THREE QUANTIFIER SENTENCES

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Abstract. We give a complete proof that all 3 quantifier sentences in the primitive notation of set theory (∈,=), are decided in ZFC, and in fact in a weak fragment of ZF without the power set axiom. We obtain information concerning witnesses of 2 quantifier formulas with one free variable. There is a 5 quantifier sentence that is not decided in ZFC (see [Fr02]).

1. Preliminaries.

Daniel Gogol [Go79] presents an argument that all 3 quantifier sentences in the primitive notation of set theory (∈,=) are decided in ZFC. In the author's words, not all of the details are presented:

"It is tedious but involves no difficulty to verify that if...". p. 5, line 14.

"This can be verified by considering all the possible cases, but is quite clear if considered carefully. So we omit what would be a very long verification." p.8, end.

We give a complete proof that all 3 quantifier sentences in set theory based on ∈,=, are decided in ZFC. In fact, we show that all sentences of somewhat higher complexity are decided in a weak fragment T of ZF without the power set axiom. We also give some strong information about witnessing 3 quantifier sentences that begin with an existential quantifier. Our main results are summarized in Theorem 11.1. We do not use ideas from [Go79].

We work entirely within the primitive language of set theory, which is standard predicate calculus with equality and □, using the standard quantifiers □,□, and the standard connectives □,□, ,□ ,□ . We assume the usual complete axioms and rules of classical logic.

We let T be the following weak set theory.

1. Extensionality.
2. Pairing.
3. Union.
4. Infinity.
5. Foundation.
6. Bounded Separation.

Pairing asserts the existence of {a,b}. Union asserts the existence of □a.

The usual formulation of Infinity in ZF is the existence of a set containing ∅ and closed under the operation that sends x to x □ {x}. Because we are only using Bounded Separation, we use the stronger version of infinity that asserts the existence of a set containing ∅ and closed under the operation that sends x,y to x □ {y}.

Foundation asserts that every nonempty set has an epsilon minimal element.

Bounded Separation asserts the existence of {x □ a: □}, where □ is a formula in which all quantifiers are bounded. I.e., all quantifiers are of the forms (□u □ v), (□u □ v), where u,v are distinct variables.

In T, we can prove the existence of a least nonempty transitive set closed under power set, which we write as V(□). In T, we can develop all of the basic facts about V(□) and its elements, as well as define subsets of V(□) and functions on V(□) by recursion.

Since the focus of the paper is on 3 quantifier sentences, we strictly follow the simplifying convention that all formulas will use at most the three distinct variables x,y,z. Letters such as u,v,w are used as metavariables over the official variables x,y,z, or variables used to carry out proof sketches that are to take place in T.

The basic formulas are the atomic formulas that use at most x,y,z, together with their negations.

The $□_0$ formulas are the quantifier free formulas that use at most x,y,z.

We use the notation □, □, □□, □□, □□, □□, □□□, □□□, □□□, □□□, etcetera, to denote the obvious prenex classes of formulas that use at most x,y,z, where the quantifiers, from left to right, read z, yz, or xyz.

We also consider some thickened complexity classes. For example, we write □□□...□□□ for the class of all conjunctions of □□ formulas. We also write □(□□□...□□□) for the class of all single existentially quantified formulas over □□□...□□□. Note that the former have at most the free variable x, and the latter are sentences. We will prove that all sentences in □(□□□...□□□) are decided in T.

We use u inc v for "u,w are incomparable"; i.e., u □ w □ w □ u □ u ≠ w. We use u comp w for "u,w are comparable"; i.e., u □ w   w □ u   u = w. Here u,v,w are metavariables over the variables x,y,z.

The complete conjunctions are the conjunctions of basic formulas, R, where

i) all conjuncts of R use two distinct variables;
ii) the xy part of R is exactly one among x □ y, y □ x, x = y, x inc y;
iii) the xz part of R is exactly one among x □ z, z □ x, x = z, x inc z;
iv) the yz part of R is exactly one among y □ z, z □ y, y = z, y inc z.

Example: x □ y □ x = z □ y inc z is a complete conjunction, which, technically, is an abbreviation for x □ y □ x = z □ y □ z □ z □ y □ y ≠ z. Note that x □ y □ x = z □ y inc z is refutable.

Throughout the paper, all conjunctions and all disjunctions are nonempty. This means that we have to take some care about degenerate cases.

A full conjunction (of basic formulas) is a conjunction that includes exactly one element from each {□,□□}, where □ is atomic. (As always, we only allow the variables x,y,z).

Note the distinction between complete conjunctions and full conjunctions.

We say that a formula □ is decided in T if and only if T proves □ or T proves □□. We say that □ is refuted in T if and only if T proves □□.

We say that two formulas □,□ are equivalent in T if and only if T proves □ □ □. We say that □ implies □ in T if and only if T proves □ □ □.

Note that the notions of “decided” and “equivalent” are particularly strong when applied to formulas with free variables.

LEMMA 1.1. For every □$_0$ formula is equivalent, in T, to a disjunction of complete conjunctions.

Proof: We first show that every full conjunction □ is equivalent to a complete conjunction. If □ is refutable then we are done by taking the complete conjunction x = y □ y = z □ x □ z. We can assume that □ is not refutable, and so □ has each u = u, u □ u. We eliminate them from □, maintaining equivalence in T. Now look at the xy part of □. If it has no positive term then the xy part of □ is equivalent to x inc y. If it has x □ y, then it has no other positive terms, and hence is equivalent to x □ y. If it has y □ x, then it has no other positive terms, and is equivalent to y □ x. It has x = y if and only if it has y = x, in which case it has no other positive terms, and is equivalent to x = y. Argue the same way for the xz part and the yz part.

By disjunctive normal form theory, the □$_0$ formula is either the negation of a tautology or tautologically equivalent to a disjunction of full conjunctions - and hence equivalent, in T, to a disjunction of complete conjunctions. Since there is a complete conjunction that is equivalent, in T, to the negation of a tautology, the proof is complete. QED

2. One existential quantifier.

We refer the reader to the A list of formulas in the Appendix.

LEMMA 2.1. Let □ be from x □ z, z □ x, x = z, x inc z, and □ be from y □ z, z □ y, y = z, y inc z. Then (□z)(□ □ □) is equivalent, in T, to a formula on the A list.

Proof: We argue by cases according to □.

case 1. x □ z. With y □ z we get truth. With z □ y we get x □ □y. With y = z we get x □ y. With y inc z we get x ≠ y by setting z = {x,{y}}.

case 2. z □ x. With y □ z we get y □ □x. With z □ y we get x □ y ≠ ∅. With y = z we get y □ x. With y inc z we get (□z □ x)(z inc x).

case 3. x = z. With y □ z we get y □ x. With z □ y we get x □ y. With y = z we get x = y. With y inc z we get x inc y.

case 4. x inc z. With y □ z we get x ≠ y by setting z = {y,{x}}. With z □ y we get (□z □ y)(z inc x). With y = z we get x inc y. With y inc z we get truth by setting z = {{x,y}}.

QED

LEMMA 2.2. Let R be a complete conjunction. Then (□z)(R(x,y,z)) is equivalent, in T, to a formula of the form x □ y □ □, y □ x □ □, x = y □ □, or x inc y □ □, where □ is a formula on the A list.

