# TANGIBLE INCOMPLETENESS INTERIM REPORT

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This Interim Report covers the state of the art Invariant Maximality, which is part 2 of the forthcoming book Tangible Incompleteness. Part 1 is Boolean Relation Theory, which appears in final form on my website https://u.osu.edu/friedman.8/foundational-adventures/boolean-relation-theory-book/

The development here of Invariant Maximality is in the most basic fundamental combinatorial directions. We plan to incorporate richer mathematical contexts.

For the quickest glance look at section 2.10.

TANGIBLE INCOMPLETENESS - book

3 - ?. ADDITIONAL CHAPTERS

## 2.1. INTRODUCTION

Invariance and Maximality, separately, figure prominently throughout mathematics as notions of great intrinsic interest and power. Our new Invariant Maximality combines these notions in seamless ways with spectacular effect. Thus we ask for a maximal object - whose existence is generally obvious via Zorn's Lemma or direct sequential construction - which also exhibits certain invariance. This combination is unexpectedly overwhelmingly explosive.

The main context for the development is the rationals with its usual order where the nonnegative integers 0,1,2,... are treated as preferred elements. More specifically, we work in the spaces Q[-n,n]k, where Q[-n,n] is the closed interval of rationals from -n to n, with preferred elements 0,1,...,n. We only use that Q[-n,n] is a countable dense linear ordering with endpoints, and -n < 0 < 1 < ... n. This is far more elemental than even addition.

LEAD STATEMENTS TEMPLATE. Given any finitely described gadget $\gamma$ in $Q[-n,n]^k$ of a certain kind, there is an inclusion maximal associate $S \subseteq Q[-n,n]^k$ of $\gamma$, which also enjoys a certain natural invariance condition.

Here the existence of an inclusion maximal associate $S \subseteq Q[-n,n]^k$ is immediate by Zorn's Lemma or by a direct sequential construction. Also the invariance condition is easily enjoyed by a multitude of $S \subseteq Q[-n,n]^k$.

However, getting both maximality and invariance in our contexts is extremely difficult, and in fact we achieve this in three separate realms (see 2.4, 2.5, 2.6), with a total of ten specific claims. But these ten lead statements are established only by extending ZFC with so called large cardinal hypotheses - and thus far more than the usual ZFC axioms for mathematics. We know that ZFC itself is insufficient, provided that ZFC is consistent.

We also know that there are specific fairly small integers k,n for which our ten lead statements are unprovable in ZFC (assuming ZFC is consistent). For the statements involving legs, emulators, duplicators, and cliques, we have a target of $Q[-3,3]^3$ for the unprovability from ZFC, whereas $Q[-n,n]^2$ can be handled within ZFC.

That a specific $Q[-n,n]^k$ is sufficient to attain unprovability from ZFC shows that we are not even using the +1 function on the positive integers.

The main invariance condition uses a specific natural function, the upper shift for N, introduced in section 2.3. Our general framework for stating relevant invariance conditions, in particular complete invariance, is presented in section 2.2 (invariance and complete invariance of subsets of ambient spaces with respect to relations).

Our ten thematic lead statements, which fall under the Lead Statement Template above, are each provable in $SRP^+$ but not in SRP, assuming SRP is consistent. In fact, we show that they are not provable from any consistent set of theorems of SRP that include $RCA_0$. See section 2.9. They are, moreover, provably equivalent to Con(SRP) over $WKL_0$ (with the implications to Con(SRP) provable in $RCA_0$).

How Tangible are these statements? They do assert the existence of an infinite subset of $Q[-n,n]^k$ and so prima facie they lack Tangibility, although the setting is countably infinite.

We address this issue in two ways. Firstly, despite the quantification over infinite sets, a totality of size the continuum, the meaning of the statements depends only on the behavior of finite objects, and not on the behavior of infinite objects. More precisely, we claim that our statements are what we call implicitly $\Pi^0_1$. This means that there are $\Pi^0_1$ sentences that are provably equivalent to our respective statements over ZFC, and in our case, a very weak fragment of ZFC called specifically $WKL_0$ (see section 2.9). $\Pi^0_1$ sentences are sentences that assert that a specific computer algorithm does not terminate. For example, FLT and Goldbach's Conjecture are obviously $\Pi^0_1$ statements. In terms of quantifiers over integers, they are viewed as $\forall$ statements. However, the Collatz Conjecture is not, and is in fact $\Pi^0_2$, referring to the quantifier combination $\forall\exists$. Collatz Conjecture can obviously be strengthened by introducing an upper bound (say exponential) in which case such Sharp Collatz Conjectures are $\Pi^0_1$. It is quite common throughout mathematics to encounter $\Pi^0_2$ and $\Pi^0_3$ conjectures ($\Pi^0_3$ is $\forall\exists\forall$) which are easily strengthened in this way to become $\forall$ conjectures, and often such $\forall$ forms are eventually proved.

