TANGIBLE INCOMPLETENESS INTERIM REPORT

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This is an entirely self contained Interim Report which reworks and expands on the material in [3].

I.e., we rework the material in sections 2.1 - 2.3 of the following TOC. For section 2.4, we will first return to the FOM.

TANGIBLE INCOMPLETENESS (book)

1. BOOLEAN RELATION THEORY

2. INVARIANT MAXIMALITY
   2.1. Introduction.
   2.2. Emulation Theory
      2.2.1. N, Z+, Q, Q[(a,b)], Order Equivalence, Emulator, Maximality
      2.2.2. (Complete) Invariance, Shift, N Shift, N Tail, N Tail Shift, N Tail-Related
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3. INDUCTIVE DOMAINS - in development
3.1. Inductive Domains in Q
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4. SEQUENTIAL CONSTRUCTIONS - in development

This material will be eventually merged with the material in

TANGIBLE MATHEMATICAL INCOMPLETENESS OF ZFC
https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/
August 16, 2018, #106, 66 pages.

which is considerably more detailed with many complete proofs.

2. INVARIANT MAXIMALITY

2.1 Introduction

Invariance and Maximality, separately, figure prominently throughout mathematics as notions of great intrinsic interest and power. The new Invariant Maximality combines these notions in seamless ways with spectacular effect. Thus we ask for a maximal object - whose existence is generally obvious via Zorn's Lemma or direct construction - which also satisfies an invariance condition(s).

This results is a developing coherent series of tangible statements which can and can only be proved using extensions of ZFC via large cardinal hypotheses.

Although these statements involve the existence of a countably infinite set satisfying natural conditions, the statements are implicitly Π^0_1. This can be seen from their logical form and the Gödel Completeness Theorem.

The foundational significance of a statement P being implicitly Π^0_1 include

a. if P can be proved in any extension of ZFC that can be demonstrably forced (forcing going back to P.J. Cohen), then P can be proved in ZFC alone. The same holds for ZFC + "there exists a strongly inaccessible cardinal". There are a very wide range of such extensions of ZFC in the literature.

b. P has the refutability property: i.e., P is demonstrably refutable in the sense that we know that if P is false then P is, in principle, provably false (in a very weak system). A hallmark of most physical theories is that they have the experimental refutability property: we know that if they are false then they are, in principle, provably false (by experimentation). Physical theories that do not have this experimental refutability property are generally marginalized by the scientific community, and in particular Nobel Prize committees.

It remains of great foundational importance to give fully natural finite forms - i.e., where all objects under consideration are finitary (although of course there are infinitely many finitary objects under consideration). These explicitly finite forms appear in Part 4 of this book, Sequential Constructions.

Invariant Maximality, in its present form, comes in three main flavors: Emulation Theory, Invariant Graph Theory, and Invariant A...A Theory. Which flavor resonates most depends on the audience.

Emulation theory with its maximal emulators, has perhaps the broadest potential appeal among minimally mathematical thinkers, the general mathematician, and the gifted high school student. This is because the outermost universal quantifier is over the very basic finite sets of rational vectors of fixed dimension, and the notion of emulator can be described informally and thematically without getting into the details (and yet the details are transparent). In fact we believe that emulation theory in dimension 3 already goes far beyond ZFC, as does invariant graph theory and invariant A...A theory - although this has yet to be established. But note that only emulation theory has such a trivial outermost quantifier. But all three are readily understood by those comfortable with undergraduate level mathematics.

In emulation theory, practically any small set E of rational vectors in low dimension leads to an interesting if not rich development. Informally, S is an emulator of E if and only if any two elements of S (taken as a whole) "look like" some two elements of E (taken as a whole). We look for inclusion maximal emulators with certain invariance properties. This will support a very effective and exciting curriculum at the mathematically gifted high school level, and the plan is to develop this for use in the main Summer School programs for students at that level. See [4].

For related reasons, emulation theory may be key for realizing the future of Invariant Maximality that I discuss below. Instead of just working with (finite) sets of rational vectors, we can work with (finite) subsets of the domain of a relational structure. There is a clear notion of emulator here.