Proof: By predicate calculus manipulations, we can pull out the xy part of R, which is x = y, x □ y, y □ x, or x inc y. The remaining part of R consists of the xz part and the yz part. Now apply Lemma 2.1. QED

LEMMA 2.3. Every □ formula is equivalent, in T, to a formula ((x □ y □ □) □ (y □ x □ □) □ (x = y □ □) □ (x inc y □ □)), where □ is a disjunction of formulas on the A list. Every □□ formula is equivalent, in T, to a formula (□y)(x □ y □ □) □ (□y)(y □ x □ □) □ (□y)(x = y □ □) □ (□y)(x inc y □ □), where □ is a disjunction of formulas on the A list.

Proof: The second claim follows immediately from the first. Let (□z)(□) be given, where □ is $□_0$. Find a disjunction of complete conjunctions, equivalent to □, according to Lemma 1.1. Then (□z)(□) is equivalent to a disjunction (□z)(□1) ... (□z)(□n), where the □'s are complete conjunctions. By Lemma 2.2, this is equivalent to a disjunction □ of formulas on the A list. Now obviously □ is equivalent to (x □ y □ □) □ (y □ x □ □) □ (x = y □ □) □ (x inc y □ □), since x □ y, y □ x, x = y, x inc y are exhaustive and mutually exclusive. QED

In light of Lemma 2.3, we wish to make four analyses, one each for each of the four clauses. These analyses are presented in sections 3 - 6.

For these analyses, it is convenient to move to the dual, and work with the (□y)(x □ y □ □), (□y)(y □ x □ □), (□y)(x = y □ □), (□y)(x inc y □ □), where □ is a conjunction of formulas from the A* list in the Appendix.

LEMMA 2.4. Every □□ formula is equivalent, in T, to a formula (□y)(x □ y □ □) (□y)(y □ x □ □) (□y)(x = y □ □) (□y)(x inc y □ □), where □ is a conjunction of formulas from the A* list. Every such disjunction is equivalent, in T, to a □□ formula.

Proof: By Lemma 2.3 and duality, each □□ formula can be put in this form, provided that we use the list of negations of formulas on the A list. The A* list does not include all of the negations of formulas on the A list, because x ≠ x is missing. But x s≠ x is not need since it is equivalent, in T, to a conjunction of formulas from the A* list.

The displayed formula is obviously equivalent to (□y)((x □ y □ □) □ (y □ x □ □) □ (x = y □ □) □ (x inc y □ □)). Since □ is □, we see that the implications are □, and hence the conjunction is also □. QED

The next four sections treat each of the four conjunctions displayed in Lemma 2.4.

3. (□y)(x □ y □ □).

Here □ is a conjunction of an A* sublist; i.e., of a sublist of the A* list.

We say that (the conjunction of) an A* sublist, □, yields □ if and only if (□y)(x □ y □ □) is equivalent, in T, to □.

We wish to compile the various □ that are yielded by the various A* sublists □. We find convenient representatives up to equivalence of the yielded □.

Clearly truth and falsity (e.g., x = x, x ≠ x) are among the □ that are yielded. We take these for granted. In light of this convention, we do not need to work with those formulas from the A* list which are refuted in T by x □ y. Also we do not need to work with those formulas from the A* list which are provable in T from x □ y since (□y)(x □ y) is provable in T. Therefore in this section we work within the following A* sublist:

x □ y = ∅;
x □ □y;
(□z □ x)(z comp y);
(□z □ y)(z comp x).

To prove each of these Lemmas, we first assume the list together with x □ y, and derive information about x and y. The information about x will be complete enough to form the yield. Then the converse is established by setting a witness for y that depends on x.

We use the same terminology, conventions, and strategy in sections 4,5,6 below.

LEMMA 3.1. x □ y = ∅, x □ □y, (□z □ y)(z comp x)) yields truth.

Proof: Set y = {x}. QED

LEMMA 3.2. x □ □y, (□z □ y)(z comp x), (□z □ x)(z comp y)) yields truth.

Proof: Set y = x □ {x}. QED

LEMMA 3.3. x □ y = ∅, (□z □ x)(z comp y) yields x = ∅. x □ y = ∅, x □ □y, (□z □ x)(z comp y)), (□z □ y)(z comp x) yields x = ∅.

Proof: Let z □ x. Since x □ y = ∅, we have z □ y. Also since x □ y, we see that z = y and y □ z are impossible. Hence x = ∅. For the reverse, set y = {∅}. QED

Here is the yield table.

1. x □ y = ∅. Yields truth.
2. x □ □y. Yields truth.
3. (□z □ x)(z comp y). Yields truth.
4. (□z □ y)(z comp x). Yields truth.
5. x □ y = ∅, x □ □y. Yields truth.
6. x □ y = ∅, (□z □ x)(z comp y). Yields x = ∅.
7. x □ y = ∅, (□z □ y)(z comp x). Yields truth.
8. x □ □y, (□z □ x)(z comp y). Yields truth.
9. x □ □y, (□z □ y)(z comp x). Yields truth.
10. (□z □ x)(z comp y), (□z □ y)(z comp x). Yields truth.
11. x □ y = ∅, x □ □y, (□z □ x)(z comp y). Yields x = ∅.
12. x □ y = ∅, x □ □y, (□z □ y)(z comp x). Yields truth.
13. x □ y = ∅, (□z □ x)(z comp y), (□z □ y)(z comp x). Yields x = ∅.
14. x □ □y, (□z □ x)(z comp y), (□z □ y)(z comp x). Yields truth.
15. x □ y = ∅, x □ □y, (□z □ x)(z comp y), (□z □ y)(z comp x). Yields x = ∅.

4. (□y)(y □ x □ □).

Recall the terminology and conventions of section 3.

We work within the following A* sublist:

x □ y = ∅
y □ □x
x comp y
(□z □ x)(z comp y)
(□z □ y)(z comp x).

We need to work with x comp y, even though it is provable, in T, from y □ x, since (□y)(y □ x) is not itself provable in T.

LEMMA 4.1. In the presence of y □ x, (□z □ y)(z comp x) is equivalent, in T, to y □ x.

Proof: Let z □ y. Then z = x and x □ z are each impossible. So z □ x. Hence y □ x. QED

In light of Lemma 4.1, we work with the following A* list.

x □ y = ∅
y □ □x

y □ x
x comp y
(□z □ x)(z comp y).

It is clear that (□y)(y □ x □ x comp y □ □) is equivalent, in T, to (□y)(y □ x □ □). Therefore, we have only to work with x comp y once. In fact, we merely have to note that x comp y yields x ≠ ∅, and then we can remove x comp y from consideration. In light of Lemma 4.2 below, we can ignore x comp y entirely. Thus we work only with the A* sublist

x □ y = ∅
y □ □x
y □ x
(□z □ x)(z comp y).

In many of these cases, we just make an obvious restatement of (□y)(y □ x □ □) in English. The cases where we can do better are presented as Lemmas.

LEMMA 4.2. x □ y = ∅ yields x ≠ ∅.

Proof: Since y □ x, we have x ≠ ∅. For the converse, set y to be a minimal element of x. QED

LEMMA 4.3. x □ y = ∅, (□z □ x)(z comp y) yields "some element of x lies in all other elements of x."

Proof: From the list, we see that y is a minimal element of x that is comparable with all elements of x. Since y is minimal, no element of x lies in y. Hence y lies in all other elements of x. Conversely, let y □ x lie in all other elements of x. Then y is epsilon minimal. QED

LEMMA 4.4. x □ y = ∅, y □ x yields ∅ □ x.

