as every element is obviously definable with itself as a parameter. In the latter case, we do allow any finite number of parameters.

c. We use "0-definable over" in order to require that there are no parameters allowed. This, and convention b, also applies to second order definability and weak second order definability. Thus we speak of "(weak) second order 0-definable over".

d. The theory of a structure always refers to the set of first order sentences true in the structure. We also consider the weak second order theory of structures.

## 2. Unique undefinable elements.

DEFINITION 2.1. N is the set of all nonnegative integers. An N-system is a structure $(N \cup \{A_0,A_1,...\},\varepsilon,...)$, where $A_0,A_1,... \subseteq N$, arranged in an infinite sequence, and $x \mathrel{\varepsilon} y \leftrightarrow x \in N \wedge y \in \{A_0,A_1,...\} \wedge x \in y$. The ... to the right of $\varepsilon$ represents countably many (possibly none) constants, relations, and functions on the domain $N \cup \{A_0,A_1,...\}$. A pure N-system is an N-system $(N \cup \{A_0,A_1,...\},\varepsilon)$.

DEFINITION 2.2. $S_r$ is the binary relation on N given by $S_r(x,y) \leftrightarrow x,y \in N \wedge y = x+1$. We use $S_f$ as a component in N-systems, where it always signifies the function on the domain given by $S_f(x) = x+1$ if $x \in N$; 0 otherwise.

DEFINITION 2.3. The relational type of a structure is the underlying set of constant, relation, and function symbols that it interprets. The theory of a structure is the set of all first order sentences that hold in the structure.

We will use N-systems with infinite relational type only to support the quantifier elimination argument.

The [Fu68] examples are N-systems $(N \cup \{A_0,A_1,...\},\varepsilon,S_r)$. [Fu68] proves that $A_0$ is the unique undefinable element, and the theory is recursive.

Here we give a self contained presentation of some somewhat simpler examples with somewhat stronger properties. These take the form of N-systems $(N \cup \{A_0,A_1,...\},\varepsilon,0,S_f)$, $(N \cup \{A_0,A_1,...\},\varepsilon,0)$, and also pure N-systems.

The use of both 0 and $S_f$ supports the very strong contrast between the unique undefinable element and the other elements, given by Theorem 2.8.

Theorem 2.9 uses N-systems $(N \cup \{A_0,A_1,...\},\varepsilon,0)$, and supports the strongest such contrast that can be expected without using function symbols (in a finite relational type), at least in terms of quantifier complexity.

Theorem 2.10 reduces to pure N-systems, with a weakened but still strong contrast.

DEFINITION 2.4. We say that $A_0,A_1,... \subseteq N$ is very independent if and only if
the following holds. Let $n \geq 1$. There exist $x \in N$ such that the $n^2$ statements $x+i \in A_j$, $0 \leq i,j \leq n-1$, have any of the $2^{n^2}$ possible truth values.

DEFINITION 2.5. We inductively define $J_i$, $i \geq 1$, as follows. $J_1 = \{0,1\}$. $J_{i+1}$ consists of the next i+1 integers after max(Ji).

DEFINITION 2.6. We say that $A_0,A_1,... \subseteq N$ is special if and only if $A_0,A_1,...$ is very independent, and the following holds.
i. For all $i,j \geq 1$, $J_i \subseteq A_j \leftrightarrow i = j$.
ii. Let X,Y be disjoint finite subsets of N, where X does not contain any interval $J_i$. There exists $i \in N$ such that $X \subseteq A_i \subseteq N\backslash Y$.

LEMMA 2.1. There exists a recursive special sequence $A_0,A_1,...$ . There is a recursive sequence $A_1,A_2,...$ such that there are continuumly many special $A_0,A_1,...$ .

Proof: We build $A_0,A_1,...$ inductively as follows. At any stage, for each i, finitely many numbers are declared in $A_i$ and finitely many numbers are declared not in $A_i$. At any stage, only finitely many $A_i$'s have any declarations.

We enumerate pairs of disjoint finite subsets of N, $(X_j,Y_j)$, $j \geq 0$, where $X_i$ does not contain any interval $J_i$. Note that no two powers of 2 lie in a common $J_i$.

At stage $i \geq 1$, we declare $J_i \subseteq A_i$. Also declare all numbers $< \min(J_i)$ to be outside of $A_i$, as long as they have not been declared to be in $A_i$. Next enforce Definition 2.3 for $A_0,...,A_i$ using powers of 2 larger than $\max(J_i)$ and any integers that have been used. Finally, enforce clause ii for $(X_j,Y_j)$ using $A_r$, where r is sufficiently large, by declaring $X_j \subseteq A_r \subseteq N\backslash Y_j$. Also declare $J_r \subseteq A_r$ and $A_r$ disjoint from $J_{r-1}\backslash Xj$. This will also enforce clause i of Definition 2.5 for $A_r$. Finally, we declare $A_0$ to exclude those numbers in $J_i$ that have not been declared to be in $A_0$.

Note that at stage i, membership in $A_i$ and $A_0$ is decided for all elements of $J_1 \cup ... \cup J_i$. Hence this construction can be carried out effectively, producing a recursive sequence $A_0,A_1,...$ .

Clearly $A_0$ excludes at least one element from every $J_i$ that is not a power of 2. Hence we can adjust $A_0$ in continuumly many ways, by inserting some of them into $A_0$, without changing $A_1,A_2,...$, so that $A_0,A_1,...$ remains special. QED

LEMMA 2.2. Suppose $A_0,A_1,...$ is very independent. Then the A's are infinite and distinct. Also clause ii of Definition 2.4, holds with "there exists" by "there exists infinitely many". If $A_0,A_1,...$ is special, then clause ii of Definition 2.6, holds with "there exists" by "there exists infinitely many".

Proof: Suppose $A_0,A_1,...$ is very independent. The first claim follows from the second claim. Suppose we have a pattern of truth values for $A_0,...,A_i$. Extend this pattern to mutually incompatible patters for $A_0,...,A_{i+1}$, for $A_0,...,A_{i+2}$, and so forth. These must be realized by different integers.

Suppose $A_0,A_1,\ldots$ is special. Let X,Y be as given. Extend X,Y to finitely many disjoint pairs X',Y', where any pair of these new pairs are incompatible, and where the X' still does not contain any $J_p$. These must be realized by different A's. QED

DEFINITION 2.7. Let $A_0,A_1,\ldots \subseteq N$. $M[A_0,A_1,\ldots]$ is the N-system $(N \cup \{A_0,A_1,\ldots\},\varepsilon,0,S_f,P,N,0,1,\ldots,A_1,A_2,\ldots)$, where $P(x) = x-1$ if $x \in N\backslash\{0\}$; 0 otherwise, $N(x) \leftrightarrow x \in N$, $0,1,\ldots$ are constants for the elements of N, and $A_1,A_2,\ldots$ are constants for elements of $\{A_1,A_2,\ldots\}$. We use $x \in N$ for the formula $N(x)$.

We now fix special $A_0,A_1,\ldots \subseteq N$. Let $D = \mathrm{dom}(M) = N \cup \{A_0,A_1,\ldots\}$. We will show that $A_0$ is undefinable in $M[A_0,A_1,\ldots]$. But first we establish a lot of definability in a reduct of $M[A_0,A_1,\ldots]$.

LEMMA 2.3. In the N-system $(N \cup \{A_0,A_1,\ldots\},\varepsilon,0,S_f)$, every element except $A_0$ is definable. N is 0-definable over this N-system.

Proof: In fact, we can do without 0. $x \in N \leftrightarrow (\exists y)(x\ \varepsilon\ y)$, $x = 0 \leftrightarrow (\exists y,z)(y \neq z \wedge S_f(y) = S(z) = x)$, and $x = A_i \leftrightarrow (\forall y)(y \in J_i \rightarrow y\ \varepsilon\ x)$, where $y \in J_i$ is expanded into a disjunction using $0,S_f$. Also $x \in N \leftrightarrow (\exists y)(x\ \varepsilon\ y)$. QED

LEMMA 2.4. $M[A_0,A_1,\ldots]$ admits quantifier elimination. I.e., every formula $\varphi$ is equivalent, over $M[A_0,A_1,\ldots]$, to a quantifier free formula $\psi$, where every free variable of $\psi$ is a free variable of $\varphi$. Here $\psi$ can be obtained effectively from $\varphi$. The first order theory of $M[A_0,A_1,\ldots]$ is recursive in the sequence $A_1,A_2,\ldots$, and does not depend on $A_0$ (as long as $A_0,A_1,\ldots$ remains special).

Proof: By standard manipulations, it suffices to eliminate the quantifier in these two forms.