However, for many discrete mathematicians, especially graph theorists, Invariant Graph Theory may resonate more. Instead of emulators of finite sets, here we look at cliques in (order invariant) graphs. We look for maximal cliques with certain invariance properties (the same invariance properties as in Emulation Theory). But I have found a somewhat surprising reaction against graphs and cliques among many mathematicians. This negative reaction is of varying degrees of intensity.

Most intense is the statement made to me by a Fields Medalist when asked if they knew what a graph is. ANSWER: No I don't, and I never want to know what a graph is.

So the interesting question here is just what did this Fields Medalist know about graphs to make them never want to know what a graph is?

The answer to this question should reveal a lot about the current mathematical environment.

For many logicians, particularly many of the model theorists, universal classes resonate the most. Computer scientists might be split between universal classes and maximal cliques. General mathematicians and non mathematicians and gifted youth would be expected to prefer maximal emulators. I expect that the further away from math the greater the affinity for emulators.

Of course the dramatic point is that the basic results of Emulation Theory, Invariant Graph Theory, and Invariant A...A Theory are unprovable in ZFC and in fact are provably equivalent to the consistency of certain large cardinal hypotheses, most commonly Con(SRP). They are implicitly Π^0_1 due to their logical form, via the Gödel Completeness Theorem.

We believe that the phenomena kicks in already in dimension 3, although this is yet to be established. Thus the reach of the Incompleteness Phenomena has been greatly extended by such totally clear and accessible thematic examples of Tangible Incompleteness.

Moreover, there is an additional way of looking at these results. Namely as **discrete forms of large cardinal hypotheses**. The analog of N Tail-Related invariance (section 2.2.2) for ordinals in large cardinal theory are very clear and very familiar in set theory. They correspond to the subtle cardinal hierarchy. So our lead statements can be viewed as very natural implicitly Π^0_1 forms of large cardinal hypotheses. Of course it is absolutely essential that the quantifiers in the statements get all rearranged. But the skeleton is essentially the same. Let's take a look at the IMC/2 of section 2.3.1. The discussion can be adapted to the IME/2 of section 2.2.3 and other variants.

IMC/2. Invariant Maximal Clique/2. Every order invariant graph on $Q[-n,n]^k$ has an N Tail-Related invariant maximal clique.

What is the relevant large cardinal hypothesis?

LARGE CARDINAL. There exists a cardinal λ such that for every $B \subseteq \lambda^k$ there exists $\alpha_0 < \ldots \alpha_n < \lambda$ such that B is $\{\alpha_0,\ldots,\alpha_n\}$ Tail-Related invariant.

Thus the λ corresponds to the $Q[-n,n]^k$. The B corresponds to the maximal clique. The $\alpha_0,\ldots,\alpha_n$ corresponds to the $0,\ldots,n$. The $\{\alpha_0,\ldots,\alpha_n\}$ Tail-Related invariance corresponds to the N Tail-Related invariance.

IMC/2 is naturally proved using LARGE CARDINAL, but with some ideas. Not immediate by any means. We let λ be a large cardinal. We weave the large cardinal into a blown up version G* of the graph G where we construct a maximal clique S* in G* in a straightforward natural way (transfinite construction of length λ). Here the vertex set of G* is $\lambda \times Q[0,1)$, as we want density of the ordering. Note that $\lambda \times \{0\}$ forms the crucial closed unbounded set in $\lambda \times Q[0,1)$, which supports the large cardinal combinatorics. We use this maximal clique S* to read off $B \subseteq \lambda^k$ and apply LARGE CARDINAL to B. We get $\alpha_0 < ... < \alpha_n$ from the conclusion of LARGE CARDINAL. We then extract an appropriate countable subsystem of $(G^*,S^*,\alpha_0,...,\alpha_n)$ which takes the form (G,S,0,...,n) where S is a maximal clique in G, after applying an isomorphism that sends each α_i to i, $0 \leq i \leq n$. Actually, we can avoid this countable subsystem construction by invoking the Downward Skolem Lowenheim theorem from model theory.

We also give an explicitly finite form by giving a finite sequential construction associated with IMC/2 and the other variants - see section 4. Thus by transitivity we have finite forms of large cardinal hypotheses. These discrete and finite forms, as they proliferate and simplify, could blur the distinction many people draw between arithmetic and set theoretic statements in terms of their intrinsic objectivity (definite truth values).