That our eight main statements are implicitly $\Pi^0_1$ is seen from their logical form, which supports a powerful use of the Gödel Completeness Theorem. The existence of the maximal subset of $Q[-n,n]^k$ corresponds to the satisfiability of a finite set of sentence in the usual first order predicate calculus with equality - said finite set effectively generated from parameters n,k.

The foundational significance of a statement P being implicitly $\Pi^0_1$ include

a. P is immune to the forcing method originated by P.J. Cohen. I.e., if P can be proved in any extension of ZFC that can be demonstrably forced, such as ZFC plus the continuum hypothesis or ZFC plus the negation of the continuum hypothesis, then P can be proved in ZFC alone. The same holds with ZFC replaced by ZFC + "there exists a strongly inaccessible cardinal", and other large cardinal hypotheses.

b. P has the refutability property: i.e., P is demonstrably refutable in the sense that we know a priori that if P is false then P is, in principle, provably false (in a very weak system). A hallmark of most physical theories is that they have the experimental refutability property: we know that if they are false then they are, in principle, provably false (by experimentation). Physical theories that do not have this experimental refutability property are generally marginalized by the scientific community, and in particular by Nobel Prize committees.

Of course this line of reasoning depends on the acceptance of the standard conception of infinite subsets of $Q[-n,n]^k$. However, there are those who question this standard conception, most commonly the constructivists, originally led by L.E.J. Brouwer.

Such conditions, although reasonably convincing, have compelled us to search hard for the strongest kind of direct Tangibility that we can have for unprovability from ZFC. This is of course for explicitly (as opposed to implicitly) $\Pi^0_1$ statements unprovable in ZFC. Attempts over the years have not been fully successful in that the extreme level of naturalness and simplicity was sacrificed somewhat.

We have recently discovered how to preserve this naturalness and simplicity by weakening the maximality condition to allow finite subsets of $Q[-n,n]^k$. In particular, we have discovered N maximality: in our ten main statements, we instead ask for a finite N maximal set. These statements are prima facie $\Pi^0_2$, involving only quantification over finite objects. By placing an obvious a priori upper bound on the size of the finite N maximal set, we are led to explicitly $\Pi^0_1$ incompleteness (see section 2.7).

Moreover, the Tangibility and explicit $\Pi^0_1$-ness are more immediate when the setting is transferred directly into the integers (see section 2.8).

We close by commenting on the three settings that we use for our statements. These are the maximal squares and Legs in 2.4, the maximal emulators in 2.5, and the maximal cliques in 2.6.

In terms of mathematical simplicity, from the point of view of the mathematically experienced, the squares (or legs) formulation is best.

The emulators approach has perhaps the broadest potential appeal among lesser experienced mathematical thinkers, including the mathematically gifted high school students. This is because the outermost universal quantifier is over the very basic finite sets of rational vectors of fixed dimension, and the notion of emulator can be described informally and thematically without getting into the details (and yet the details are transparent).

In emulation theory, practically any small set E of rational vectors in low dimension leads to an interesting if not rich development. Informally, S is an emulator of E if and only if any two elements of S (taken as a whole) "look like" some two elements of E (taken as a whole). We look for inclusion maximal emulators with certain invariance properties. This will support a very effective and exciting curriculum at the mathematically gifted high school level, and the plan is to develop this for use in the main Summer School programs for students at that level. See [Fr19b].

For many discrete mathematicians, especially graph theorists, the graph/clique approach may resonate more. Instead of emulators of finite sets, here we look at cliques in (order invariant) graphs. But I have found a somewhat surprising reaction against graphs and cliques among many mathematicians. This negative reaction is of varying degrees of intensity.

Most intense is the statement made to me by a Fields Medalist when asked if they knew what a graph is. ANSWER: No I don't, and I never want to know what a graph is.

Other important mathematicians have expressed extreme distaste for cliques as they connote extreme social snobbery.

Of course relations and legs are closely related to graphs and cliques, and so we thereby accommodate different kinds of mathematical thinkers. Emulators and duplicators are a much more substantial variation, with special advantages educationally.