1. $(\exists x \in N)(x \pm= s_1 \wedge \ldots \wedge x \pm= s_n \wedge t_1 \pm\varepsilon\ w_1 \wedge \ldots \wedge t_m \pm\varepsilon\ w_m)$.
2. $(\exists x \notin N)(x \pm= s_1 \wedge \ldots \wedge x \pm= s_n \wedge t_1 \pm\varepsilon\ x \wedge \ldots \wedge t_m \pm\varepsilon\ x)$.

where $n+m \geq 1$, $\pm=$ is $=$ or $\neq$, $\pm\varepsilon$ is $\varepsilon$ or $\neg\varepsilon$, and $s_1,\ldots,s_n,t_1,\ldots,t_m,w_1,\ldots,w_m$ are terms.

We now treat form 1. We can assume that all occurrences of x are displayed. If there is at least one equation, then we can immediately eliminate the quantifier. So we assume that there are no equations. Also, we can assume that x does not appear in $s_1,\ldots,s_n,w_1,\ldots,w_m$, and x appears in $t_1,\ldots,t_m$. So we arrive at the form

$$(\exists x \in N)(x \neq y_1+a_1,\ldots,y_n+a_n,b_1,\ldots,b_m \wedge x+c_1 \varepsilon u_1 \wedge \ldots \wedge x+c_r \varepsilon u_r \wedge x+d_1 \neg\varepsilon v_1 \wedge \ldots \wedge x+d_p \neg\varepsilon v_p)$$

where $a_1,\ldots,a_n,c_1,\ldots,c_r,d_1,\ldots,d_p$ are integers, and $b_1,\ldots,b_m$ are nonnegative integers, with the obvious formalization using $0,S_f,P$. Also $y_1,\ldots,y_n$ are variables other than x, and $u_1,\ldots,u_r,v_1,\ldots,v_p$ are variables other than x or among the constants $A_1,A_2,\ldots$ .

By splitting on the predicate N, and splitting on equations, we arrive at the convenient form

$$y_1,\ldots,y_n \in N \wedge u_1,\ldots,u_r,v_1,\ldots,v_p \notin N \wedge x \geq -c_1,\ldots,-c_r,-d_1,\ldots,-d_p \wedge$$
$$(\exists x \in N)(x \neq y_1+a_1,\ldots,y_n+a_n,b_1,\ldots,b_m \wedge x+c_1 \varepsilon u_1 \wedge \ldots \wedge x+c_r \varepsilon u_r \wedge x+d_1 \neg\varepsilon v_1 \wedge \ldots \wedge x+d_p \neg\varepsilon v_p).$$

Using the first claim of Lemma 2.2, it is clear that this formula is false if and only if either the conjunction before the quantifier is false, or there is a conflict in the membership and non membership clauses. The only conflict can be in the form of a pair $x+c \ \varepsilon \ u$, $x+c \ \neg\varepsilon \ v$, where $u = v$. The existence of a conflict is expressed as a quantifier free formula. Hence we have eliminated the quantifier.

We next treat form 2. We can assume that $s_1,\ldots,s_n$ are variables other than x, or among the constants $A_1,A_2,\ldots$ . If there is an equation then we can immediately eliminate the quantifier. So we assume there are no equations. We can also split on the predicate N. We arrive at the form

$$s_1,\ldots,s_n \notin N \wedge t_1,\ldots,t_m,u_1,\ldots,u_r \in N \wedge t_1 \neq \ldots \neq t_m \neq u_1 \neq \ldots \neq u_r \wedge$$
$$(\exists x \notin N)(x \neq s_1,\ldots,s_n \wedge t_1,\ldots,t_m \varepsilon x \wedge u_1,\ldots,u_r \neg\varepsilon x).$$

Using the second claim of Lemma 2.2, it is clear that this formula is false only if either the conjunction before the quantifier is false, or the membership clause requires that x contain some $J_i$. Now since m is a definite number, the only relevant $J_i$ are of cardinality at most m, and there are only finitely many of these intervals. Hence this formula is false only if a certain quantifier free condition holds.

Now if the membership clause requires that x contain some $J_i$, then x is among $A_1,...,A_m$, and we have eliminated the quantifier, using the constants $A_1,...,A_m$. Hence we have eliminated the quantifier. QED

LEMMA 2.5. $A_0$ is undefinable in $M[A_0,A_1,...]$.

Proof: Suppose $\varphi(x)$ defines $A_0$ in $M[A_0,A_1,...]$. By Lemma 2.4, we can assume that $\varphi(x)$ is in disjunctive normal form, with at most the variable x, and defines $A_0$ in $M[A_0,A_1,...]$.

Since $\varphi(A_0)$ holds, one of its disjuncts $\psi(A_0)$ holds. Evidently, $A_0$ is unique such that $\psi(A_0)$ holds. Thus $A_0$ is unique such that

$s_1,...,s_n \in A_0 \wedge t_1,...,t_m \neg\in A_0 \wedge A_0 = A_{i\_1},...,A_{i\_p} \wedge A_0 \neq A_{j\_1},...,A_{j\_q}$.

where $s_1,...,s_n,t_1,..,t_m$ are closed terms. We can assume that $p = 0$ and $j_1,...,j_q \neq 0$. Thus $A_0$ is unique such that

$a_1,...,a_n \in A_0 \wedge b_1,...,b_m \neg\in A_0$.

where $a_1 \neq ... \neq a_n \neq b_1 \neq ... \neq b_m$ are specific integers.

If $\{a_1,...,a_n\}$ contains some $J_i$, then $A_0$ is some $A_i$, $i \geq 1$. This is impossible, as the A's are distinct. If $\{a_1,...,a_n\}$ does not contains any $J_i$, then $A_0$ cannot be unique with the above. To see this, let $v \in A_0$, $v > \max(a_1,...,a_n,b_1,...,b_m)+1$. There must be an $A_j$ such that $a_1,...,a_n \in A_j \wedge b_1,....,b_m \notin A_j \wedge v \notin A_j$, and this $A_j$ cannot be $A_0$. QED

THEOREM 2.6. Let $A_0,A_1,... \subseteq N$ be special. $A_0$ is the unique undefinable element in $M[A_0,A_1,...]$. The first order theory of $M[A_0,A_1,...]$ is recursive in the sequence $A_1,A_2,...$ . This is also true of the N system $(N \cup \{A_0,A_1,...\},\varepsilon,0,S_f)$.

Proof: By Lemmas 2.2, 2.3, 2.4, 2.5. QED

THEOREM 2.7. There is a recursive sequence $A_0,A_1,... \subseteq N$, without repetition, such that the N-system $(N \cup \{A_0,A_1,...\},\varepsilon,0,S_f)$ has the unique undefinable element $A_0$ and recursive theory. Furthermore, there are continuumly many $B \subseteq N$ such that the replacement of $A_0$ with B results in an N-system with unique undefinable element B, the same theory, and a structure not isomorphic to the original.

Proof: Use any recursive special sequence $A_0,A_1,\ldots \subseteq N$. Apply Lemmas 2.1, 2.4, and 2.6. Especially note the independence of the theory and the undefinable element under changes of the first term in a special sequence from Lemma 2.4. This will be used in Theorems 2.8 - 2.10 below. QED

We now obtain some more refined information.

DEFINITION 2.8. In first order predicate calculus, the $\Sigma_0$ and $\Pi_0$ formulas are the quantifier free formulas. The $\Sigma_n$ ($\Pi_n$) formulas, $n \geq 1$, are the prenex formulas with at most n quantifiers, with either no quantifiers or where the first quantifier is existential (universal). The $\Sigma_n$* ($\Pi_n$*) formulas, $n \geq 0$, are the prenex formulas with at most n alternating blocks of quantifiers, with either no quantifiers or where the first block of quantifies is existential (universal).

DEFINITION 2.9. An element is $\Sigma_n$ ($\Pi_n$, $\Sigma_n$*, $\Pi_n$*) definable in a structure if and only if it is the unique element satisfying a $\Sigma_n$ ($\Pi_n$, $\Sigma_n$*, $\Pi_n$*) formula with no parameters.

THEOREM 2.8. There is a recursive sequence $A_0,A_1,\ldots \subseteq N$ without repetition, such that in the N-system $(N \cup \{A_0,A_1,\ldots\},\varepsilon,0,S_f)$, the unique $\Sigma_0$ undefinable element $A_0$ is undefinable, and the theory is recursive. Furthermore, there are continuumly many $B \subseteq N$ such that the replacement of $A_0$ with B results in an N-system with unique $\Sigma_0$ undefinable element B, the same theory, and a structure not isomorphic to the original.

Proof: Use any recursive special sequence $A_0,A_1,\ldots \subseteq N$. By Theorem 2.7 and the equivalence $x = A_i \leftrightarrow J_i \subseteq A_i$, going back to Lemma 2.3. The inclusion is written as a conjunction of atomic formulas using 0, $S_f$, and $\varepsilon$. QED

Note the very strong contrast between $A_0$ and the remaining elements. Theorem 2.8 fails if we use only $S_f$. For the [Fu68] examples, even if we use $0,S_f$, we have to replace $\Sigma_0$ with $\Pi_1$ because of the use of periodic sets.