Proof: From the list, we see that y = ∅. Hence ∅ □ x. For the converse, set y = ∅. QED

LEMMA 4.5. y □ □x, (□z □ x)(z comp y) yields "x has an epsilon maximum element".

Proof: Let z □ x. Then z = y or z □ y. So y is an (the) epsilon maximum element. For the converse, set y be the epsilon maximum element. QED

LEMMA 4.6. x □ y = ∅, y □ □x, (□z □ x)(z comp y) yields "x is a singleton".

Proof: As in Lemma 4.5, y must be an epsilon maximal element of x. By x □ y = ∅, we see that x = {y}. QED

LEMMA 4.7. x □ y = ∅, y □ □x, y □ x yields "∅ is an epsilon maximal element of x".

Proof: From the list, we see that y = ∅, and so ∅ □ x. Also ∅ is not in □x. For the converse, set y = ∅. QED

LEMMA 4.8. x □ y = ∅, y □ □x, y □ x, (□z □ x)(z comp y), yields x = {∅}.

Proof: From the list we obtain y = ∅. From Lemma 4.6, we have x = {y}. For the converse, set y = ∅. QED

LEMMA 4.9. x □ y = ∅, y □ □x yields "some epsilon minimal element of x is an epsilon maximal element of x".

Proof: By x □ y = ∅, y is an epsilon minimal element of x. By y □ □x, y is an epsilon maximal element of x. The converse is also immediate. QED

LEMMA 4.10. x □ y = ∅, y □ x, (□z □ x)(z comp y) yields "∅ □ x is an element of all other elements of x".

Proof: By Lemma 4.3, we see that y is a minimal element of x that is comparable with all elements of x. Since y □ x, we have y = ∅. Conversely, set y = ∅. QED

LEMMA 4.11. y □ □x, y □ x, (□z □ x)(z comp y) yields "x is of the form b □ {b}".

Proof: By Lemma 4.5, we have that y is the maximum element of x; i.e., x □ y □ {y} and y □ x. By y □ x, we have x = y □ {y}. QED

Here is the yield table.

1. x □ y = ∅. Yields x ≠ ∅.
2. y □ □x. Yields "x has an epsilon maximal element".
3. y □ x. Yields "some subset of x lies in x".
4. (□z □ x)(z comp y). Yields "some element of x is comparable with all elements of x".

5. x y = ∅, y x. Yields "some epsilon minimal element of x is an epsilon maximal element of x".
6. x y = ∅, y x. Yields ∅ x.
7. x y = ∅, (z x)(z comp y). Yields "some element of x lies in all other elements of x."
8. y x, y x. Yields "some subset of x is an epsilon maximal element of x".
9. y x, (z x)(z comp y). Yields "x has an epsilon maximum element".
10. y x, (z x)(z comp y). Yields "there is a subset of x lying in x which is comparable with every element of x".
11. x y = ∅, y x, (z x)(z comp y). Yields "x is a singleton".
12. x y = ∅, y x, y x. Yields "∅ is an epsilon maximal element of x".
13. x y = ∅, y x, (z x)(z comp y). Yields "∅ x is an element of all other elements of x".
14. y x, y x, (z x)(z comp y). Yields "x is of the form b {b}".
15. x y = ∅, y x, y x, (z x)(z comp y). Yields x = {∅}.

In 4,10, comparable means comp.

5. (y)(x = y).

(y)(x = y) is logically equivalent to [y/x]. So we need only consider the [y/x], where is a conjunction of formulas from the A* list. We first determine the [y/x] where is on the A* list, that are not decided in T. By inspection, this can only be x = ∅. So x = ∅ is the only nontrivial yield, the others being truth and falsity.

6. (y)(x inc y).

Recall the terminology and conventions of section 3.

We work with the following A* sublist:

x y = ∅
x y
y x
(z x)(z comp y)
(z y)(z comp x)).

LEMMA 6.1. x y = ∅, x y, y x yields truth.

Proof: Set y = {{{x}}}. QED

LEMMA 6.2. x □ □y, y □ □x, (□z □ x)(z comp y)) yields truth.

Proof: Set y = x □ {{{x}}}. QED

LEMMA 6.3. x □ y = ∅, y □ □x, (□z □ y)(z comp x)) yields truth.

Proof: Set y = {{x}}. QED

LEMMA 6.4. x □ y = ∅, (□z □ x)(z comp y) yields “all elements of x have a common element”. Same if x □ □y is appended.

Proof: Let z □ x. Now z = y, z □ y are impossible. Hence y □ z. So y lies in every element of x. For the converse, first suppose that x = ∅. Set y = {{{∅}}}. Now suppose that x ≠ ∅. Let y lie in every element of x. Hence x □ y = ∅. Also, y □ x since y lies in any element of x. In addition, x □ y, x = y, x □ □y are impossible since y lies in an element of x. Therefore x inc y. QED

LEMMA 6.5. The following is provable in T. Let |x| ≥ 2. Then some proper subset of x does not lie in x and does not lie in □x.

Proof: There is a conceptually straightforward proof that, unfortunately, is not carried out in T. First suppose that x has an element u of highest rank. Then by rank considerations, {u} □ x,□x. Now suppose that x has no element of highest rank. Let u □ x. Then rk(x\{u}) = rk(x), and so again by rank considerations, x\{u} □ x,□x. We now present a proof formalizable within T. Let |x| ≥ 2.

case 1. □x ≠ ∅. Let y □ □x. Let y □ z □ x. Suppose x\{y} □ x. If x\{y} ≠ y then x\{y} □ x\{y}. Hence x\{y} = y. Also since z ≠ y, we have z □ x\{y}, and so z □ y. This is a contradiction. We have established that x\{y} □ x.

Now suppose x\{y} □ □x, and write x\{y} □ w □ x. Then w = y, for otherwise, w □ x\{y} □ w. So x\{y} □ y □ x. Now z □ x\{y} □ y □ z, which is a contradiction. Hence x\{y} □ □x.

case 2. □x = ∅. Let y be an epsilon minimal element of x. We claim that x\{y} □ x. Suppose x\{y} □ x. Since |x| ≥ 2, x\{y} must have an element which lies in x. Therefore x\{y} is not an epsilon minimal element of x. Hence x\{y} ≠ y. Since x\{y} is an element of x other than y, clearly x\{y} □ x\{y}. This is a contradiction. So x\{y} □ x.

QED

LEMMA 6.6. x □ □y, (□z □ y)(z comp x) yields x ≠ ∅,{∅}.

Proof: let z □ y. Then z = x contradicts x inc y, and x □ z contradicts x □ □y. So z □ x. Hence y □ x. Hence y must be a proper subset of x that does not lie in x. Hence x ≠ ∅,{∅}.

For the converse, first assume |x| = 1 and set y = ∅. Since x = {u}, u ≠ ∅, we have x inc y.

Now assume |x| ≥ 2. By Lemma 6.5, set y to be any proper subset of x that does not lie in x. QED

LEMMA 6.7. x □ y = ∅, y □ □x, (□z □ y)(z comp x) yields truth.

Proof: Set y = {{x}}. QED

LEMMA 6.8. y □ □x, (□z □ x)(z comp y), (□z □ y)(z comp x) yields truth.

Proof: Set y = x □ {{x}}. QED

LEMMA 6.9. x □ y = ∅, x □ □y, (□z □ y)(z comp x) yields x ≠ ∅ □ ∅ □ x.