We do expect Invariant Maximality to take on a life of its own, and not just take on the role of discrete forms of large cardinal hypotheses.

INVARIANT MAXIMALITY OF THE FUTURE?

1. There will be a vast mathematical subject called Invariant Maximality which cuts across virtually the whole of mathematics. Weak forms of Invariant Maximality, where the invariance is weakened, or where the dimension is extremely low, and so forth, can be treated with various levels of logical power well within ZFC, ranging from RCA_0 through ZC and beyond. However, stronger forms even in low dimensions such as 3 or 4 cannot be handled in ZFC.

2. Because of the basic informal nature of the notions, particularly in Emulation Theory, we will have crude

scientific interpretations such as cloning and growth in biology, and dynamic atomic combination into molecules in chemistry. In the process of making these far fetched crude interpretations less crude and less far fetched, additional nicely motivated mathematical challenges will emerge for Invariant Maximality of intrinsic mathematical interest.

3. In light of the fundamental basic combinatorial essence of Invariant Maximality, there will be powerful applications to mathematical areas that have arisen well before Invariant Maximality. It does not seem realistic for this to happen quickly. A particularly likely place for this to happen is the breaking of the major logjams in computational complexity theory. Of course, I do not know how to apply Invariant Maximality to complexity theory, but over the long run it is likely, in my opinion, to happen. Especially in the more likely form of having first proofs of complexity results using Invariant Maximality, where perhaps upon examination, the use of large cardinals is later removed.

This kind of thing has already happened with work of Richard Laver in connection with left distributive algebras. The main result using extremely large cardinals was removed in favor of ordinary finitary reasoning. There do remain some more technical Π^0_2 statements proved with extremely large cardinals, where there is no known finitary, or even ZFC, proof. However, no ZFC independence has emerged from this Laver originated development. See[1].

4. Invariant Maximality will gain considerable momentum as a great thematic subject of intrinsic interest in its own right. This will be the driving force towards those powerful applications. Namely, in the course of developing Invariant Maximality all over mathematics, new tools of a fundamental nature will need to be developed, and some of these new tools will have unexpected applications to subjects like complexity theory.

5. Invariant Maximality - currently through Emulation Theory - has an intricate totally elementary character, where practically any small randomly generated sets of tuples in low dimensions leads to substantial intricacies which enlist a substantial mathematical mind without requiring any significant mathematical knowledge. In this way, Invariant Maximality will have a revolut9ionary impact on Mathematically Gifted Youth Programs. We have begun

writing about exactly what we have in mind here in the first of a series, [4], dealing with the educational side of Emulation Theory.

6. Presently, Invariant Maximality is confined to the context of sets of rational vectors of fixed dimension, with Emulation Theory, Invariant Graph Theory, and Invariant A...A Theory. We can already say how we can deal with more general contexts.

Let M = (D,rel) consist of a domain D together with finitely many relations of varying numbers of arguments on D. S is an r-emulator of $E \subseteq D^k$ if and only if every $x \in S^r$ is M equivalent to some $y \in E^r$. (M equivalent means satisfies the same atomic formulas). S is a maximal r-emulator of $E \subseteq D^k$ if and only if for all $x \in D^k$, $S \cup. \{x\}$ is not an r-emulator of E.

We will at a later date go on to develop a general theory of suitable invariance notions that lifts the material in section 2.2 to this general context.

We regard this still as a rather limited - but obviously still quite rich - expansion of Emulation Theory. More radical expansions would involve expanding the maximality notion. Here it is based on the obvious inclusion relation. But in general we want to use other notions of refinements. We would look for maximally refined emulators - emulators with no further refined emulators - with invariance properties.

2.2. Emulation Theory

Emulation Theory has these two lead statements. For the required definitions see sections 2.2.1 - 2.2.3.

IME/1. Invariant Maximal Emulation/1. Every finite subset of $Q[-n,n]^k$ has a completely N Tail Shift invariant maximal emulator.

IME/2. Invariant Maximal Emulation/2. Every finite subset of $Q[-n,n]^k$ has an N Tail-Related invariant maximal emulator.