## 2.2. PRELIMINARIES

DEFINITION 2.2.1. N, $Z^+$, Q are, respectively, the set of nonnegative integers, the set of positive integers, and the set of rationals. We use i,j,k,n,m,r,s,t with or without subscripts for positive integers, unless otherwise indicated. We use p,q with or without subscripts for rational numbers, unless otherwise indicated. We use A,B,C,D,E,K,S,T,U,V,W with or without subscripts for sets unless otherwise indicated. For p,q $\in$ Q, Q[p,q] is the closed interval of rationals {r: p $\leq$ r $\leq$ q}. $(x_1,...,x_r)$ is the concatenation of the finite sequences $x_1,...,x_s$. A* is the set of all finite sequences from A.

We base all of our ten lead statements in sections 2.4, 2.5, 2.6, on the intervals Q[-n,n], and the spaces Q[-n,n]k. We use only that Q[-n,n] is a countable dense linear ordering with a left and right endpoint, with the distinguished elements 0,1,...,n.

Invariance and complete invariance play a central role in the development. We use the following general framework.

DEFINITION 2.2.1. Let R be a relation, in the sense of an arbitrary set of ordered pairs. $S \subseteq X$ is R invariant if and only if for all $x,y \in X$ with $x\ R\ y$, $x \in S \rightarrow y \in S$. $S \subseteq X$ is completely R invariant if and only if for all $x,y \in X$ with $x\ R\ y$, $x \in S \leftrightarrow y \in S$.

Note that we are treating X as the ambient space for $E \subseteq X$, because of the condition $x,y \in X$. Two important special cases are where R is a function and where R is an equivalence relation. Functions are treated as relations in the standard way. In the latter case, invariant and completely invariant are the same. More generally, invariant and completely invariant are the same for symmetric relations R.

The most important equivalence relation for us is as follows.

DEFINITION 2.2.2. Order equivalence is the relation on Q* given as follows. $x,y \in Q^k$ are order equivalent if and only if x,y are of the same length, and for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$. $E \subseteq Q[-n,n]^k$ is order invariant if and only if $E \subseteq Q[-n,n]^k$ is order equivalence invariant according to Definition 2.2.1.

THEOREM 2.2.1. There are finitely many order invariant $E \subseteq Q[-n,n]^k$. The number can be bounded by a triple exponential in k.

THEOREM 2.2.2. The order invariant $E \subseteq Q[-n,n]^k$ are the $E \subseteq Q[-n,n]^k$ defined by a Boolean combination of inequalities between variables $v_1,\ldots,v_k$, without parameters. They are also the k-dimensional 0-definable sets in (Q,<) intersected with $Q[-n,n]^k$.

## 2.3. UPPER SHIFT FOR N

DEFINITION 2.3.1. The upper shift for N maps Q* into Q* as follows. For $x \in Q^*$, USH/N(x) is obtained by adding 1 to all coordinates of x greater than all coordinates of x outside N.

The notion

$S \subseteq Q[-n,n]^k$ is completely USH/N invariant

plays a crucial role in our development. For the mathematically gifted high school student setting we focus mainly on $k = 2$ and $n = 1$, where this notion is particularly easy to explain. See section 2.5 for more on this setting.

THEOREM 2.3.1. $S \subseteq Q[-1,1]^2$ is completely USH/N invariant if and only if the following hold.
i. $(0,0) \in S \leftrightarrow (1,1) \in S$.
ii. For all $p < 0$, $(p,0) \in S \leftrightarrow (p,1) \in S$,
iii. For all $p < 0$, $(0,p) \in S \leftrightarrow (1,p) \in S$.

## 2.4. MAXIMAL SQUARES, CUBES, LEGS

DEFINITION 2.4.1. A square in E is a subset $A^2 \subseteq E$. An r-cube in E is a subset $A^r \subseteq E$. A leg in E is an A such that $A^2 \subseteq E$. An r-leg in E is an A such that $A^r \subseteq E$.

Obviously squares are just 2-cubes and legs are just 2-legs.

We consider squares in the $E \subseteq Q[-n,n]^k$, and r-cubes in the $E \subseteq Q[-n,n]^k$. For odd k, the only square in $E \subseteq Q[-n,n]^k$ and the only leg in $E \subseteq Q[-n,n]^k$ is $\emptyset$. For r not dividing k, the only r-cube in $Q[-n,n]^k$ and the only r-leg in $Q[-n,n]^k$ is $\emptyset$. Thus the only interesting cases of squares and legs is for even k, and the only interesting cases of r-cubes and r-legs is when $r|k$.