Proof: By the proof of Lemma 6.6, y must be a proper subset of x that does not lie in x. Hence y = ∅. Since x inc y, we have x ≠ ∅ and ∅ □ x. For the converse, set y = ∅. QED

LEMMA 6.10. x □ y = ∅, y □ □x, (□z □ x)(z comp y) yields x = ∅.

Proof: Let z □ x. Now z = y is impossible by x inc y, z □ y is impossible by x □ y = ∅, and y □ z is impossible by y □ □x. So x = ∅. For the converse, set y = {{∅}}. QED

LEMMA 6.11. x □ y = ∅, (□z □ x)(z comp y), (□z □ y)(z comp x) yields x = ∅. Same with y □ □x.

Proof: As in the proof of Lemma 6.4, we get that y lies in every element of x. Let z □ y. Now z = x and z □ x are each impossible. Therefore x □ z. If x ≠ ∅ then write x □ z □ y □ w □ x, a contradiction. Therefore x = ∅. Now suppose x = ∅. Set y = {{∅}}. QED

LEMMA 6.12. x □ □y, y □ □x, (□z □ y)(z comp x) yields x ≠ ∅,{∅},{{∅}}.

Proof: By the proof of Lemma 6.6, x≠ ∅,{∅}, and y is a proper subset of x not in x. Also x ≠ {{∅}}, because otherwise y = ∅ and y □ □x. Now suppose x ≠ ∅,{∅},{{∅}}. If |x| = 1 then take y = ∅. If |x| ≥ 2 then by Lemma 6.5, take y to be a proper subset of x not lying in x and not lying in □x. QED

LEMMA 6.13. x □ □y, (□z □ x)(z comp y), (□z □ y)(z comp x) yields "there is a proper subset y of x not lying in x, which is an element of every element of x\y".

Proof: By the proof of Lemma 6.6, we get that y is a proper subset of x not lying in x. Let z □ x\y. Now z □ y, z = y are impossible. Hence y □ z. For the converse, let y be a proper subset of x not lying in x, which is an element of every element of x\y. We must verify (□z □ x)(z comp y). Let z □ x. If z □ y then y □ z. QED

LEMMA 6.14. x □ y = ∅, x □ □y, y □ □x, (□z □ y)(z comp x) yields x ≠ ∅, ∅ □ x, ∅ □ □x.

Proof: By Lemma 6.9, we get x ≠ ∅, ∅ □ x, y = ∅. Hence ∅ □ □x. Conversely, set y = ∅. QED

LEMMA 6.15. x □ y = ∅, x □ □y, (□z □ x)(z comp y), (□z □ y)(z comp x) yields falsity.

Proof: By Lemma 6.11, we get x = ∅. Let z □ y. Now z = x, z □ x are impossible. Hence ∅ □ z, contradiction. QED

LEMMA 6.16. x □ □y, y □ □x, (□z □ x)(z comp y), (□z □ y)(z comp x) yields falsity.

Proof: By Lemma 6.13, we get y is a proper subset of x not lying in x, which is an element of every element of x\y. In particular, y is an element of some element of x, violating y □ □x. QED

Here is the yield table.

1. x □ y = ∅. Yields truth.
2. x □ □y. Yields truth.
3. y □ □x. Yields truth.
4. (□z □ x)(z comp y). Yields truth.
5. (□z □ y)(z comp x). Yields truth.
6. x □ y = ∅, x □ □y. Yields truth.
7. x □ y = ∅, y □ □x. Yields truth.
8. x □ y = ∅, (□z □ x)(z comp y). Yields "all elements of x have a common element".
9. x □ y = ∅, (□z □ y)(z comp x). Yields truth.
10. x □ □y, y □ □x. Yields truth.
11. x □ □y, (□z □ x)(z comp y). Yields truth.
12. x □ □y, (□z □ y)(z comp x). Yields x ≠ ∅,{∅}.
13. y □ □x, (□z □ x)(z comp y). Yields truth.
14. y □ □x, (□z □ y)(z comp x). Yields truth.
15. (□z □ x)(z comp y), (□z □ y)(z comp x). Yields truth.
16. x □ y = ∅, x □ □y, y □ □x. Yields truth.
17. x □ y = ∅, x □ □y, (□z □ x)(z comp y). Yields "all elements of x have a common element".
18. x □ y = ∅, x □ □y, (□z □ y)(z comp x). Yields x ≠ ∅ □ ∅ □ x.
19. x □ y = ∅, y □ □x, (□z □ x)(z comp y). Yields x = ∅.
20. x □ y = ∅, y □ □x, (□z □ y)(z comp x). Yields truth.
21. x □ y = ∅, (□z □ x)(z comp y), (□z □ y)(z comp x). Yields x = ∅.
22. x □ □y, y □ □x, (□z □ x)(z comp y). Yields truth.
23. x □ □y, y □ □x, (□z □ y)(z comp x). Yields x ≠ ∅,{∅},{{∅}}.
24. x □ □y, (□z □ x)(z comp y), (□z □ y)(z comp x). Yields "there is a proper subset y of x, not lying in x, which is an element of every element of x\y".
25. y □ □x, (□z □ x)(z comp y), (□z □ y)(z comp x). Yields truth.
26. x □ y = ∅, x □ □y, y □ □x, (□z □ x)(z comp y). Yields falsity.
27. x □ y = ∅, x □ □y, y □ □x, (□z □ y)(z comp x). Yields x ≠ ∅, ∅ □ x, ∅ □ □x.
28. x □ y = ∅, x □ □y, (□z □ x)(z comp y), (□z □ y)(z comp x). Yields falsity.

29. x □ y = ∅, y □ □x, (□z □ x)(z comp y), (□z □ y)(z comp x). Yields x = ∅.
30. x □ □y, y □ □x, (□z □ x)(z comp y), (□z □ y)(z comp x). Yields falsity.
31. x □ y = ∅, x □ □y, y □ □x, (□z □ x)(z comp y), (□z □ y)(z comp x). Yields falsity.

7. The combined list.

The combined list from sections 3 - 6, of the formulas that are yielded, is presented in the Appendix. We do not list truth and falsity, as these are understood. The list is segregated, with the B,C,D,E list consisting of the yields from section 3,4,5,6, respectively.

LEMMA 7.1. Every □□ formula is decided in T, or is equivalent, in T, to a disjunction $\square_1$ $\square_2$ $\square_3$ $\square_4$, where $\square_1$ is on the B list, $\square_2$ is on the C list, $\square_3$ is on the D list, $\square_4$ is on the E list, and where 1,2, or 3 of these disjuncts may be missing. Furthermore, every such disjunction is equivalent, in T, to a □□ formula.

Proof: Let □ be □□. By Lemma 2.4, □ is equivalent to (□y)(x □ y □ □) (□y)(y □ x □ □) (□y)(x = y □ □) (□y)(x inc y □ □), where □ is a conjunction of formulas from the A* list. By the construction of the combined list, we see that □ is equivalent to $\square_1$ $\square_2$ $\square_3$ $\square_4$, where $\square_1$ is on the B list or is decided, $\square_2$ is on the C list or is decided, $\square_3$ is on the C list or is decided, and $\square_4$ is on the D list or is decided. If any of the □'s are decided positively, then □ is decided. If all of the □'s are decided, then □ is decided. The remaining case is where some □ is not decided, and no □ is positively decided. Remove the □'s that are negatively decided.