THEOREM 2.9. There is a recursive sequence $A_0,A_1,\ldots \subseteq N$ without repetition, such that in the N-system $(N \cup \{A_0,A_1,\ldots\},\varepsilon,0)$, the unique $\Sigma_1$* undefinable element $A_0$ is undefinable, and the theory is recursive. Furthermore, there are continuumly many $B \subseteq N$ such that the replacement of A0 with B results in an N-system with unique $\Sigma_1$* undefinable element B, the same theory, and a structure not isomorphic to the original.

Proof: Use any recursive special sequence $A_0,A_1,... \subseteq N$. Let $M = (N \cup \{A_0,A_1,...\},0,S_f,\varepsilon)$. We modify M to the N-system M' $= (N \cup \{B_0,B_1,...\},\varepsilon,0)$ so that the unique $\Sigma_1$* undefinable element $B_0$ is undefinable and the theory is recursive. We set $B_0,B_2,B_4,...$ to be $A_0+3,A_1+3,A_2+3,...$ . The remaining $B_1,B_3,B_5,...$ consists of

i. $\{0\}$, $\{0,1\}$, $\{0,1,2\}$.
ii. $\{1,n,n+1\}$, $n \geq 2$.
iii. $\{2,n,n+1,n+3\}$, $n \geq 3$.

We make the following definitions in M'.

$x = 1 \leftrightarrow x \neq 0 \wedge (\exists y \neq z)(0,x \ \varepsilon\ y,z)$.
$x = 2 \leftrightarrow x \neq 0,1 \wedge (\exists y)(0,1,x \ \varepsilon\ y)$.
$|x-y| = 1 \leftrightarrow (\exists z)(1,x,y \ \varepsilon\ z) \vee \{x,y\} \in \{\{0,1\},\{1,2\}\}$.
$y = x+1 \leftrightarrow (\exists w)(2,x,y,z \ \varepsilon\ w \wedge |x-y| = 1) \vee (x,y) \in \{(0,1),(1,2),(2,3)\}$.

Note that by the definition of $y = x+1$ above in M', we see that each $n \in N$ is $\Sigma_1$* definable in M'. Hence $B_1,B_3,B_5,...$ are each $\Sigma_1$* definable in M'. Furthermore, the definition in M' of $B_{2i}$ by $J_i+3 \subseteq B_{2i}$ is also $\Sigma_1$* definition.

Thus we have shown that in M', every element other than $B_0$ is $\Sigma_1$* definable.

Now suppose $B_0$ is definable element of M'. We can interpret M' in M in the following way. The new domain consists of the points $A_0,A_1,...$ and tuples for the i-iii above, with the obvious modification of $\varepsilon'$ of $\varepsilon$.

It is now clear that $B_0 = A_0+3$ is 0-definable over M. Therefore $A_0$ is 0-definable over M, and so $A_0$ is a definable element of M. This is a contradiction.

These mutual interpretations also show that the theories effectively reduce to each other. QED

THEOREM 2.10. There is a recursive sequence $A_0,A_1,... \subseteq N$ without repetition, such that in the pure N-system $(N \cup \{A_0,A_1,...\},\varepsilon)$, the unique $\Sigma_2$* undefinable element $A_0$ is undefinable, and the theory is recursive. Furthermore, there are continuumly many B ⊆ N such that the replacement of A0 with B results in a pure N-system with unique $\Sigma_2$* undefinable element B, the same theory, and a structure not isomorphic to the original.

Proof: By the same construction given in Theorem 2.9. It suffices to show that in M', we can give a $\Sigma_2$* definition of 0 without using 0. Note that in M', x = 0 ↔ (∃y,z,w)(x ∈ y,z,w ∧ the distinct y,z,w each have at most 3 ε predecessors). QED

## 3. Cohen generic sets and forcing.

We use N for the set of all nonnegative integers, and $2^N$ for the set of all functions from N into {0,1}. We use N# for the set of all nonempty finite sequences of elements of $2^N$. We use N## for the set of all infinite sequences of elements of $2^N$. All sequences are indexed from 0.

We use lth(x) for the length of sequences. Lengths of elements of N# are positive integers. Lengths of elements of N## are ∞.

We identify the elements of N# of length 1 with $2^N$. For $\alpha$ ∈ N# ∪ N##, we use $\alpha^{(n)} \in 2^N$ for the characteristic function of the n-th Turing jump of $\alpha$. We define $\alpha^{(0)} = \alpha$.

A condition is a nonempty finite sequence of nonempty finite sequences from {0,1}, indexed from 0. A k-condition is a condition indexed from 0 through k, and has length k+1. Let C[∞] be the set of all conditions. Let C[k] be the set of all k-conditions.

We define the partial ordering ≤* on C[∞] ∪ N# ∪ N## as follows. Let x,y ∈ C[∞] ∪ N# ∪ N##. Then

x ≤* y if and only if every $x_i$ is included in $y_i$.

Note that x ≤* y implies that lth(x) ≤ lth(y). Also note that we have used the inclusion relation among nonempty finite and infinite sequences from {0,1}. I.e., (0,1,0) ⊆ (0,1,0,1,1).

NOTE: In many treatments of general forcing over models of set theory, p ≤ q means that the condition p is at least as strong than the condition q. But here x ≤* y for conditions means that the condition y is at least as strong as the condition x.

For p ∈ C[k*], 0 ≤ k* ≤ ∞, define res(p;n) to be p restricted to {0,...,n}. I.e., res(p;n) consists of the terms of p indexed from 0 through min(n,k).

Let 0 ≤ k* ≤ ∞. We say that E ⊆ C[k*] is dense if and only if (∀x ∈ C[k*])(∃y ∈ C[k*])(x ≤* y).

Let x ∈ N#, α ∈ N# ∪ N##. We say that x is α generic if and only if for every dense E ⊆ C[lth(x)-1] recursive in α, there exists y ∈ E such that y ≤* x.

LEMMA 3.1. Let α ∈ N# ∪ N## and 0 ≤ k* ≤ ∞. Every dense E ⊆ C[k*] that is
r.e. (recursively enumerable) in α contains a dense E' ⊆ C[k*] that is recursive in α.

Proof: Let α,k* be as given, and E ⊆ C[k*] be dense and r.e. in α. Let $p_0,p_1,...$ be a recursive enumeration without repetition of C[k*]. We compute G:C[k*] → E as follows. We use an enumeration the elements of E without repetition by an algorithm recursive in α. Take $G(p_i)$ to be the first element $p_j$ of E so enumerated, with the property that $p_i \leq^* p_j \wedge j \geq i$.

Obviously rng(G) is dense and included in E. To see that rng(G) is recursive in α, note that $p_j \in \text{rng}(G) \leftrightarrow (\exists i \leq j)(G(p_i) = p_j)$. QED

LEMMA 3.2. Let α ∈ N# ∪ N##. There exists α generic $x \leq_T \alpha^{(2)}$ of every length from 0 through ∞. Suppose x is α generic. Each permutation of x that moves only finitely many terms, each nonempty finite subsequence of x, each tail of x, and each finite change of the 0's and 1's that appear in x, is α generic.

Proof: Let α ∈ N# ∪ N##. We use an enumeration $E_0,E_1,...$ of the relevant dense sets recursive in α. We can obviously arrange for this enumeration to be recursive in $\alpha^{(2)}$. I.e., $\{(i,j): i \in E_j\} \leq_T \alpha^{(2)}$. An α generic x can be constructed inductively in the obvious way, recursively in this enumeration. Therefore the α generic x can be constructed recursively in $\alpha^{(2)}$.

Now let x be α generic, lth(x) = k*, 0 ≤ k* ≤ ∞. Let π be a permutation x that moves only finitely many terms. Let E ⊆

C[k*]. Then $\pi^{-1}(E) \subseteq C[k*]$ is dense and recursive in $\alpha$. Let $p \leq^* x$, $p \in \pi^{-1}(E)$. Then $\pi(p) \leq^* \pi(x)$, $\pi(p) \in E$, and so $\pi(x)$ is $\alpha$ generic.

Let $y = (x_{i\_1},\ldots,x_{i\_n})$, $i_1 < \ldots < i_n$. Let $E \subseteq C[n]$ be dense, $E \leq_T \alpha$. Use $E' \subseteq C[k*] = \{p \in C[k*]: (p_{i\_1},\ldots,p_{i\_n}) \in E\}$.

For the fourth claim, by the previous claim, we can assume that $k* = \infty$. $y = (x_i,x_{i+1},\ldots)$. Let $E \subseteq C[\infty]$ be dense, $E \leq_T \alpha$. Use $E' = \{p \in C[\infty]$: the tail of p starting at position i lies in E or is empty$\}$.