DEFINITION 2.4.2. A maximal square, leg, r-cube, r-leg S in $E \subseteq Q[-n,n]^k$ is a square, leg, r-cube, r-leg in $E \subseteq Q[-n,n]^k$ where every superset of S that is a square, leg, r-cube, r-leg in $E \subseteq Q[-n,n]^k$ is S.

THEOREM 2.4.1. ($RCA_0$) The maximal legs in $E \subseteq Q[-n,n]^k$ are the sides of the maximal squares in $E \subseteq Q[-n,n]^k$. The maximal r-legs in $E \subseteq Q[-n,n]^k$ are the sides of the maximal r-cubes in $E \subseteq Q[-n,n]^k$.

THEOREM 2.4.2. ($RCA_0$) Every subset of every $Q[-n,n]^k$ has a maximal square and maximal leg, and for all r, has a maximal r-cube and maximal r-leg.

THEOREM 2.4.2. The following are provably equivalent over $RCA_0$.
i. In every order invariant subset of $Q[-1,1]^2$, every square (leg) extends to a maximal square (leg).
ii. In every subset of every $Q[-n,n]^k$, for all r, every r-cube (r-leg) extends to a maximal r-cube (r-leg).
iii. $ACA_0$.

Now look what happens when we combine maximality with complete invariance.

INVARIANT MAXIMAL SQUARES. IMS. Every order invariant subset of every $Q[-n,n]^k$ has a completely USH/N invariant maximal square.

INVARIANT MAXIMAL CUBES. IMC. Every order invariant subset of every $Q[-n,n]^k$ has, for all r, a completely USH/N invariant maximal r-cube.

INVARIANT MAXIMAL LEGS. IML. Every order invariant subset of every $Q[-n,n]^k$ has a completely USH/N invariant maximal leg.

INVARIANT MAXIMAL LEGS*. IML*. Every order invariant subset of every $Q[-n,n]^k$ has, for all r, a completely USH/N invariant maximal r-leg.

THEOREM 2.4.3. IMS, IMC, IML, IML* are implicitly $\Pi^0_1$ via the Gödel Completeness Theorem. IMS, IMC, IML, IML* are provably equivalent to Con(SRP) over $WKL_0$. The implication to Con(SRP) is provable in $RCA_0$. IMS, IMC, IML, IML* are not provable in any consistent set of theorems of SRP that include $RCA_0$. In particular IMS, IMC, IML, IML* are not provable in ZFC provided ZFC is consistent.

## 2.5. MAXIMAL EMULATORS, DUPLICATORS

The Maximal Emulator formulation does not rely on order invariant sets as does the maximal square formulation. We find that this is a distinct advantage for the mathematically gifted high school student setting. However, order equivalence is used in the definition of emulators. We believe that this tradeoff is highly advantageous for the systematic study of special examples, something very effective for the mathematically gifted high school student setting. But for the professional mathematician, the maximal square (and leg) formulation is more straightforwardly mathematical.

The Maximal Duplicator formulation is an obvious variant of the Maximal Emulator formulation that serve the same educational purpose. When looking at the details of the presentations, we have concluded that the Maximal Emulator formulation is a bit preferable to the Maximal Duplicator formulation.

The Maximal Clique formulation of section 2.6 is particularly attractive for graph theorists.

DEFINITION 2.5.1. S is an emulator of $E \subseteq Q[-n,n]^k$ if and only if $S \subseteq Q[-n,n]^k$ and every element of $S^2$ is order equivalent to an element of $E^2$. S is a duplicator of $E \subseteq Q[-n,n]^k$ if and only if $S \subseteq Q[-n,n]^k$ and every element of $S^2$ is order equivalent to an element of $E^2$ and vice versa. I.e., $E^2$ and $S^2$ are the same up to order equivalence.

DEFINITION 2.5.2. S is a maximal emulator (duplicator) of $E \subseteq Q[-n,n]^k$ if and only if S is an emulator (duplicator) of $E \subseteq Q[-n,n]^k$ and every superset of S that is an emulator (duplicator) of $E \subseteq Q[-n,n]^k$ is S.

THEOREM 2.5.1. ($RCA_0$) Every subset of $Q[-n,n]^k$ has the same emulators and duplicators as one of its finite subsets. We can put an a priori upper bound on the size of the subset as a double exponential in k.