In IME/1 we use complete invariance with respect to the N Tail Shift function $f:Q^k \to Q^k$. This is perhaps the simplest notion of invariance that we can use for Emulation Theory.

In IME/2, we use invariance (complete is automatic here) with respect to the equivalence relation N Tail-Related on Q^k. This is a bit more involved in that its definition refers to order equivalence.

IME/1 and IME/2 are of approximately equal merit, and we expect people to differ on their preference. IME/2 immediately implies IME/1 as N Tail-Related invariance immediately implies complete N Tail Shift invariance.

NOTE: I am going to look again at mere N Tail Shift invariance (without complete). The last time I looked there was a real problem in getting the unprovability in ZFC.

We see that IME/1,2 remain unchanged (over RCA_0) if we drop "finite". However we prefer to keep "finite" to emphasize the tangible nature of IME/1,2.

THEOREM 2.2.3.1. IME/1,2 are implicitly Π^0_1 by the Gödel Completeness Theorem. IME/1,2 are each provably equivalent to Con(SRP) over WKL_0. The forward implication is provable in RCA_0.

In section 2.2.4 we determine which uniform shifts can be used to replace the uniform shift "N Tail Shift" in IMME/1. This is called "shift usability". We see that "N Tail Shift" is distinguished among the usables in certain natural senses.

In section 2.2.5, we take up the IMME/2 usability of relations. We determine which N shift/order invariant relations can be used to replace the N shift/order invariant relation "N Tail-Related" in IMME/2. We show that there is a maximum such relation, and it is N Tail-Related.

2.2.1. N, Z^+, Q, Q[(a,b)], Order Equivalence, Emulator, Maximality

DEFINITION 2.2.1.1. N, Z^+, Q are, respectively, the set of all
nonnegative integers, the set of positive integers, and the

rationals. We use n,m,r,s,t,i,j,k, with or without subscripts, for positive integers unless otherwise indicated. We use a,b,c,d,p,q, with or without subscripts, for rationals unless otherwise indicated. A rational interval is an interval in the extended rationals $Q \cup \{-\infty,\infty\}$. We designate rational intervals by $Q(a,b),Q(a,b],Q[a,b),Q[a,b]$, $a,b \in Q \cup \{-\infty,\infty\}$, for $Q \cup \{-\infty,\infty\}$ intersected with $(a,b),(a,b],[a,b),[a,b]$, respectively. We use I,J, with or without subscripts, for rational intervals. Note that $Q(-\infty,\infty) = Q$ and $Q[-\infty,\infty] = Q \cup \{-\infty,\infty\}$. We use $\cup.$ for disjoint union.
I.e., $A \cup. B$ is $A \cup B$ if A,B are disjoint; undefined otherwise.
$(x_1,\ldots,x_r)$ is the concatenation of the finite sequences $x_1,\ldots,x_s$.

DEFINITION 2.2.1.2. $x,y \in Q^k$ are order equivalent if and only if for all $i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$.

There is an illuminating way to look at the notion of order equivalence: it can be viewed as (Q,<) equivalence, in the sense that x,y obey the same atomic formulas.

DEFINITION 2.2.1.3. S is an emulator of $E \subseteq I^k$ if and only if every element of S^2 is order equivalent to an element of E^2. (Note that S2,E2 ⊆ I2k by concatenation). S is a maximal emulator of $E \subseteq I^k$ if and only if there is no $S \cup. \{x\} \subseteq I^k$ which is an emulator of $E \subseteq I^k$.

Here we are using $E \subseteq I^k$ to indicate that the source E is drawn from the ambient space I^k. Note that the emulators of E do not depend on the ambient space for E, but in general, the maximal emulators of E do depend on the ambient space for E.

The emulators of E themselves are naturally viewed as having the same ambient space I^k as the source. This is important because the ambient space is used for the invariance conditions. See section 2.2.2 for a discussion of invariance.

THEOREM 2.2.1.1. (RCA_0) Every $E \subseteq I^k$ has a finite $E' \subseteq I^k$ with the same emulators and therefore the same maximal emulators. Furthermore we can impose an upper bound on the cardinality of E' that is double exponential in the dimension k.