For the fifth claim, from dense E, we use the dense set E' obtained by making changes in the 0's and 1's at the same places on the elements of E. QED

LEMMA 3.3. Let $\alpha \in N\# \cup N\#\#$. Suppose x is $\alpha$ generic and $f \in 2^N$ is $(\alpha,x)$ generic. Then $(f,x)$ is $\alpha$ generic. If $lth(x) < \infty$ then $(x,f)$ is $\alpha$ generic.

Proof: Let $\alpha,x,f$ be as given. Let $lth(x) = k*$, $0 \leq k* \leq \infty$. Let $E \subseteq C[1+k*]$ be dense, $E \leq_T \alpha$. We want to find $(p,q) \in E$ such that $(p,q) \leq^* (f,x)$.

Let $p \in C[1]$. Since E is dense, $A = \{q \in C[k*]: (\exists p' \in C[1])(p \leq^* p' \wedge (p',q) \in E\} \subseteq C[k*]$ is dense, and r.e. in $\alpha$. By Lemma 3.1, let $A' \subseteq A$ be dense, $A' \leq_T \alpha$. Since x is $\alpha$ generic, let $q \leq^* x$, $p \leq^* p'$, $(p',q) \in E$.

Thus we have shown that $B = \{p \in C[1]: (\exists q \leq^* x)((p,q) \in E)\} \subseteq C[1]$ is dense, and r.e. in $(\alpha,x)$. By Lemma 2.1, let $B' \subseteq B$ be dense, $B' \leq_T (\alpha,x)$. Since f is $(x,\alpha)$ generic, let $p \leq^* f$, $q \leq^* x$, $(p,q) \in E$. Then $(p,q) \leq^* (f,x)$. QED

We now introduce forcing (so called weak forcing) in a convenient form for our purposes. We use the arithmetical language L[0,prim(F),R], where prim(F) is an effective listing of all function symbols with primitive recursion names, whose intended meanings are the functions primitive recursive in the unknown function F. Here F is a unary function symbol, and R is a binary relation symbol. We use variables $v_0,v_1,\ldots$ over N, connectives $\neg,\wedge,\vee$, quantifiers $\forall,\exists$, and =.

For $x \in N\#\#$, we define $BIN(x) \subseteq N^2$ by $BIN(x)(n,m) \leftrightarrow x_m(n) = 1$.

The only interpretations of L[0,prim(F),R] that we consider are of the form (N,0,prim($\alpha$),BIN(x)), where $\alpha$:N → N, x ∈ N##, and prim($\alpha$) lists the functions primitive recursive in $\alpha$, according to the primitive recursion names used in prim(F).

We inductively define forcing as follows. Let p ∈ C[∞], $\alpha$:N → N and $n_0,n_1,\ldots$ ∈ N be eventually constant.

p ||= s = t[$\alpha,n_0,n_1,\ldots$] if and only if (N,0,prim($\alpha$)) |= s = t[$n_0,n_1,\ldots$].

p ||= R(s,t)[$\alpha,n_0,n_1,\ldots$] if and only if $p_j(i) = 1$, where i,j are the values of s,t, respectively, in the structure (N,0,prim($\alpha$)), at the assignment $n_0,n_1,\ldots$ .

p ||= ¬$\varphi$[$\alpha,n_0,n_1,\ldots$] if and only if for no q ≥* p does q ||= $\varphi$[$\alpha,n_0,n_1,\ldots$].

p ||= $\varphi \wedge \psi$[$\alpha,n_0,n_1,\ldots$] if and only if p ||= $\varphi$[$\alpha,n_0,n_1,\ldots$] ∧ p ||= $\psi$[$\alpha,n_0,n_1,\ldots$].

p ||= $\varphi \vee \psi$[$\alpha,n_0,n_1,\ldots$] if and only if for all q ≥* p there exists r ≥* q such that r ||= $\varphi$[$\alpha,n_0,n_1,\ldots$] ∨ r ||= $\psi$[$\alpha,n_0,n_1,\ldots$].

p ||= ($\forall v_i$)($\varphi$)[$\alpha,n_0,n_1,\ldots$] if and only if for all m ∈ N, p ||= $\varphi$[$\alpha,n_0,\ldots,n_{i-1},m,n_{i+1},\ldots$].

p ||= ($\exists v_i$)($\varphi$)[$\alpha,n_0,n_1,\ldots$] if and only if for all q ≥* p there exists r ≥* q and m ∈ N such that r ||= $\varphi$[$\alpha,n_0,\ldots,n_{i-1},m,n_s,\ldots$].

For the first connection between forcing and truth, it is convenient to use a strengthening of $\alpha$ genericity. Let $\alpha$:N → N. We say that x ∈ N## is strongly $\alpha$ generic if and only if for all n, x is $\alpha^{(n)}$ generic.

We state the key Lemmas without proof, as they are well known.

THEOREM 3.4. Let $\alpha$:N → N, $\varphi$ in L[0,prim(F),R], x be strongly $\alpha$ generic, and $n_0,n_1,\ldots$ ∈ N be eventually constant. (N,0,prim($\alpha$),BIN(x)) |= $\varphi$[$n_0,n_1,\ldots$] if and only if ($\exists$p ≤* x)(p ||- $\varphi$[$\alpha,n_0,n_1,\ldots$]).

THEOREM 3.5. Let $\alpha$:N → N, $\varphi$ in L[0,prim(F),R], and $n_0,n_1,\ldots$ ∈ N be eventually constant. p ||- $\varphi$[$\alpha,n_0,n_1,\ldots$] if and only if for all strongly $\alpha$ generic x, p $\leq^*$ x $\rightarrow$ (N,0,prim($\alpha$),BIN(x)) |= $\varphi[\alpha,n_0,n_1,...]$.

In the proofs of Theorems 3.4, 3.5, we use that forcing for a given $\alpha,\varphi$ is arithmetic in $\alpha$.

We now use Theorem 3.5 to get complexity information as follows. For $\varphi$ in L[0,prim(F),R], we define c($\varphi$) as the least number of alternations of quantifiers of any $\psi$ in L[0,prim(F),R], where

i. $\psi$ is in prenex form, and the free variables of $\psi$ are among the free variables of $\varphi$.
ii. $RCA_0$ proves that $\varphi \leftrightarrow \psi$ holds in all interpretations (N,0,prim($\alpha$),BIN(x)) of L[0,prim(F),R].

Note that c($\varphi$) depends only on $\psi$, and not on any function $\alpha$:N $\rightarrow$ N. We say that $\psi$ is a reduced form of $\varphi$ if the above holds.

THEOREM 3.6. Let $\alpha$:N $\rightarrow$ N. The relation p ||- $\varphi[\alpha,n_0,n_1,...]$ with $\alpha$ fixed, and p,$\varphi$,$n_0,n_1,...$ varying, subject to the restriction that c($\varphi$) $\leq$ k, and $n_0,n_1,...$ be eventually constant, is recursive in $\alpha^{(k+1)}$.

Proof: Let $\alpha$:N $\rightarrow$ N. This is clear if we further restrict $\varphi$ to be prenex with at most k quantifiers. In the forcing, each universal quantifier gives rise to a single universal quantifier, and each existential quantifier gives rise to a universal quantifier followed by two existential quantifiers. Thus the forcing relation in question, under this restriction on $\varphi$, is recursive in $\alpha^{(k+1)}$.

Now every c($\varphi$) $\leq$ k is equivalent to a reduced form $\psi$, no matter what $\alpha$,BIN(x),,$n_0,n_1,...$ is used, and $\psi$ is prenex with at most k quantifiers. Hence by Theorem 3.5, forcing for $\varphi,\psi$ are equivalent. Furthermore, there is an obvious procedure for constructing reduced forms that is recursive in $0^{(1)} \leq_T \alpha^{(k+1)}$. QED

Armed with Theorem 3.6, we now go back and refine Theorems 3.4, 3.5.

THEOREM 3.7. Let $\alpha$:N $\rightarrow$ N, $\varphi$ in L[0,prim(F),R], c($\varphi$) $\leq$ k, x be $\alpha^{(k+2)}$ generic, and $n_0,n_1,... \in$ N be eventually constant. (N,0,prim($\alpha$),BIN(x)) |= $\varphi[n_0,n_1,...]$ if and only if ($\exists$p $\leq^*$ x)(p ||- $\varphi[\alpha,n_0,n_1,...]$).

THEOREM 3.8. Let $\alpha:N \to N$, $\varphi$ in L[0,prim(F),R], $c(\varphi) \leq k$, and $n_0,n_1,... \in N$ be eventually constant. p ||- $\varphi[\alpha,n_0,n_1,...]$ if and only if for all $\alpha^{(k+2)}$ generic x, p ≤* x → (N,0,prim($\alpha$),BIN(x)) |= $\varphi[\alpha,n_0,n_1,...]$.

Here $\alpha^{(k+2)}$ comes about from the use of certain dense sets of conditions, defined in terms of forcing, with an added existential quantifier. Since by Theorem 2.6, the relevant forcing is recursive in $\alpha^{(k+1)}$, the dense sets used are recursive in $\alpha^{(k+2)}$.