From Theorem 2.5.1 we see that our statements about emulators and duplicators can be equivalently phrased in terms of all subsets or all finite subsets. We will henceforth state them in terms of all subsets, where the reader understands that only finite subsets matter.

THEOREM 2.5.2. ($RCA_0$) Every subset of every $Q[-n,n]^k$ has a maximal emulator (duplicator).

THEOREM 2.5.3. The following are provably equivalent over $RCA_0$.
i. For every subset of $Q[-1,1]^2$, every emulator (duplicator) extends to a maximal emulator (duplicator).
ii. For every subset of every $Q[-n,n]^k$, every emulator (duplicator) extends to a maximal emulator (duplicator).
iii. $ACA_0$.

We now combine maximality with complete invariance.

INVARIANT MAXIMAL EMULATORS. IME. Every subset of every $Q[-n,n]^k$ has a completely USH/N invariant maximal emulator.

INVARIANT MAXIMAL DUPLICATORS. IMD. Every subset of every $Q[-n,n]^k$ has a completely USH/N invariant maximal duplicator.

We now introduce a parameter r as we did with r-cubes.

DEFINITION 2.5.3. S is an r-emulator of $E \subseteq Q[-n,n]^k$ if and only if $S \subseteq Q[-n,n]^k$ and every element of $S^r$ is order equivalent to an element of $E^r$. S is an r-duplicator of $E \subseteq Q[-n,n]^k$ if and only if $S \subseteq Q[-n,n]^k$ and every element of $S^r$ is order equivalent to an element of $E^r$ and vice versa. I.e., $E^r$ and $S^r$ are the same up to order equivalence.

DEFINITION 2.5.4. S is a maximal r=emulator (r-duplicator) of $E \subseteq Q[-n,n]^k$ if and only if S is an r-emulator (r-duplicator) of $E \subseteq Q[-n,n]^k$ and every superset of S that is an r-emulator (r-duplicator) of $E \subseteq Q[-n,n]^k$ is S.

INVARIANT MAXIMAL MULTI EMULATORS. IMME. Every subset of every $Q[-n,n]^k$ has, for all r, a completely USH/N invariant maximal r-emulator.

INVARIANT MAXIMAL MULTI DUPLICATORS. IMMD. Every subset of every $Q[-n,n]^k$ has, for all r, a completely USH/N invariant maximal r-duplicator.

THEOREM 2.5.4. IME, IMD, IMME, IMMD are implicitly $\Pi^0_1$ via the Gödel Completeness Theorem. IME, IMD, IMME, IMMD are provably equivalent to Con(SRP) over $WKL_0$. The implication to Con(SRP) is provable in $RCA_0$. IME, IME, IMME, IMMD are not provable in any consistent set of theorems of SRP that include $RCA_0$. In particular IME, IME, IMME, IMMD are not provable in ZFC provided ZFC is consistent.

## 2.6. MAXIMAL CLIQUES, HYPERCLIQUES

DEFINITION 2.6.1. G is a graph if and only if G = (V,E), where V is the set of vertices, and $E \subseteq V^2$ is the set of edges, and where E is required to be irreflexive and symmetric. G is an order invariant graph on $Q[-n,n]^k$ if and only if $V = Q[-n,n]^k$ and $E \subseteq Q[-n,n]^{2k}$ is order invariant.

DEFINITION 2.6.2. H is an r-graph if and only if H = (V,E), where V is the set of vertices, and $E \subseteq V^r$ is the set of edges, where every element of E has distinct coordinates, and all of its permutations lie in E. S is an r-clique of H if and only if $S \subseteq V$ and every r-tuple of distinct elements of S lies in E. H is an order invariant r-graph on $Q[-n,n]^k$ if and only if $V = Q[-n,n]^k$ and $E \subseteq Q[-n,n]^{rk}$ is order invariant.

DEFINITION 2.6.3. S is a clique (r-clique) in G (H) if and only if $S \subseteq V$ and every ordered pair (r-tuple) of distinct elements of S lies in E. S is a maximal clique (r-clique) in G (H) if and only if S is a clique (r-clique) in G (H) where every superset of S that is a clique (r-clique)in G (H) is S.

Note that a graph is a 2-graph and a clique is a 2-clique.

THEOREM 2.6.1. (RCA0) Every r-graph on $Q[-n,n]^k$ has a maximal r-clique.

