THE LOGICAL STRENGTH OF MATHEMATICAL STATEMENTS

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Oct. 15, 1975

We announce a new approach to metamathematics by describing how it applies to what is called "analysis on Euclidean space." Analysis on Euclidean space, generally speaking, is the study of point sets in Euclidean n-space and functions on Euclidean n-space. Though this includes several complex variables and other topics, we shall use the term "real variables."

We have developed a formal system RV_1 whose language is based on variables over the set $\mathbb{R}^*$ of nonempty finite sequences of reals, variables over subsets of $\mathbb{R}^*$, variables over n-ary partial functions on $\mathbb{R}^*$ (each $n > 0$), nonlogical symbols $+, \cdot, -, 1/, |\ |$, lth, 0,1, $=, \in, <$, N, and the special logical symbol D for "being defined." The axioms consist of some simple miscellaneous axioms, equality axioms, explicit definitions, axioms of a normed Archimedian ordered field, definition by finite Σ and Π, induction for sequences of reals, every Cauchy sequence converges, and the limit function of a sequence of functions exists.

Virtually any statement of real variables can be translated in an obvious and direct manner into the language of RV_1. Furthermore, the axioms of RV_1 are unavoidable in any fully rigorous treatment of the necessary basics of real variables. In addition, RV_1 is equiconsistent with Peano arithmetic (PA), and suffices to naturally prove a considerable portion of real variables.

For various assertions A of real variables, we consider the formal system $RV_1 + A^*$ (where A^* is the translation of A into the language of RV_1). We determine the logical strength of $RV_1 + A^*$ in the sense of establishing its equiconsistency with one of the standard formal systems studied by logicians. The ordinal of $RV_1 + A^*$ can be defined outright.

In this way, we say that we have determined the logical strength and ordinal strength of the mathematical assertion A relative to the base theory RV_1. One basic and interesting example is the case of the least upper bound principle. Its logical strength relative to RV_1 is given by $ID_{<\omega}$ (the theory of finitely iterated ~~one~~ generalized inductive definitions), and so we may conclude that no formalism codifying the necessary basics of real variables together with the least upper bound principle can be proved consistent by contemporary predicative methods.

We have discovered that virtually all of the principal recursive ordinals named by logicians are the ordinal strengths of commonplace assertions in real variables relative to RV_1. We have also discovered that logical strength often organizes assertions in real variables into mathematically recognizable groups, so that certain informal distinctions among types of real variables are formally delineated.