Theorem 3.8 uses a stronger notion of complexity than is required (involving provability in $RCA_0$), and the k+2 is also much sharper than what is needed. But some complexity notion and upper bound are needed. Theorem 3.8 is in a readily usable form, even though it is considerably stronger than what we need.

## 4. Cohen generic constructions.

We fix a strictly increasing recursive function $\gamma:N \to N$. We need $\gamma$ to be reasonably fast growing. For example $\gamma(0) = 8$, $\gamma(k+1) \geq \gamma(k)^2$ easily suffices.

LEMMA 4.1. Let $\alpha \in 2^N$. There exist $f_0,f_1,... \in 2^N$ such that for all $1 \leq i \leq k < \infty$,
i. $f_i \leq_T \alpha^{(\gamma(2i-1))}$.
ii. $(f_0,f_i,...,f_k)$ is $\alpha^{(\gamma(2i-2))}$ generic.
iii. $(f_0,f_1,...) \leq_T \alpha^{(\omega)}$.

Proof: Set $f_1 \leq_T \alpha^{(\gamma(1))}$ to be $\alpha^{(\gamma(0))}$ generic. Suppose $f_1,...,f_k$ have been defined, where for all $1 \leq i \leq k$, $f_i \leq_T \alpha^{(\gamma(2i-1))}$ is $\alpha^{(\gamma(2i-2))}$ generic. Set $f_{k+1} \leq_T \alpha^{(\gamma(2k+1))}$ to be $\alpha^{(\gamma(2k))}$ generic.

Let $i \geq 1$. We prove by induction on $k \geq i$ that $(f_i,...,f_k)$ is $\alpha^{(\gamma(2i-2))}$ generic.

Suppose $(f_i,...,f_k)$ is $\alpha^{(\gamma(2i-2))}$ generic, where $k \geq i$. Now $f_{k+1}$ is $\alpha^{(\gamma(2k))}$ generic and $f_i,...,f_k \leq_T \alpha^{(\gamma(2k-1))}$. Hence $f_{k+1}$ is $(\alpha^{(\gamma(2i-2))},f_i,...,f_k)$ generic. By Lemma 2.3, $(f_i,...,f_k,f_{k+1})$ is $\alpha^{(\gamma(2i-2))}$ generic.

By standard recursion theoretic uniformity considerations, the natural implementation of the above has $(f_1,f_2,...) \leq_T 0^{(\omega)}$.

We now choose $f_0$ such that for all $1 \leq i \leq k < \infty$, $(f_0,f_i,...,f_k)$ is $\alpha^{(\gamma(2i-2))}$ generic. This requires that $f_0$ meet a schedule of dense sets, the whole of which is recursive in $\alpha^{(\omega)}$. Therefore we can find $f_0 \leq_T 0^{(\omega)}$. QED

LEMMA 4.2. Let $\alpha \in 2^N$ and $f_0,f_1,\ldots \in 2^N$. Suppose for all $k \geq 0$, $(f_0,\ldots,f_k)$ is $\alpha$ generic. There exists $g_0,g_1,g_2,\ldots \in 2^N$ such that
a. For all $i \geq 0$, $\{n: f_i(n) \neq g_i(n)\}$ is finite.
b. $(g_0,g_1,\ldots)$ is $\alpha$ generic.
c. $f_0 = g_0$.
Furthermore, $(g_0,g_1,\ldots)$ can be obtained recursively from $(\alpha^{(2)},f_0,f_1,\ldots)$, uniformly in $(f_0,f_1,\ldots)$ and $\alpha$.

Proof: Let $\alpha,f_0,f_1,\ldots$ be as given. Let $E_0,E_1,\ldots \subseteq C[\infty]$ enumerate all dense sets recursive in $\alpha$. We set $g_0 = f_0$ and $p_0 \leq^* g_0$. Suppose $g_0,\ldots,g_k$ and $p_0 \leq^* \ldots \leq^* p_k \in C[\infty]$ have been defined, so that for all $0 \leq i \leq k$,

i. $\{n: f_i(n) \neq g_i(n)\}$ is finite.
ii. $res(p_i;i) \leq^* (g_0,\ldots,g_i)$.
iii. If $i \geq 1$ then each $E_i$ has an element $\leq^* p_i$.
iv. $f_0 = g_0$.

We continue by defining $g_{k+1}$ and $p_{k+1} \geq^* p_k$ such that $g_0,\ldots,g_{k+1}$ and $p_0 \leq^* p_1 \leq^* \ldots \leq^* p_{k+1}$ have properties i - iii.

By Lemma 3.2, $(g_0,\ldots,g_k,f_{k+1})$ is $\alpha$ generic. Clearly $A = \{res(q;k): q \in E_{k+1} \wedge p_k \leq^* q\} \subseteq C[k]$ is dense and r.e. in $\alpha$. By Lemma 3.1, let $A' \subseteq A$ be dense, where $A' \leq_T \alpha$. Let $res(q;k) \leq (g_0,\ldots,g_k)$, $q \in E_{k+1}$, $p_k \leq^* q$. Set $g_{k+1}$ to be $f_{k+1}$ changed at finitely many places so that $res(q;k+1) \leq^* (g_0,\ldots,g_k,g_{k+1})$. Set $p_{k+1} = q$.

We have defined $g_0,g_1,\ldots \in 2^N$ and $p_0 \leq^* p_1 \leq^* \ldots$ with properties i - iii for all $i \geq 0$. It is clear by iii that the limit of the $p_i$ is an $\alpha$ generic element of N##. It is clear by ii that this limit must be $(g_0,g_1,\ldots)$. QED

LEMMA 4.3. Let $\alpha \in 2^N$. There exist $f_0,f_1,\ldots \in 2^N$ such that for all $i \geq 1$,
i. $f_i \leq_T \alpha^{(\gamma(2i-1))}$.
ii. $(f_0,f_i,f_{i+1},\ldots)$ is $\alpha^{(\gamma(2i-2))}$ generic.
iii. $(f_0,f_1,\ldots) \leq_T \alpha^{(\omega)}$.

Proof: Let $\alpha \in 2^N$. We start with the $f_0,f_1,\ldots$ as given by Lemma 4.1. Since the finite initial segments are $\alpha^{(\gamma(0))}$ generic, we can apply Lemma 4.2. Make finite changes to each of $f_1,\ldots$ so that the resulting $f_0,f_1',\ldots$ is $\alpha^{(\gamma(0))}$

generic. By Lemmas 3.2, 4.1, the finite initial segments of $f_0, f_2', f_3', \ldots$ are $\alpha^{(\gamma(2))}$ generic. By Lemma 4.2, make finite changes to each term so that the resulting $f_0, f_2'', f_3'', \ldots$ is $\alpha^{(\gamma(2))}$ generic. Continue in this way for infinitely steps, resulting in $(f_0, f_1', f_2'', \ldots)$, where the term with index i differs from the original $f_i$ at only finitely many places. The uniformity in Lemma 4.2 guarantees that $(f_0, f_1', f_2'', \ldots) \leq_T (f_0, f_1, \ldots) \leq_T \alpha^{(\omega)}$. QED

LEMMA 4.4. Let $\alpha \in 2^N$. There exist $f_0, f_1, \ldots \in 2^N$ such that
i. Lemma 4.3 holds.
ii. There are continuumly many $h \in 2^N$ such that for all $i \geq 1$, $(h, f_i, f_{i+1}, \ldots)$ is $\alpha^{(\gamma(2i-2))}$ generic.

Proof: Let $\alpha \in 2^N$, and let $f_0, f_1, \ldots \in 2^N$ be given by Lemma 4.3. If h is $\alpha^{(\omega)}$ generic then h is as required in ii. QED

## 5. Unique weak second order undefinable elements.

DEFINITION 5.1. Weak second order logic extends first order logic with quantifiers over finite relations (i.e., with relations with finitely many tuples) of fixed arity on the domain.

Weak second order logic is substantially stronger than first order logic, but dramatically weaker than second order logic. See section 7.

Obviously, any unique undefinable element that is weak second order undefinable is also the unique second order undefinable element.

In section 2, we focused on the condition: there is a unique undefinable element. In this section, we will focus on the stronger condition: the unique undefinable element is weak second order undefinable.

Note that this stronger condition represents a strengthening of the contrast between the unique undefinable element and the other elements.

In the [Fu68] examples, and our examples from section 2, the unique undefinable element is weak second order undefinable. This follows from the remarks at the end of section 2, and the following observation.

THEOREM 5.1. In any finite relational type, for any $n \geq 0$, any unique $\Sigma_n$* undefinable element is weak second order definable.

Proof: Let M,n be as hypothesized. We show that the set of all $\Sigma_n$* definable $x \in dom(M)$ is defined by a weak second order formula over M without parameters. We can treat blocks of like quantifiers by finite sets, although there is the problem that we really need finite sequences. This can be handled easily by treating blocks of like quantifiers by finite relations. Since n is standard, this involves a standard number of alternating weak second order quantifiers.