2.2.2. (Complete) Invariance, Shift, N Shift, N Tail, N Tail Shift, N Tail-Related

DEFINITION 2.2.2.1. Let R be a relation (i.e., a set of ordered pairs) and S ⊆ X. S is R invariant if and only if for all x,y ∈ X with x R y, we have x ∈ S → y ∈ S. S is completely R invariant if and only if for all x,y ∈ X with x R y, we have x ∈ S ↔ y ∈ S. We treat functions as sets of ordered pairs. Note that X is used as the ambient space for S.

A particularly important application of invariance, especially for section 2.3, is order invariance.

DEFINITION 2.2.2.2. S ⊆ I^k is order invariant if and only if for all order equivalent x,y ∈ I^k, x ∈ S ↔ y ∈ S.

There is some basic useful general information about (complete) invariance.

THEOREM 2.2.2.1. (RCA_0) Let R be a relation and S ⊆ X.
i. S is (completely) R invariant if and only if S is (completely) R ∩ X2 invariant.
ii. S is completely R invariant if and only if S is R' invariant, where R' is defined by x R* y ↔ (x R y ∨ y R x).
iii. S is R invariant if and only if S is R* invariant, where R* is the transitive closure of R ∩ X^2.
iv. S is completely R invariant if and only if S is R** invariant, where R** is the least equivalence relation on X containing R ∩ X^2.

There is a convenient generalization to sets of relations R which does not really break new ground (see Theorem 2.2.2.2), but which is expositionally useful.

DEFINITION 2.2.2.3. Let K be a set of relations and S ⊆ X. S is K invariant if and only if for all R ∈ K and x,y ∈ X with x R y, we have x ∈ S → y ∈ S. S is completely K invariant if and only if for all R ∈ K and x,y ∈ X with x R y, we have x ∈ S ↔ y ∈ S.

An interesting phenomena that arises in sections 2.2.4 and 2.2.5 is that certain sets of functions and relations are usable if and only if every element is usable.

THEOREM 2.2.2.2. Let K be a set of relations and S ⊆ X. S is (completely) K invariant if and only if S is (completely) ∪K invariant.

DEFINITION 2.2.2.4. Let $x \in Q^k$. The fractional coordinates of x are the coordinates of x that are not integers. The N Tail Shift of x as the result of adding 1 to all nonnegative integer coordinates greater than all fractional coordinates.

We have enough definitions to support our first of two lead statement in Emulation Theory. However, we make some additional motivating definitions that we use in sections 2.2.3 - 2.2.5.

DEFINITION 2.2.2.5. Let $x \in Q^k$. The Shift of x results in adding 1 to all coordinates. The N Shift of x is the result of adding 1 to all nonnegative integer coordinates (and leaving the other coordinates fixed). The N Tail of x consists of the nonnegative integer coordinates greater than all fractional coordinates (held in place).

Thus the N Tail Shift of x results in adding 1 to its N Tail.

DEFINITION 2.2.2.6. $x,y \in Q^k$ are N Tail-Related if and only if x,y are order equivalent and are identical off of their N Tails.

THEOREM 2.2.2.3. (RCA_0) N Tail-Related on Q^k forms an equivalence relation on Q^k that includes N Tail Shift on Q^k.

2.2.3. Invariant Maximal Emulation

IME (bad). Invariant Maximal Emulation (bad). Every finite subset of $Q[-n,n]^k$ has a completely Shift invariant maximal emulator.

IME (bad). Invariant Maximal Emulation (bad). Every finite subset of $Q[-n,n]^k$ has a completely N Shift invariant maximal emulator.

IME/1. Invariant Maximal Emulation/1. Every finite subset of $Q[-n,n]^k$ has a completely N Tail Shift invariant maximal emulator.

IME/2. Invariant Maximal Emulation/2. Every finite subset of $Q[-n,n]^k$ has an N Tail-Related invariant maximal emulator.

Note that these statements remain unchanged (over RCA_0) if we drop "finite", by Theorem 2.2.1.1. Also note that by Theorem 2.2.2.2, IME/2 immediately implies IME/1 over RCA_0.

THEOREM 2.2.3.1. The two bad IME are refutable in RCA_0.