By the construction of the B,C,D, and E lists, such a disjunction must be equivalent to a disjunction of the form (□y)(x □ y □ □1) (□y)(x □ y □ □2) (□y)(x = y □ □3) (□y)(x inc y □ □4), where the □'s are conjunctions from the A* list, and where 1,2, or 3 of these disjuncts may be missing. As in the proof of Lemma 2.4, we see that this disjunction is equivalent, in T, to an □□ formula. QED

LEMMA 7.2. Every □□ ... □□ formula is decided in T, or is equivalent, in T, to a disjunction of formulas from the combined list.

Proof: By Lemma 7.1. QED

It is now convenient to take the dual and work with the negations of formulas from the combined list. This combined starred list is presented in the Appendix. Again, the list is segregated into the B*, C*, D*, and E* lists.

LEMMA 7.3. Every □□ formula is decided in T, or is equivalent, in T, to a conjunction □$_1$ □ □$_2$ □ □$_3$ □ □$_4$, where □$_1$ is on the B* list, □$_2$ is on the C* list, □$_3$ is on the D* list, □$_4$ is on the E* list, and where 1,2, or 3 of the conjuncts may be missing. Furthermore, every such conjunction is equivalent, in T, to a □□ formula.

Proof: This is the dual of Lemma 7.1. QED

For the rest of this section, we consider the complexity class □□□...□□□ rather than □□. We return to a discussion of □□ (□□) in section 9.

For this reason, it is convenient to present the combined starred list as the F list without segregation, and with repetitions removed. See the Appendix for the presentation.

The realizability of a list of formulas with at most x free is identified with the sentence obtained by placing (□x) in front of the conjunction of that list of formulas. This results in a sentence.

LEMMA 7.4. Every □□□...□□□ formula is decided in T, or equivalent, in T, to a conjunction of formulas from the F list. Every formula on the F list is equivalent, in T, to a □□ formula.

Proof: By Lemma 7.3. QED

LEMMA 7.5. Suppose that an F sublist contains F2, F18, or F20. Then its realizability is decided in T. Realizibility of the F sublist implies, in T, realizability of the F sublist by x = ∅, {∅}, or {{∅}}.

Proof: Any of F2, F18, F20 imply that x be among ∅, {∅}, {{∅}}. Hence the realizability of any given F sublist containing any of F2, F18, F20 is equivalent to realizability by at least one of ∅, {∅, {{∅}}. However, realizability by any one of these three sets is decided in T because the truth value of any given formula on the F list with x taken to be any one of these three sets is decided in T. QED

LEMMA 7.6. Suppose that for every conjunction of formulas from the F list with F2, F18, F20 removed, realizability is decided in T. Then every □(□□...□□□) sentence is decided in T.

Proof: Immediate from Lemmas 7.4 and 7.5. QED

This means that we can remove F2, F18, F20 from consideration. We call the result of this removal, the F′ list, and it appeears in the Appendix.

8. Realizable F′ sublists. □(□□...□□□) sentences.

We will show the following in T.

1. The F′ list with F19 removed is realizable.
2. The F′ list with F5,F8 removed is realizable.
3. The F′ list with F5 removed is not realizable.
4. The F′ list with F8 removed is not realizable.
5. F19,F5 imply x = ∅. F19,F8 imply x = ∅.

Item 5 is obvious by inspection.

LEMMA 8.1. The following is provable in T. The F′ list is not realizable.

Proof: F1, F8, F19 are obviously incompatible. QED

We consider the following list of conditions on x.

G LIST

G1. ∅ □ x, ∅ □ □x, x □ V(□).
G2. Every y □ x is an element of an infinite subset of x whose elements are epsilon incomparable and pairwise disjoint.
G3. x has no epsilon maximal element.
G4. No subset of x lies in x.

LEMMA 8.2. The following is provable in T. Let x satisfy the conditions on the G list. Then x realizes the formulas on the F′ list without F19.

Proof: F_1 is from G_1. F_3 from G_3. F_4 from G_2. F_5 from G_4. F_6 from G_3. F_7 from G_2. F_8 from G_1. F_9 from G_3. F_{10} from G_3. F_{11} from G_4. F_{12} from G_2. F_{13} from G_3. F_{14} from G_1. F_{15} from G_3. F_{16} from G_3. F_{17} from G_2. F_{22} from G_1. Finally, we verify F_{21}. Suppose the statement in question is true. Let y be such a proper subset of x. If y is infinite then it cannot be an element of any element of x\y, since all elements of x are finite. Hence y is finite. Let u be an infinite subset of x whose elements are pairwise disjoint. Let v,w $\square$ u be distinct and not lie in y. Since v,w are disjoint elements of x\y, y cannot lie in every element of x\y. QED

The Fibonacci set sequence is defined as follows. $x_0 = \emptyset$, $x_1 = \emptyset$, $x_{n+2} = \{x_n,x_{n+1}\}$.

We write FIB_1 for the set of all odd numbered terms of the Fibonacci set sequence, and FIB_2 for the set of all positive even number terms of the Fibonacci set sequence. I.e., $FIB_1 = \{x_1,x_3,...\}$, and $FIB_2 = \{x_2,x_4,...\}$.

LEMMA 8.3. The following is provable in T. FIB_2 satisfies all conditions on the G list.

Proof: In the Fibonacci set sequence, by induction, the x_n, $n \geq 1$, are distinct elements of $V(\square)$.

G1 is obvious. Note that for all even $n \geq 2$, $x_n \square x_{n+2}$. This verifies G3 Also for all even $n \geq 2$, $x_n \square x_{n+4} = \emptyset$, $x_n \square x_{n+4} \square x_n$. This establishes G2. Finally, x_2 is not a subset of x. Also for even $n \geq 2$, x_n is not a subset of x since $x_{n-1} \square x$. This establishes G4. QED

LEMMA 8.4. The following is provable in T. The F' list without F19 is realizable. Furthermore, it is realized by $x = FIB_2$.

Proof: Immediate from Lemmas 8.2 and 8.3. QED

We now show in T that the F' list with F5, F8 removed is realizable.

We consider the following list of conditions on x.

H1. $\emptyset \square x$, $\emptyset \square \square x$, $x \square V(\square)$.

H2. Every y □ x is an element of an infinite subset of x whose elements are epsilon incomparable and pairwise disjoint.
H3. x has no epsilon maximal element.

LEMMA 8.5. The following is provable in T. Let x satisfy the conditions on the H list. Then x realizes the formulas on the F′ list without F5, F8.

Proof: F_1 is from H_1. F_3 from H_3. F_4 from H_2. F_6 from H_3. F_7 from H_2. F_9 from H_3. F_{10} from H_3. F_{11} from H_2. F_{12} from H_2. F_{13} from H_3. F_{14} from H_2. F15 from H_3. F_{16} from H_3. F_{17} from H_2. F_{19} from H_1. F_{22} from H_1. Finally, we verify F_{21} exactly as in the proof of Lemma 8.2. QED

LEMMA 8.6. The following is provable in T. FIB_1 satisfies all conditions on the H list. The F′ list without F5, F8 is realizable by FIB_1.

Proof: Since x1 = ∅ and ∅ □ x3, we see that H1 holds. The rest of the proof follows those of Lemmas 8.3 and 8.4. QED

LEMMA 8.7. The following is provable in T. The F′ list without F5 is not realizable. The F′ list without F8 is not realizable.

Proof: The F′ list without F5,F8 implies ∅ □ x, using F1 and F18. Hence the F′ list without F8 is not realizable, using F5 (∅ is a subset of x lying in x), and the F′ list without F5 is not realizable, using F8. QED

This completes the verification of items 1-5 at the beginning of this section.