We do have to handle the quantifier free part. We can assume that all quantifier free parts are in disjunctive normal form. We have access to arithmetic on finite initial segments of N in the usual way in weak second order logic. Hence we have access to Gödel numbers for terms, atomic formulas, multiple conjunctions of atomic and negated atomic formulas, and, finally, multiple disjunctions of conjunctions of atomic and negated atomic formulas. We can define assignments and finite truth sets in weak second order logic.

This creates a weak second order definition of "x is $\Sigma_n$* definable". Therefore the unique $\Sigma_n$* undefinable element, if it exists, must be defined in weak second order logic. QED

We now use the development from sections 3,4 to build N-systems, of finite relational type, whose unique undefinable element is weak second order undefinable. In fact, we build these N-systems so as to interpret any countable structure in a countable relational type.

DEFINITION 5.2. Let $f_0,f_1,\ldots \in 2^N$ be distinct with infinitely many 1's, and $T \subseteq N^2$. Define $J(f_0,f_1,\ldots,T)$ to be the N-system

$$(N \cup \{A_0,A_1,\ldots\},\varepsilon,+,\bullet,T)$$

where each $A_i = \{n: f_i(n) = 1\}$, $+,\bullet$ are the addition and multiplication functions on N, with the default value 0 if not all arguments lie in N. Fix $\alpha \in 2^N$, where $\alpha =_T T$.

LEMMA 5.2. There exists a strictly increasing recursive function $\gamma:N \to N$ such that the following holds. Let $\varphi(v)$ be a formula in weak second order logic, in the language

$\varepsilon,+,\bullet,T$, with at most $k \geq 4$ quantifiers. There exists a (forcing) sentence $\psi$ in $L(0,prim(\alpha),R)$, $c(\psi) \leq \gamma(2k-2)-8$ such that the following holds. Let $f_0,f_1,... \in 2^N$, where $f_1,...,f_{k-1} \leq_T \alpha^{(\gamma(2k-3))}$. Then $\varphi(f_0)$ holds in $J(f_0,f_1,...,T)$ if and only if $\psi$ holds in $(N,0,prim(\alpha),BIN(f_0,f_k,f_{k+1},...))$.

Proof: This is a straightforward transfer from weak second order statements $\varphi(f_0)$ over the structure $J(f_0,f_1,...,T)$ to corresponding statements in the forcing structure $(N,0,prim(\alpha),BIN(f_0,f_k,f_{k+1},...))$, where $f_1,...,f_{k-1}$ are fixed, and $f_0,f_k,f_{k+1},...$ vary. The complexity of the resulting forcing statement is bounded solely in terms of the complexity bound for $\varphi$ and the recursion theoretic complexity of the fixed $f_1,...,f_{k-1}$. QED

We now fix $\gamma:N \to N$ as given by Lemma 5.2. We also fix $f_0,f_1,... \in 2^N$ as given by Lemma 4.4. Fix $M = J(f_0,f_1,...,T)$.

LEMMA 5.3. In $J(f_0,f_1,...,T)$, every element other than $f_0$ is definable.

Proof: 0 is definable in $J(f_0,f_1,...,T)$ by $x = 0 \leftrightarrow x+x = x$, and 1 is definable by $x = 1 \leftrightarrow x\bullet x = x \wedge x \neq 0$. From $0,1,+$, every element of N is definable in $J(f_0,f_1,...)$. Also N is 0-definable over M by $x \in N \leftrightarrow x+0 = x$. By standard techniques, $\alpha$ is 0-definable over $(N,+,\bullet,T)$, and hence $\alpha$ is 0-definable over $J(f_0,f_1,...,T)$. Since the functions $f_1,f_2,...$ are each arithmetical in $\alpha$, they are each 0-definable as functions over M. Therefore, $A_0,A_1,...$ are each definable in M as elements, using $\varepsilon$. QED

LEMMA 5.4. The element $f_0$ is not definable in $J(f_0,f_1,...,T)$ in weak second order logic.

Proof: Let $f_0$ be defined in M as a point, by the weak second order logic formula $\varphi(v)$, with at most $k \geq 4$ quantifiers (including both first and weak second order quantifiers). Let $\psi$ be the sentence given by Lemma 5.2. We have $c(\psi) \leq \gamma(2k-2)-8$.

By Lemma 4.3, $f_1,...,f_{k-1} \leq_T \alpha^{(\gamma(2k-3))}$, and $(f_0,f_k,f_{k+1},...)$ is $\alpha^{(\gamma(2k-2))}$ generic.

By assumption, $\varphi(f_0)$ holds in $J(f_0,f_1,...,T)$. By Lemma 4.3, $\psi$ holds in $(N,0,prim(\alpha),BIN(f_0,f_k,f_{k+1},...))$. By Theorem 3.7, let $p \in C[\infty]$, $p \leq^* (f_0,f_k,f_{k+1},...)$, $p \mid\mid= \psi[\alpha,0,0,...]$. Note that $f_0,f_k,f_{k+1},...$ forms a dense subset of $2^N$. Let $n \geq k$ be such that $p \leq^* (f_n,f_k,...,f_{n-1},f_0,f_{n+1},...)$.

By Lemma 3.2, $(f_n,f_k,...,f_{n-1},f_0,f_{n+1},...)$ is also $\alpha^{(\gamma(2k-2))}$ generic. By Theorem 3.8, $\psi$ holds in $(N,0,prim(\alpha),BIN(f_n,f_k,...,f_{n-1},f_0,f_{n+1},...))$. By Lemma 5.2, $\varphi(f_n)$ holds in $J(f_n,f_1,...,f_{n-1},f_0,f_{n+1},...,T)$. But $J(f_0,f_1,...,T)$ and $J(f_n,f_1,...,f_{n-1},f_0,f_{n+1},...,T)$ are identical. Therefore $\varphi(f_n)$ holds in $J(f_0,f_1,...,T)$. This contradicts that $f_0$ is defined over $J(f_0,f_1,...,T)$ by $\varphi(v)$, since $f_n \neq f_0$. QED

THEROEM 5.5. Let $T \subseteq N^2$. There exists an infinite sequence $A_0,A_1,... \subseteq N$ without repetition, $(A_0,A_1,...) \leq_T T^{(\omega)}$, where in the N-system $(N \cup \{A_0,A_1,...\},\varepsilon,+,\bullet,T)$, the unique undefinable element $A_0$ is weak second order undefinable. Furthermore, there are continuumly many $B \subseteq N$ such that the replacement of A0 with B results in an N-system where the unique undefinable element B is weak second order undefinable, the weak second order theory remains unchanged, and the structure is not isomorphic to the original.

Proof: Let $\alpha \in 2^N$, $\alpha =_T T$, and $f_0,f_1,...$ be given by Lemma 4.3. Set $M = (N \cup \{A_0,A_1,...\},\varepsilon,+,\bullet,T) = J(f_0,f_1,...,T)$. The first claim follows from Lemmas 5.3, 5.4. For the second claim, change f0 to any h according to Lemma 4.4. Let $\varphi$ be a weak second order sentence in the language of M that holds in $J(f_0,f_1,...,T)$. As in the proof of Lemma 5.4, let k be large, and look at "$\varphi$ holds in $J(f_0,f_1,...,T)$" as a forcing statement about the sufficiently generic $(f_0,f_k,f_{k+1},...)$ over $(f_1,...,f_{k-1})$. Let $p \leq^* (f_0,f_k,f_{k+1},...)$ force "$\varphi$ holds in $J(f_0,f_1,...,T)$". Using the denseness of $f_0,f_k,f_{k+1},...$, let $n \geq k,lth(p)+2$, so that $p \leq^* (f_n,f_k,f_{k+1},...,f_{n-1},h,f_{n+1},...)$. Since $(f_n,f_k,f_{k+1},...,f_{n-1},h,f_{n+1},...)$ is sufficiently generic over $(f_1,...,f_{k-1})$, we have "$\varphi$ holds in $J(f_n,f_1,...,f_{n-1},h,f_{n+1},...)$". But $J(f_n,f_1,...,f_{n-1},h,f_{n+1},...) = J(h,f_1,f_2,...)$, and we are done. QED

In the next section, we convert to N-systems $(N \cup \{A_0,A_1,...\},\varepsilon)$ and bipartite graphs and atomic inclusions.

## 6. Bipartite graphs and atomic inclusions.

We first reduce the N-systems from section 5 to pure N-systems. The general process uses the 0-definability of < on N, and so it cannot be directly used for the section 2 examples. Recall that reductions were already given in Theorems 2.7 - 2.9, avoiding any use of <.