THEOREM 2.2.3.2. IME/1,2 are implicitly Π^0_1 via Gödel's Completeness Theorem. IME/1,2 are provably equivalent to Con(SRP) over WKL_0. The forward implications are provable in RCA_0.

2.2.4. r-Emulation, N Order Equivalence, Uniform Shifts, Shift Usability

We now investigate what alternatives to the N Tail Shift can be used in IME/1. There appear to be interesting open technical issues yet to be resolved. However, we have resolved these issues if we use the following very natural strengthening of IME/1.

DEFINITION 2.2.4.1. S is an r-emulator of $E \subseteq I^k$ if and only if every $x \in S^r$ is order equivalent to some $y \in E^r$. S is a maximal r-emulator of $E \subseteq I^k$ if and only if no $S \cup. \{x\} \subseteq I^k$ is an r-emulator of $E \subseteq I^k$.

Note that an emulator is a 2-emulator.

IMME/1. Invariant Maximal Multi Emulation/1. Every finite subset of $Q[-n,n]^k$ has a completely N Tail Shift invariant maximal r-emulator.

IMME/2. Invariant Maximal Multi Emulation/2. Every finite subset of $Q[-n,n]^k$ has an N Tail-Related invariant maximal r-emulator.

IMME/1 with r = 2 and IME/1 are the same. IMME/2 with r = 2 and IME/2 are the same.

THEOREM 2.2.4.1. IMME/1,2 are implicitly Π^0_1 via Gödel's Completeness Theorem. IMME1,2 are provably equivalent to Con(SRP) over WKL_0. The forward implications is provable in RCA_0.

DEFINITION 2.2.4.2. Let $f:Q^k \to Q^k$. f is a shift if and only if for all $x \in Q^k$, f(x) adds 1 to zero or more coordinates of x. $f:Q^k \to Q^k$ is an N shift if and only if for all $x \in Q^k$, f(x) adds 1 to zero or more nonnegative integer coordinates of x.

Note that there are uncountably many shifts on Q^k, and even uncountably many N shifts on Q^k. Thus we want to impose a uniformity condition on the shifts on Q^k. For this purpose, we use the following natural equivalence relation.

DEFINITION 2.2.4.3. $x,y \in Q^k$ are N order equivalent if and only if x,y are order equivalent and for all $i \leq k$, $x_i \in N \leftrightarrow y_i \in N$.

Note that order equivalence can be viewed as (Q,<) equivalence (as remarked right after Definition 2.2.1.2). Here N order equivalence can be viewed as (Q,<,N) equivalence.

Informally, a uniform shift is a shift where the choice of positions of coordinates that are shifted is the same for "similar" $x \in Q^k$.

DEFINITION 2.2.4.4. $f:Q^k \to Q^k$ is a uniform shift if and only if f is a shift where for all N order equivalent $x,y \in Q^k$ and $i \leq k$, $f(x)_i = x_i+1 \leftrightarrow f(y)_i = y_i+1$.

THEOREM 2.2.4.2. For each k there are finitely many uniform shifts on Q^k. The number is bounded above by a triple exponential in k.

DEFINITION 2.2.4.5. $f:Q^k \to Q^k$ is IME/1, IMME/1 usable if and only if we can replace N Tail Shift in IME/1, IMME/1, respectively, with f.

Which uniform shifts are IME/1 usable? We are not quite sure, as there are some open technical issues. Which uniform shifts are IMME/1 usable?

DEFINITION 2.2.4.6. An N subtail of $x \in Q^k$ is a set B of coordinates of x, held in place, where every $x_j \geq x_i \in B$ lies in $B \cap N$. An N subtail shift on Q^k is an $f:Q^k \to Q^k$ where each f(x) is obtained by adding 1 to an N subtail of x.

IMMESU. Invariant Maximal Multi Emulation Shift Usability. A uniform shift on Q^k is IMME/1 usable if and only if it is an N subtail shift.

IMMESU*. Invariant Maximal Multi Emulation Shift Usability*. A set of uniform shifts on Q^k is IMME/1 usable if and only if it consists of N subtail shifts.