LEMMA 8.8. The following is provable in T. An F′ sublist is realizable if and only if ((it does not contain F19) or (it does not contain F5 and it does not contain F8) or (it is realizable by x = ∅)).

Proof: We use items 1-5. Suppose the F′ sublist □ is realizable. If F19 is not in □ then we are done. We assume F19 is in □. If F5,F8 are both not in □, then we are done. We assume F5 in □ or F8 in □. By 5, we see that realizability of □ implies the realizability of □ by x = ∅. This completes the forward direction.

Conversely, suppose the displayed disjunction holds. If □ does not contain F19 then □ is realizable by 1. If □ does not contain either of F5,F8, then □ is realizable by 2. QED

LEMMA 8.9. The realizability of any given F′ sublist is equivalent, in T, to its realizability by x = ∅, FIB_1, or FIB_2. The realizability of any given F sublist is equivalent, in T, to its realizility by x = ∅, {∅}, {{∅}}, FIB_1, or FIB_2.

Proof: The first claim is by item 6 and Lemmas 8.4 and 8.6. The second claim follows from the first claim and Lemma 7.5. QED

THEOREM 8.10. Every □(□□□...□□□) sentence is decided in T. The realizability of any given □□□...□□□ formula is equivalent, in T, to its realizibility by x = ∅, {∅}, {{∅}}, FIB_1, or FIB_2.

Proof: By Lemmas 7.4, 7.5, and 8.9. QED

9. □□□ sentences.

LEMMA 9.1. Every □□ formula is either decided in T or is equivalent, in T, to a disjunction of formulas on the combined list presented before Lemma 7.1. Every □□□ sentence is either decided in T, or is equivalent, in T, to a disjunction of sentences of the form (□x)(□), where □ is a formula on the combined list.

Proof: The first claim is by Lemma 7.1. The second claim follows immediately. QED

LEMMA 9.2. Every sentence of the form (□x)(□), where □ is a formula on the combined list, is decided in T. The following is provable in T. The realizability of any formula on the combined list is equivalent, in T, to its realizability by x = ∅, {∅}, {{∅}}, or {{{∅}}}.

Proof: We examine each entry on the combined list and give a realizing value for x.

B1. x = ∅. ∅.
C1. x ≠ ∅. {∅}.
C2. x has an epsilon maximal element. {∅}.
C3. Some element of x is comparable with all elements of x. {∅}.

C4. Some subset of x lies in x. {∅}.
C5. Some epsilon minimal element of x is an epsilon maximal element of x. {∅}.
C6. Some element of x lies in all other elements of x. {∅}.
C7. ∅ □ x. {∅}.
C8. x has an epsilon maximum element. {∅}.
C9. Some subset of x is an epsilon maximal element of x. {∅}.
C10. There is a subset of x lying in x which is comparable with every element of x. {∅}.
C11. x is a singleton. {∅}.
C12. ∅ is an epsilon maximal element of x. {∅}.
C13. ∅ □ x is an element of all other elements of x. {∅}.
C14. x is of the form b □ {b}. {∅}.
C15. x = {∅}. {∅}.
D1. x = ∅. ∅.
E1. All elements of x have a common element. ∅.
E2. x ≠ ∅,{∅}. {{∅}}.
E3. x ≠ ∅ □ ∅ □ x. ∅.
E4. x = ∅. ∅.
E5. x ≠ ∅,{∅},{{∅}}. {{{∅}}}.
E6. There is a proper subset y of x, not lying in x, which is an element of every element of x\y. {{∅}}.
E7. x ≠ ∅, ∅ □ x, ∅ □ □x. {{{∅}}}.

QED

THEOREM 9.3. Every □□□ sentence is decided in T. The following is provable in T. The realizability of any given □□ formula is equivalent, in T, to its realizability by x = ∅, {∅}, {{∅}}, or {{{∅}}}.

Proof: By Lemmas 9.1 and 9.2. QED

We conjecture that every □(□□□...□□□) sentence is decided in T.

10. □□□...□□□ formulas. □(□□□...□□□) sentences. □□□ sentences.

We use Lemma 2.2 and the A list.

LEMMA 10.1. Every □□ formula is equivalent, in T, to a disjunction of formulas of the form
(□y)(x □ y □ □)
(□y)(y □ x □ □)
(□y)(x = y □ □)

(y)(x inc y)
where is on the A list.

Proof: Any formula is a disjunction of the formulas treated in Lemma 2.2. So the formulas are of the form (y)(), where is a disjunction of formulas of the four forms displayed in Lemma 2.2. QED

LEMMA 10.2. Every formula of the form (y)(x y), where is on the A list, is decided in T or is equivalent, in T, to x ≠ ∅.

Proof: If is among x = x, x = y, x ≠ y, x y, y x, y x, x inc y, then (y)(x y) is obviously decided. (y)(x y x y ≠ ∅) is equivalent to x ≠ ∅, by setting y = x {x}. (y)(x y x y) holds by setting y = {x,{x}}. (y)((z x)(z inc y)) is equivalent to x ≠ ∅ by setting y = {{u}}, where u x. (y)(z y)(z inc x)) holds by setting y = {{{x}}}. QED

LEMMA 10.3. Every formula of the form (y)(y x), where is on the A list, is decided in T or is equivalent, in T, to one of the following:
x ≠ ∅
"some element of x lies in x"
"there exist two distinct epsilon incomparable elements of x"
"x is not transitive".

Proof: If is among x = y, x ≠ y, x y, x inc y, then (y)(y x) is obviously decided. (y)(y x x = x) and (y)(y x y x) are obviously equivalent to x ≠ ∅. (y)(y x x y ≠ ∅) is equivalent to x ≠ ∅ by taking y to be an epsilon minimal element of x. (y)(y x y x) is equivalent to "some element of x lies in x". (y)(y x (z x)(z inc y)) is equivalent to "there exist two epsilon incomparable elements of x". (y)(y x (z y)(z inc x)) is equivalent to (y)(y x (z y)(z x)) which is equivalent to "x is not transitive". QED

LEMMA 10.4. Every formula of the form (y)(x = y), where is on the A list, is decided in T or is equivalent, in T, to x ≠ ∅.

Proof: (y)(x = y) is equivalent to [y/x]. By inspection, the [y/x], where is on the A list, is either decided or equivalent to x ≠ ∅. QED

LEMMA 10.5. Every formula of the form (□y)(x inc y □ □), where □ is on the A list, is decided in T or is equivalent, in T, to “x is not transitive”.

Proof: If □ is among x = x, x = y, x ≠ y, x □ y, y □ x, x inc y, then (□y)(x inc y □ □) is decided. (□y)(x inc y □ x □ y ≠ ∅) holds by setting y = {{x}}. (□y)(x inc y □ x □ □y) holds by setting y = {{x}}. (□y)(x inc y □ y □ □x) is equivalent to (□y)(y □ □x □ y □ x) which is equivalent to “x is not transitive”. (□y)(x inc y □ (□z □ x)(z inc y)) is equivalent to x ≠ ∅ by setting y = {{x}}. (□y)(x inc y □ (□z □ y)(z inc x)) holds by setting y = {{{x}}}. QED

Here is the combined list obtained from the four Lemmas.

K LIST

K1. x ≠ ∅
K2. Some element of x lies in □x.
K3. There exist two distinct incomparable elements of x.
K4. x is not transitive.