DEFINITION 6.1. For each $k \geq 1$, let $\sigma_1,\ldots,\sigma_p$ be an enumeration without repetition of the order types of k-tuples. Clearly $p \leq k^k$. We define $code(n_1,\ldots,n_k) = \{r,\ldots,r+4,n_1+r+5,\ldots,n_k+r+5\}$, where the order type of $n_1,\ldots,n_k$ is $\sigma_r$.

Note that $code(n_1,\ldots,n_k)$ is of cardinality > 5. This will be useful.

DEFINITION 6.2. Let M be an N-system $(N \cup \{A_0,A_1,\ldots,\},\varepsilon,T_1,\ldots,T_n)$. $M\#[A_0,A_1,\ldots] = (N \cup K \cup \{A_0,A_1,\ldots\},\varepsilon)$, where K is defined as follows. First let $T_1',\ldots,T_n'$ be the adjustment of $T_1,\ldots,T_n$, where we replace the items that are functions by the corresponding relation on N. Now let $T = T_1' \times \ldots \times T_n'$. If $k = 0$, set $T = N$. K consists of the sets

1. $\{n,n+1\}$, $n \in N$.
2. $\{n,n+1,m,m+1,m+2)$, where $n+3 \leq m$.
3. $code(n_1,\ldots,n_k)$, where $(n_1,\ldots,n_k) \in T$.

LEMMA 6.1. Let M be an N-system $(N \cup \{A_0,A_1,\ldots\},\varepsilon,T_1,\ldots,T_n)$, where the A's are infinite and distinct, and < on N is 0-definable over M. $N,\{A_0,A_1,\ldots\},T_1,\ldots,T_n$ are 0-definable over M. Every element of N is definable in M. N, and < on N, are 0-definable over M#. Every element of $N \cup K$ is definable in M. $N,K,(A_0,A_1,\ldots\}$ are 0-definable over M#. $T_1,\ldots,T_n$ are 0-definable over M#. For all $i \geq 0$, $A_i$ is definable (weak second order definable) in M# if and only if $A_i$ is definable (weak second order definable) in M. The weak second order theories of M,M# are mutually recursively reducible.

Proof: Let M be as given. The first two claims are about M, and are immediately verified. In the next six paragraphs, all definability is in M# and all 0-definability is over M#.

Note that $|n-m\} = 1 \leftrightarrow (\exists x)$(x has exactly the $\varepsilon$ predecessors n,m). We refer to this condition as adjacency.

$n+2 \leq m \leftrightarrow (\exists x)$(x has exactly 5 predecessors a,b,c,d,e, the only adjacent pairs being {a,b},{c,d},{d,e}, and $n = b \wedge m = c$).

$n < m \leftrightarrow (\exists r)(|m-r| = 1 \wedge m+2 \leq m)$.

Hence < on N is 0-definable and so N is 0-definable. Also every element of N ∪ K is definable, since all elements of K have finitely many ε predecessors.

K is 0-definable as the set of all elements with a largest ε predecessor (this uses that the A's are infinite). Since N is 0-definable, $\{A_0,A_1,...\}$ is 0-definable.

Note that by the set of codes thrown into K, T is also 0-definable. Hence $T_1,...,T_n$ are 0-definable.

For the equivalence, note that we have 0-defined M over M#. Hence 0-definability over M implies 0-definability over M#.

For the converse, Definition 6.2 provides an interpretation of M# over M without parameters, where the new elements thrown in are tuples of standard integer lengths that represent "sets" of standard integer size, and are defined in terms of < on N. Hence 0-definability over M implies 0-definability over M#.

These mutual interpretations are also clearly sufficient for mutually reductions of the weak second order theories of M,M#. QED

THEOREM 6.2. Let $T_1,T_2,...$ be relations on N. There exists an infinite sequence without repetition, $A_0,A_1,... \subseteq N$, $(A_0,A_1,...) \leq_T (T_1,T_2,...)^{(\omega)}$, where the unique undefinable element $A_0$ in $(N \cup \{A_0,A_1,...\},\varepsilon)$ is weak second order undefinable, and $T_1,T_2,...$ are 0-definable in $(N \cup \{A_0,A_1,...\},\varepsilon\}$. Furthermore, there are continuumly many $B \subseteq N$ such that the replacement of $A_0$ with B results in an N-system where the unique undefinable element B is weak second order undefinable, the weak second order theory remains unchanged, and the structure is not isomorphic to the original.

Proof: Let $T \subseteq N^2$ be such that $(T_1,T_2,...) =_T T$. Then $T_1,T_2,...$ are 0-definable over $(N,+,\bullet,T)$. Let $M = (N \cup \{A_0,A_1,...\},\varepsilon,+,\bullet,T)$ be given by Theorem 5.5. The required pure N-system is M#. Apply Theorem 6.1. QED

We now convert pure N-systems to a bipartite graphs and atomic inclusions. Here we will not be using <, and so the reduction applies to examples from section 2.

There are a few essentially equivalent definitions of bipartite graph. Most convenient for present purposes is the following.

DEFINITION 6.3. A bipartite graph is a triple (A,B,R), where $A \cap B = \emptyset$, $A,B \neq \emptyset$, and $R \subseteq A \times B$.

Model theoretically, we can view (A,B,R) as a two sorted structure, with domains A,B, together with a binary relation R from A to B. We can also equivalently treat this as a one sorted structure $(A \cup B,A,B,E)$, where A,B are unary relations on $A \cup B$, and R is a binary relation on $A \cup B$.

DEFINITION 6.4. Let M be a pure N-system. M(bpg) is the bipartite graph $(N,\{A_0,A_1,...\},\varepsilon)$.

DEFINITION 6.5. An inclusion is a family K of nonempty sets under $\subseteq$. An atomic inclusion is an inclusion V, where every element of V is the union of singletons lying in V.

Obviously, an atomic inclusion is a special kind of poset (with a kind of extensionality).

DEFINITION 6.6. Let M be a pure N-system. M(ati) is the atomic inclusion $\{A_0,A_1,...\} \cup \{\{n\}: n \in \cup_i A_i\}$ under $\subseteq$.

LEMMA 6.3. Let M be a pure N-system, where the A's are nonempty and have union N. For all $x \in dom(M)$ and $n \geq 1$, x is definable ($\Sigma_2$* definable) in M(bpg) if and only if x is definable ($\Sigma_2$* definable) in M. The first order (weak second order) theories of M and M(bpg) are mutually recursively reducible.

Proof: Let M be as given. Then N is $\Sigma_1$ and $\Pi_1$ 0-definable over M, as "having an $\varepsilon$ successor" and "having no $\varepsilon$ predecessor". This 0-defines M(bpg) over M in both $\Sigma_1$ and $\Pi_1$ form. QED

LEMMA 6.4. Let M be a pure N-system, where the A's each have at least two elements, and $\cup_i A_i = N$. For all $x \in N$, x is definable in M if and only if $\{x\}$ is definable in M(ati). For all $i \geq 0$, $A_i$ is definable in M if and only if $A_i$ is definable in M(ati). The first order (weak second order) theories of M and M(ati) are mutually recursively reducible.

Proof: Let M be as given. Then N is 0-definable over M, and so we can 0-define an isomorphic copy of M(ati) over M. Also, sitting in M(ati), we can 0-define an isomorphic copy of M by interpreting N as the minimal elements, $\{A_0,A_1,...\}$ as the rest of the elements, and x ε y as "x is minimal and x ⊆≠ y". This is sufficient to establish the claims. QED

LEMMA 6.5. Let M be a pure N-system arising from the proof of Theorem 2.10. For all n ∈ N, {n} is $\Sigma_2$* definable in M(ati). For all i ≥ 1, $A_i$ is $\Sigma_2$* definable in M(ati). $A_0$ is undefinable in M(ati). M(ati) has recursive theory.

Proof: Let M be as given. The last two claims are clear because we can 0-define an isomorphic copy of M(ati) over M using that N is 0-definable over M.

For the first two claims, we can obviously 0-define an isomorphic copy of M in M(ati) in the obvious way, but x ε y is not interpreted as x ⊆≠ y, but rather as (x ⊆≠ y ∧ x is minimal). Thus the interpretation of ε in M(ati) introduces a universal quantifier.

In M(ati), we define {0} by x = {0} ↔ (∃y,z)(x ⊆≠ y ⊆≠ z ∧ z has at most two distinct proper subsets), giving a $\Sigma_2$* definition of {0} in M(ati).

We now go back to the proof of Theorem 2.9. We repeat these definitions using the $\Pi_1$* interpretation of ε. QED

THEOREM 6.6. Let $T_1,T_2,...$ be relations on N. There exists a bipartite graph (N,B,R) where the unique undefinable element is weak second order undefinable, and $T_1,T_2,...$ are 0-definable over (N,R,R). There exists an atomic inclusion (V,⊆) where the unique undefinable element is weak second order undefinable, and $(N,T_1,T_2,...)$ is interpretable in (V,⊆) without parameters.