THEOREM 2.2.4.3. The forward implications in IMMESU, IMMESU* are provable in RCA0. The reverse implications in IMMESU, IMMESU* are provably equivalent to Con(SRP) over WKL0. IMMEU → Con(SRP) and IMMESU* → Con(SRP) are provable in RCA0.

We can now pick out N Tail Shift from among the IMME usable.

COROLLARY 2.2.4.4. N Tail Shift on Q^k is
i. The most moving IMME/1 usable uniform shift on Q^k.
ii. The pointwise coordinatewise maximum IMME/1 usable uniform shift on Q^k.

Let $H:Q^k \to Q^k$ be given by $H(x) = x+1$ if $x_1 = \ldots = x_k \notin N$; x otherwise.

THEOREM 2.2.4.5. H is an IME/1 uniform shift on Q^k that is not an N subtail shift.

However we do have something in the vicinity of IMMEU for IME.

IMESU. Invariant Maximal Emulation Shift Usability. A uniform N shift on Q^k is IME/1 usable if and only if it is an N subtail shift.

IMESU*. Invariant Maximal Emulation Shift Usability*. A set of uniform N shifts on Q^k is IME/1 usable if and only if it consists of N subtail shifts.

THEOREM 2.2.4.6. The forward implications in IMESU, IMESU* are provable in RCA_0. The reverse implications in IMESU, IMESU* are provably equivalent to Con(SRP) over WKL_0. IMESU → Con(SRP) and IMESU* → Con(SRP) are provable in RCA_0.

COROLLARY 2.2.4.7. A set of uniform shifts on Q^k is IMME usable if and only if each element is IMME usable. A set of uniform N shifts on Q^k is IME usable if and only if each element is IME usable.

The obvious proof of Corollary 2.2.4.7 is in WKL_0 + Con(SRP). We conjecture that Corollary 2.2.4.7 is provably equivalent to Con(SRP) over WKL_0.

2.2.5. Relation Usability

DEFINITION 2.2.5.1. $R \subseteq Q^k \times Q^k$ is IME/2, IMME/2 usable if and only if we can replace N Tail-Related in IME/2, IMME/2, respectively, with R. We make the same definition for sets of $R \subseteq Q^k \times Q^k$.

Recall that N Tail-Related on Q^k is the equivalence relation defined in Definition 2.2.2.6.

THEOREM 2.2.5.1. (RCA_0) N Tail-Related on Q^k is the least equivalence relation containing all N subtail shifts.

DEFINITION 2.2.5.2. Let R be a relation on Q^k. R is N order invariant if and only if it is N order invariant as a subset of Q^{2k}. R is N Shift invariant if and only if the N Shift of every element of $R \subseteq Q^{2k}$ is in R. (This is the usual invariance notion for functions, and it is important that this not be complete invariance). R is N shift/order invariant if and only if R is N shift invariant and N order invariant.

IMMERU. Invariant Maximal Multi Emulation Relation Usability. N Tail-Related on Q^k is the largest N shift/order invariant IMME usable relation on Q^k.

IMMERU*. Invariant Maximal Multi Emulation Relation Usability*. A set of N shift/order invariant relations is IMME usable if and only if all of its elements are subsets of N Tail-Related on Q^k.

THEOREM 2.2.5.1. The forward implications in IMMERU and IMMERU* are provably equivalent to Con(SRP) over WKL_0. The forward implication in IMERU* is provable in RCA_0. IMEU → Con(SRP) and IMEU* → Con(SRP) are provable in RCA_0.

COROLLARY 2.2.5.2. A set of N shift/order invariant relations is IMME usable if and only if each element is IMME usable.

The forward direction of Corollary 2.2.5.1 is provable in RCA0 and the reverse direction is provable in WKL0 + Con(SRP). We do not know if Corollary 2.2.5.2 is provable in ZFC.

2.3. Invariant Graph Theory

Invariant Graph Theory is an alternative to Invariant Emulation Theory. It is a bit stronger than Invariant Graph Theory, although both arrive at the same place metamathematically. Namely at Con(SRP).