LEMMA 10.6. Every □□ formula is decided in T or is equivalent, in T, to a disjunction of formulas on the K list.

Proof: By Lemmas 10.1 - 10.5. QED

LEMMA 10.7. Every □□□...□□□ formula is decided in T or is equivalent, in T, to a disjunction of conjunctions from the K list. Every □(□□□...□□□) sentence is decided in T or equivalent, in T, to a disjunction of sentences of the form (□x)(□), where □ is a conjunction of formulas on the K list.

Proof: By Lemma 10.6, every □□□...□□□ formula is decided or equivalent to a conjunction of disjunctions of formulas on the K list. Therefore it is decided or equivalent to a disjunction of conjunctions of formulas on the K list. When we place the quantifier (□x) in front, we can take the disjunctions out. QED

LEMMA 10.8. The following is provable in T. The entire K list is realized by x = FIB_1.

Proof: By inspection. QED

THEOREM 10.9. Every □(□□□...□□□) sentence is decided in T. The realizability of any given □□□...□□□ formula is eqvuivalent, in T, to its realizability by x = FIB_1.

Proof: By Lemmas 10.7 and 10.8. By Lemma 10.8, every one of the sentences used in Lemma 10.7 of the form (□x)(□), where □ is a conjunction of formulas on the A list, is realized by x = FIB_1. QED

We now come to □□□. We list the negations of the formulas on the K list.

K* LIST

K1*. x = ∅
K2*. No element of x lies in □x.
K3*. Any two distinct elements of x are epsilon comparable.
K4*. x is transitive.

LEMMA 10.10. Every □□ formula is decided in T or is equivalent, in T, to a conjunction of formulas on the K* list.

Proof: By Lemma 10.6 using duality. QED

THEOREM 10.11. Every □□□ sentence is decided in T. The following is provable in T. The realizability of any given □□ formula is equivalent, in T, to its realizability by x = ∅.

Proof: Obviously the entire K* list is realized by x = ∅. QED

11. 2,3 quantifiers. Additional results and remarks.

THEOREM 11.1. Every 3 quantifier sentence is decided in T. There are 6 sets such that every realizable 2 quantifier formula is realized by one of them. The realizability of any given 2 quantifier formula is equivalent, in T, to its realizability by x = ∅, {∅}, {{∅}}, {{{∅}}}, FIB_1, or FIB_2.

Proof: In predicate calculus, every formula is equivalent to a formula in prenex form with the same number of quantifiers, where the quantifiers in the prenex form use distinct variables. Hence in our context, the 3 quantifier sentences are logically equivalent to sentences in the 8 prenex classes with by a triple of letters from □,□. By Theorem 8.10, every □(□□□...□□□) sentence is decided in T, and every realizable

□□□...□□□ formula is realized within the displayed list. By Theorem 9.3, every □□□ sentence is decided in T, and every realizable □□ formula is realized within the displayed list. By Theorem 10.9, every □(□□□...□□□) sentence is decided in T, and every realizable □□□...□□□ formula is realized within the displayed list. By Theorem 10.11, every □□□ sentence is decided in T, and every realizable □□ formula is realized within the displayed list. Thus we see that every □□□, □□□, □□□, and □□□ sentence is decided in T. Hence every 3 quantifier sentence is decided in T (by taking negations). Also, note that every 2 quantifier formula is □□, □□, □□, or □□, and each of these cases has been covered for the realizability claim. QED

We now use Lemma 7.3 to give an example of a □□□...□□□ formula that is not equivalent to a □□ formula. We will use the combined starred list just before Lemma 7.3.

Let □ = "∅ □ x □ x is not a singleton".

LEMMA 11.2. □ is equivalent, in T, to a □□□□□ formula. □ is not decided in T.

Proof: □ is equivalent to (□y □ x)(□z □ y) □ (□y)(□z)(y □ x □ (z □ x □ z ≠ y)). Note that □ holds of ∅ and fails of {∅}. QED

In the following Lemma, we use equivalence in the set theoretic sense, and not in the sense of T.

LEMMA 11.3. □ is not equivalent to a conjunction $\square_1$ □ $\square_2$ □ $\square_3$ □ $\square_4$, where $\square_1$ is from the B* list, $\square_2$ is from the C* list, $\square_3$ is from the D* list, $\square_4$ is from the E* list, where 1,2, or 3 of these □ may be missing.

Proof: Assume $\square_1$ □ $\square_2$ □ $\square_3$ □ $\square_4$ is such a conjunction that is equivalent to □. Since □ holds of ∅, $\square_1$ and $\square_3$ must be missing. So we are left with a conjunction of a formula from the C* list and a formula from the E* list, where perhaps one of these is also missing.

We can eliminate many formulas on the C* and E* lists on the grounds that they are not implied by □. I.e., the formula from the list fails at a set at which □ holds. Here are the eliminations and the reasons.

C1*. x = ∅. □({{∅},{{∅}}}).
C2*. x has no epsilon maximal element. □({{∅},{{∅}}}).
C3*. No element of x is comparable with all elements of x. □({{∅},{{∅}}}).
C4*. No subset of x lies in x. □({{∅},{{∅}}}).
C5*. No epsilon minimal element of x is an epsilon maximal element of x. □({{∅},{{{∅}}}}).
C6*. No element of x lies in all other elements of x. □({{∅},{{∅}}}).
C7*. ∅ □ x.
C8*. x does not have an epsilon maximum element. □({{∅},{{∅}}}).
C9*. No subset of x is an epsilon maximal element of x. □({{∅},{{∅}}}).
C10*. There is no subset of x lying in x which is comparable with every element of x. □({{∅},{{∅}}}).
C11*. x is not a singleton.
C12*. ∅ is not an epsilon maximal element of x.
C13*. ∅ □ x or ∅ is not an element of all other elements of x.
C14*. x is not of the form b □ {b}.
C15*. x ≠ {∅}.
E1*. There is no set that lies in all elements of x. □({{∅},{∅,{∅}}}).
E2*. x = ∅ or x = {∅}. □({{∅},{{∅}}}).
E3*. x = ∅ or ∅ □ x. □({{∅},{{∅}}}).
E4*. x ≠ ∅. □(∅).
E5*. x = ∅, {∅}, or {{∅}}. □({{∅},{{∅}}}).
E6*. The following is false. There is a proper subset y of x, not lying in x, which is an element of every element of x\y. □({{∅},{∅,{∅}}}). Set y = ∅.
E7*. x = ∅ or ∅ □ x or ∅ □ □x. □({{{∅}}}).

We are left with the following much smaller list.

C7*. ∅ □ x.
C11*. x is not a singleton.
C12*. ∅ is not an epsilon maximal element of x.
C13*. ∅ □ x or ∅ is not an element of all other elements of x.
C14*. x is not of the form b □ {b}.
C15*. x ≠ {∅}.

In particular, we have eliminated all formulas in the E* list. But now we discard these remaining formulas on the grounds that they do not imply □:

C7*. ∅ □ x. □□({{∅}}).
C11*. x is not a singleton. □□({∅,{∅}}).
C12*. ∅ is not an epsilon maximal element of x. □□({{∅}}).
C13*. ∅ □ x or ∅ is not an element of all other elements of x. □□({{∅}}).
C14*. x is not of the form b □ {b}. □□({∅,{{∅}}}).
C15*. x ≠ {∅}. □□({{∅}}).

QED

In the second claim of the following Theorem, we use equivalence in the set theoretic sense, and not in the sense of T.