Proof: By Theorem 6.2 and Lemmas 6.3, 6.4. QED

THEOREM 6.7. There exists a recursive bipartite graph where the unique $\Sigma_2$* undefinable element is undefinable, and the theory is recursive. There is a recursive atomic inclusion where the unique $\Sigma_2$* undefinable element is undefinable, and the theory is recursive.

Proof: By Theorem 2.9 and Lemma 6.5. QED

## 7. Unique second order undefinable elements.

A structure is said to be rigid if and only if every automorphism is the identity.

LEMMA 7.1. Every countable structure in a countable relational type is the unique structure up to isomorphism satisfying some sentence $\varphi$ of the infinitary language $L_{\omega\_1\omega}$. In every rigid countable structure in a countable relational type, every element is defined by some formula $\psi(v)$ of $L_{\omega\_1\omega}$.

Proof: For the first claim, see [Sc64], [Sc65]. For the second claim, let M be countable and rigid, and let d $\in$ dom(M). Let $\varphi(c)$ be such that (M,d) is the only structure satisfying $\varphi$ up to isomorphism, where c is a new constant symbol. For the definition of d in M, we use $\varphi(v)$, the result of replacing c by the variable v. Suppose $\varphi(d')$ holds in M. Then $\varphi(c)$ holds in (M,d'). Therefore d' = d. QED

THEOREM 7.2. Assume ZFC + "there exists a parameterless $\Sigma^1_\infty$ well ordering of $\wp$(N)". In every countable rigid structure in a finite relational type, every element is second order definable.

Proof: Assume there exists a $\Sigma^1_k$ well ordering R of $\wp$(N). Let M be a countable rigid structure in a finite relational type, with domain D. If D is finite, then every element of D is definable, and we are done. So we assume that D is countably infinite.

We first place a well ordering on M that is second order definable without parameters over M. I.e., second order 0-definable over M.

For $d_1,d_2 \in$ D, we write $d_1 <^* d_2$ if and only if there is a code (subset of N) for a countable infinitary formula that defines $d_1$, that is an R predecessor of every code for a countable infinitary formula that defines $d_2$.

To see that <* is second order 0-definable over M, we can add coding apparatus for countable infinitary formulas to M, by first adding a well ordering P of D with no limit point, then arithmetic with respect to this well ordering P of type $\omega$. We can then code countable infinitary formulas by codes for decorated well founded trees, and unique satisfaction relations that apply these formulas to M. We can use the arithmetic and copy of $\omega$ via P, to compare these codes using R via a second order definition. This process of comparison is clearly independent of the choice of P. So we merely have to ask for some P for which the comparison in question holds.

From <*, we obviously have a well ordering T of type $\omega$ whose domain is some $A \subseteq D$, which is second order 0-definable over M. I.e., let T be the restriction of <* to the points in <* with only finitely many predecessors in <*. We use (A,T) as a copy of $\omega$, for use with R.

We let $W \subseteq A^2$ be R least with the following property (for this purpose, we use a standard pairing function on A). The cross sections $W_a$ include a code for a countable infinitary definition of every $d \in D$ over M. Then it is clear that W is second order 0-definable over M.

Let $d \in D$. A second order definition of d in M can be recovered easily from W, by choosing $b \in A$ such that $W_b$ is a code for a countable infinitary definition of d over M. The second order 0-definition of d is described by taking the code $W_b$ and apply it to M to obtain d. This second order definition has only the parameter b. However, b itself is second order definable over M since $b \in A$ is merely a point in the well ordering T of type $\omega$. QED

LEMAM 7.3. Let M be a structure in any relational type, with a unique second order undefinable element. Then M is rigid.

Proof: Let M be as given. Suppose $f:M \to M$ is an isomorphism where $f(d) \neq d$. Then d,f(d) are not second order definable. Hence d,f(d) are second order undefinable. This violates uniqueness. QED

THEOREM 7.4. Assume ZFC + "there exists a parameterless $\Sigma^1_\infty$ well ordering of $\wp(N)$". There is no structure in a finite relational type with a unique second order undefinable element.

Proof: By Lemma 7.3, such a structure is rigid. Now apply Theorem 7.2. QED

THEOREM 7.5. If ZFC is consistent then ZFC does not prove the existence of a structure in a finite relational type with a unique second order undefinable element. If ZFC + "there exists a measurable cardinal" is consistent, then ZFC + "there exists a measurable cardinal" does not prove the existence of a structure in a finite relational type with a unique second order undefinable element.

Proof: If ZFC is consistent then ZFC + V = L is consistent, and so ZFC + "there is a paramterless $\Sigma^1_\infty$ well ordering of $\wp$(N)" is consistent, by [Go40]. Now apply Theorem 7.4. If ZFC + "there exists a measurable cardinal" is consistent then ZFC + "there is a parameterless $\Sigma^1_3$ well ordering of $\wp$(N)" is consistent, by [Si71]. Apply Theorem 7.4. QED

## 8. Dimension preservation.

THEOREM 8.1. Let M be a structure in an arbitrary relational type, with a unique undefinable element. Let $F:dom(M)^n \to dom(M)^m$ be M definable with parameters. If F is onto then $n \geq m$. If F is one-one then $n \leq m$.

Proof: Let M,n,m be as given, with dom(M) = D, and where u is the unique undefinable element. If D is finite then the results are trivial by counting. So we assume that D is infinite.

Suppose the first claim is true. I.e., if $F:dom(M)^n \to dom(M)^m$ is M definable with parameters and onto, then $n \geq m$. Then the second claim is true by the following argument. Let $F:dom(M)^n \to dom(M)^m$ be M definable with parameters and one-one. Define $G:dom(M)^m \to dom(M)^n$ by $G(x) = F^{-1}(x)$ if it exists; $\{u\}^n$ otherwise.

To establish the first claim, we prove a stronger result. Suppose $F_1,\ldots,F_k:D^n \to D^{n+1}$, where $F_1,\ldots,F_k$ are M definable with parameters. Then $\cup_i rng(F_i) \neq D^{n+1}$.

The basis case is n = 1. Let $F_1,\ldots,F_k:D \to D^2$, where $F_1,\ldots,F_k$ are M definable with parameters, where $\cup_i rng(F_i) = D^2$. All of the parameters used must be M definable except u. So we can assume that u is the only parameter used in the definitions. But then we can ask if u can be replaced with another parameter for all of the definitions at once. If the answer is yes, then we don't need any parameters. If the answer is no, then u is M definable, which is impossible.

Hence we have killed all of the parameters, and now assume that $F_1,\ldots,F_k$ are M 0-definable. Each $rng(F_i)$ contains at most one element of $D^2$ that uses u (as a coordinate), since $dom(F_i) = D$. Hence $\cup_i rng(F_i)$ contains at most finitely many elements of $D^2$ that use u. But since D is infinite, there are infinitely many elements of $D^2$ that use u. This is a contradiction.

Now suppose that the statement is true for fixed $n \geq 1$. Let $G_1,...,G_r:D^{n+1} \to D^{n+2}$, where $G_1,...,G_r$ are M definable with parameters, where $\cup_i rng(G_i) = D^{n+2}$. As above, we kill all of the parameters, and assume that $G_1,...,G_r$ are M definable without parameters.

We claim that $G_1,...,G_r$ map the $x \in D^{n+1}$ using u onto the $y \in D^{n+2}$ using u. This is because $G_1,...,G_r$ map the $x \in D^{n+1}$ not using u into the $y \in D^{n+2}$ not using u, and $G_1,...,G_r$ is onto.

Now the $G_1,...,G_r$ give rise to maps $H_1,...,H_t$ defined on $D^n$ by fixing various coordinates to be u. Here $H_1,...,H_t$ are M definable with parameter u, and $\cup_i rng(H_i)$ includes $D^{n+1} \times \{u\}$. Thus we obtain maps $J_1,...,J_t:D^n \to D^{n+1}$, M definable with parameter u, where $\cup_i rng(J_i) = D^{n+1}$. This contradicts the induction hypothesis. QED

THEOREM 8.2. Let M be a structure in an arbitrary relational type, with a unique undefinable element. Let $F_1,...,F_r:dom(M)^n \to dom(M)^{n+1}$ be M definable with parameters. Then $\cup_i F_i \neq dom(M)^{n+1}$.

Proof: This was established in the proof of Theorem 8.1. QED

THEOREM 8.3. Let M be a structure in an arbitrary relational type, with a unique undefinable element. There is no group operation on dom(M) that is M definable with parameters.

Proof: Let M be as given, with dom(M) = D, where u is the unique undefinable element. Let $\bullet$ be a group operation on D that is M definable with parameters. As in the proof of Theorem 7.1, we can kill parameters, and assume that $\bullet$ is M definable without parameters. Let $d \neq 1,u$. Then $u\bullet d \neq u$, and $u\bullet d,d$ are definable. Hence $u = (u\bullet d)\bullet d^{-1}$ is definable. This is a contradiction. QED

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