2.3.1. (Hyper) Graphs, Order Invariant (Hyper) Graphs, (Hyper) Cliques, Maximal (Hyper) Cliques

DEFINITION 2.3.1.1. $S \subseteq I^k$ is order invariant if and only if for all order equivalent $x,y \in I^k$, $x \in S \leftrightarrow y \in S$.

DEFINITION 2.3.1.2. A graph is a pair $G = (V,E)$, where V is the set of vertices, $E \subseteq V^2$ the set of edges, where it is required that for all $v,w \in V$, $v \neg E v$ and ($v E w \leftrightarrow w E v$). v,w are adjacent if and only if v E w. We say that G is a graph on V. A clique in G is a subset of V where any two distinct elements are adjacent. A maximal clique in G is a clique S in G such that for all $v \in V$, $S \cup. \{v\}$ is not a clique in G. An order invariant graph on I^k is a graph (I^k,E) where $E \subseteq I^k$ is order invariant (here I^k is the ambient space for E).

Recall the r-emulators in Emulation Theory. The analog in Invariant Graph Theory is hypergraphs, or r-graphs.

DEFINITION 2.3.1.3. An r-graph is a pair $G = (V,E)$, where V is the set of vertices, $E \subseteq V^r$ the set of hyper edges, where it is required that for all $x \in E$, $x_1,...,x_r$ are distinct, and every permutation of x lies in E. We say that G is an r-graph on V. An r-clique in G is an S ⊆ V where any r-tuple of distinct elements of S lies in E. A maximal r-clique in G is an r-clique S in G such that for all v ∈ V, S ∪. {v} is not an r-clique in G. An order invariant r-graph on I^k is an r-graph (I^k,E) where $E \subseteq (I^k)^r$ is an order invariant subset of I^{kr} (the ambient space).

IMC/1. Every order invariant graph on $Q[-n,n]^k$ has a completely N Tail Shift invariant clique.

IMC/2. Every order invariant graph on $Q[-n,n]^k$ has an N Tail-Telated invariant clique.

By Theorem 2.2.2.2, IME/2 immediately implies IME/1 over RCA_0.

THEOREM 2.3.1.1. IMC/1,2 are implicitly Π^0_1 via Gödel's Completeness Theorem. IMC/1,2 are provably equivalent to Con(SRP) over WKL_0. The forward implication is provable in RCA_0.

IMHC/1. Invariant Maximal Hyper Clique/1. Every order invariant r-graph on $Q[-n,n]^k$ has a completely N Tail Shift invariant r-clique.

IMHC/2. Invariant Maximal Hyper Clique/2. Every order invariant r-graph on $Q[-n,n]^k$ has an N Tail-Related invariant r-clique.

THEOREM 2.2.3.1. IMHC/1,2 are implicitly Π^0_1 via Gödel's Completeness Theorem. IMHC/1,2 are provably equivalent to Con(SRP) over WKL_0. The forward implication is provable in RCA_0.

2.3.2. Invariant Maximal (Hyper) Cliques

IMC/1. Every order invariant graph on $Q[-n,n]^k$ has a completely N Tail Shift invariant clique.

IMC/2. Every order invariant graph on $Q[-n,n]^k$ has an N Tail-Telated invariant clique.

By Theorem 2.2.2.2, IME/2 immediately implies IME/1 over RCA_0.

THEOREM 2.3.1.1. IMC/1,2 are implicitly Π^0_1 via Gödel's Completeness Theorem. IMC/1,2 are provably equivalent to Con(SRP) over WKL_0. The forward implication is provable in RCA_0.

IMHC/1. Invariant Maximal Hyper Clique/1. Every order invariant r-graph on $Q[-n,n]^k$ has a completely N Tail Shift invariant r-clique.

IMHC/2. Invariant Maximal Hyper Clique/2. Every order invariant r-graph on $Q[-n,n]^k$ has an N Tail-Related invariant r-clique.

THEOREM 2.2.3.1. IMHC/1,2 are implicitly Π^0_1 via Gödel's Completeness Theorem. IMHC/1,2 are provably equivalent to Con(SRP) over WKL_0. The forward implication is provable in RCA_0.

2.3.3. Usability

Usability for IMC/1,2 and IMHC/1,2 for functions and relations is defined exactly as they were for IME/1,2 and IMME/1,2. The results are identical.

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