THEOREM 11.4. "∅ □ x □ x is not a singleton" is equivalent, in T, to a □□□□□ formula. However, it is not equivalent to a □□ formula.

Proof: By Lemmas 11.2, 11.3, and 7.3. QED

As remarked in section 1, we have discovered an example of a 5 quantifier sentence which is not decided in ZFC and is, in fact, equivalent to "there exists a subtle cardinal" over ZFC. In fact, the example is in class □□□□□. See [Fr01], [Fr02].

[Go79], p.3, conjectures that all 7 quantifier sentences are provable or refutable in ZF. The 5 quantifier result obviously gives a very strong refutation of this conjecture, even for ZFC + V = L.

Furthermore, [Go79] remarks that the axiom of choice has 8 quantifiers, as displayed in the footnote on p. 3:

$$(\square x_1)((\square x_2)(\square x_3)(x_2 \square x_1 \square x_3 \square x_1 \square (\square x_4)(x_4 \square x_2 \quad x_4 \square x_3)) \square (\square x_5)(\square x_6)(x_6 \square x_1 \square (\square x_7)(x_7 \square x_6 \square x_7 \square x_5 \square (\square x_3)(x_8 \square x_6 \square x_8 \square x_5 \square x_3 = x_7)))).$$

The last two x_3's are typographical errors, so this should read:

$$(\square x_1)((\square x_2)(\square x_3)(x_2 \square x_1 \square x_3 \square x_1 \square (\square x_4)(x_4 \square x_2 \quad x_4 \square x_3)) \square (\square x_5)(\square x_6)(x_6 \square x_1 \square (\square x_7)(x_7 \square x_6 \square x_7 \square x_5 \square (\square x_8)(x_8 \square x_6 \square x_8 \square x_5 \square x_8 = x_7)))).$$

In fact, this standard version of the axiom of choice can be put into 7 quantifier form by the following quantifier manipulations:

□(□□□ □ □□□□)
□(□□□ □□□□)
□□□□□□□□.

This already refutes the conjecture made on p. 3 of [Go79] that all 7 quantifier sentences can be decided in ZF.

We conjecture that all 4 quantifier sentences in set theory with □,= are decided in ZF, perhaps even in T plus the power set axiom.

APPENDIX

A LIST

A1. x = x.
A2. x = y.
A3. x ≠ y.
A4. x □ y.
A5. y □ x.
A6. x □ y ≠ ∅.
A7. x □ □y.
A8. y □ □x.
A9. x inc y.
A10. (□z □ x)(z inc y).
A11. (□z □ y)(z inc x).

A* LIST
dual to A list

A1*. x = y;
A2*. x ≠ y;
A3*. x □ y;
A4*. y □ x;
A5*. x □ y = ∅;
A6*. x □ □y;
A7*. y □ □x;
A8*. x comp y;
A9*. (□z □ x)(z comp y);
A10*. (□z □ y)(z comp x).

COMBINED LIST
segregated list of yields from sections 3-6

B1. x = ∅.
C1. x ≠ ∅.
C2. x has an epsilon maximal element.
C3. Some element of x is comparable with all elements of x.
C4. Some subset of x lies in x.
C5. Some epsilon minimal element of x is an epsilon maximal element of x.
C6. Some element of x lies in all other elements of x.
C7. ∅ □ x.
C8. x has an epsilon maximum element.
C9. Some subset of x is an epsilon maximal element of x.
C10. There is a subset of x lying in x which is comparable with every element of x.
C11. x is a singleton.
C12. ∅ is an epsilon maximal element of x.
C13. ∅ □ x is an element of all other elements of x.
C14. x is of the form b □ {b}.
C15. x = {∅}.
D1. x = ∅
E1. All elements of x have a common element.
E2. x ≠ ∅,{∅}.
E3. x ≠ ∅ □ ∅ □ x.
E4. x = ∅.
E5. x ≠ ∅,{∅},{{∅}}.
E6. There is a proper subset y of x, not lying in x, which is an element of every element of x\y.
E7. x ≠ ∅, ∅ □ x, ∅ □ □x.

In C3,C10, we mean comparable in the sense of comp.

COMBINED STARRED LIST
dual to combined list

B1*. x ≠ ∅.
C1*. x = ∅.
C2*. x has no epsilon maximal element.
C3*. No element of x is comparable with all elements of x.
C4*. No subset of x lies in x.
C5*. No epsilon minimal element of x is an epsilon maximal element of x.
C6*. No element of x lies in all other elements of x.
C7*. ∅ □ x.
C8*. x has no epsilon maximum element.

C9*. No subset of x is an epsilon maximal element of x.
C10*. There is no subset of x lying in x which is comparable with every element of x.
C11*. x is not a singleton.
C12*. ∅ is not an epsilon maximal element of x.
C13*. ∅ □ x or ∅ is not an element of all other elements of x.
C14*. x is not of the form b □ {b}.
C15*. x ≠ {∅}.
D1*. x ≠ ∅.
E1*. There is no set that lies in all elements of x.
E2*. x = ∅   x = {∅}.
E3*. x = ∅   ∅ □ x.
E4*. x ≠ ∅.
E5*. x = ∅   x = {∅}   x = {{∅}}.
E6*. The following is false. There is a proper subset y of x, not lying in x, which is an element of every element of x\y.
E7*. x = ∅   ∅ □ x   ∅ □ □x.

In C3*,C10*, we mean comparable in the sense of comp.

F LIST
unsegregated combined starred list

F1. x ≠ ∅.
F2. x = ∅.
F3. x has no epsilon maximal element.
F4. No element of x is comparable with all elements of x.
F5. No subset of x lies in x.
F6. No epsilon minimal element of x is an epsilon maximal element of x.
F7. No element of x lies in all other elements of x.
F8. ∅ □ x.
F9. x has no epsilon maximum element.
F10. No subset of x is an epsilon maximal element of x.
F11. There is no subset of x lying in x which is comparable with every element of x.
F12. x is not a singleton.
F13. ∅ is not an epsilon maximal element of x.
F14. ∅ □ x or ∅ is not an element of all other elements of x.
F15. x is not of the form b □ {b}.
F16. x ≠ {∅}.
F17. There is no set that lies in all elements of x.
F18. x = ∅   x = {∅}.
F19. x = ∅   ∅ □ x.

F20. x = ∅   x = {∅}   x = {{∅}}.
F21. The following is false. There is a proper subset y of x, not lying in x, which is an element of every element of x\y.
F22. x = ∅   ∅ □ x   ∅ □ □x.

F′ LIST
pruned F list

F1. x ≠ ∅.
F3. x has no epsilon maximal element.
F4. No element of x is comparable with all elements of x.
F5. No subset of x lies in x.
F6. No epsilon minimal element of x is an epsilon maximal element of x.
F7. No element of x lies in all other elements of x.
F8. ∅ □ x.
F9. x does not have an epsilon maximum element.
F10. No subset of x is an epsilon maximal element of x.
F11. There is no subset of x lying in x which is comparable with every element of x.
F12. x is not a singleton.
F13. ∅ is not an epsilon maximal element of x.
F14. ∅ □ x or ∅ is not an element of all other elements of x.
F15. x is not of the form b □ {b}.
F16. x ≠ {∅}.
F17. There is no set that lies in all elements of x.
F19. x = ∅ or ∅ □ x.
F21. The following is false. There is a proper subset y of x, not lying in x, which is an element of every element of x\y.
F22. x = ∅ or ∅ □ x or ∅ □ □x.

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