THEOREM 2.5.2. The following are provably equivalent over $RCA_0$.
i. In every order invariant graph on $Q[-1,1]^2$, every clique is contained in a maximal clique.
ii. In every r-graph on every $Q[-n,n]^k$, every r-clique is contained in a maximal r-clique.

We now combine maximality with complete invariance.

INVARIANT MAXIMAL CLIQUES. IMCL. Every order invariant graph on every $Q[-n,n]^k$ has a completely USH/N invariant maximal clique.

INVARIANT MAXIMAL HYPERCLIQUES. IMH. Every order invariant r-graph on every $Q[-n,n]^k$ has a completely USH/N invariant maximal r-clique.

THEOREM 2.5.3. IMCL, IMH are implicitly $\Pi^0_1$ via the Gödel Completeness Theorem. IMCL, IMH are provably equivalent to Con(SRP) over $WKL_0$. The implication to Con(SRP) is provable in $RCA_0$. IMCL, IMH are not provable in any consistent set of theorems of SRP that include $RCA_0$. In particular IMCL, IMH are not provable in ZFC provided ZFC is consistent.

## 2.7. N MAXIMALITY

The use of maximality in our statements generally forces the maximal object to be infinite. However, there is a natural weakening of maximality to N maximality, and it is easy to see a priori that finite N maximal objects exist. This leads to explicitly $\Pi^0_1$ weakenings of our statements.

DEFINITION 2.7.1. $x,y \in Q^k$ are N equivalent if and only if for all $1 \leq i \leq k$, $x_i = y_i \in N \vee x_i,y_i \notin N$. $S,S' \subseteq Q^k$ are N equivalent if and only if every element of S is N equivalent to an element of S' and vice versa.

THEOREM 2.7.1. N equivalence is an equivalence relation on $Q^k$ with infinitely many equivalence classes. N equivalence restricted to $Q[-n,n]^k$ has finitely many equivalence classes. N equivalence on the subsets of $Q^k$ is an equivalence relation on the subsets of $Q^k$ with $\omega$ many equivalence classes. N equivalence restricted to the subsets of $Q[-n,n]^k$ has finitely many equivalence classes.

DEFINITION 2.7.2. Let P be a property of subsets of $Q[-n,n]^k$. S is maximal for P if and only if S has P, and every superset of S with P is S. S is N maximal for P if and only if S has P, and every superset of S with P is N equivalent to S.

We have presented ten examples of Tangible Incompleteness in sections 2.4, 2.5, 2.6 (also see 2.10A). In 2.10B we present the corresponding weakenings by replacing "maximal" by "finite N maximal". This removes the infinite objects from our statements. See 2.10B where this replacement is made.

THEOREM 2.7.1. In all ten statements with "finite N maximal" (see 2.10B), there is an a priori upper bound on the size of the N maximal object which is double exponential in k,r. Using quantifier elimination on (Q,<), this puts all ten of the above in explicitly $\Pi^0_1$ form.

THEOREM 2.7.2. All ten in 2.10B are provably equivalent to Con(SRP) over EFA.

## 2.8. IN THE INTEGERS, rN MAXIMALITY, USH(rN)

The explicitly finite nature of the ten statements in 2.10B allows us to move from the rationals with $Q[-n,n]^k$ to the integers with $[-kt,kt]^k$, provided t >> k. The >> is rather tame. We can use, for example, t > (8k)!, as we do in 2.10D.

All of the definitions we have used on the $Q[-n,n]^k$ are transferred over without change to the $[-kt,kt]^k$, except that N needs to be replaced by tN. Here are the two uses of tN.

DEFINITION 2.8.1. The upper shift for tN maps Q* into Q* as follows. For x ∈ Q*, USH/tN(x) is obtained by adding t to all coordinates of x greater than all coordinates of x outside tN.

DEFINITION 2.8.2. x,y ∈ Qk are tN equivalent if and only if for all 1 ≤ i ≤ m, $x_i = y_i \in tN \vee x_i,y_i \notin tN$. $S,S' \subseteq Q^k$ are tN equivalent if and only if every element of S is tN equivalent to an element of S' and vice versa.

Note that we still take USH/tN(x) from Q* into Q* and define tN equivalence for elements of Q* and subsets of the $Q^k$, even though we only use the former from Z* into Z* and the latter for subsets of the $Z^k$. This is for uniformity with the previous N case.

So we assume t >> k, work in $[-kt,kt]^k$, replace USH/N by USH/tN, and replace "finite N maximal" by "tN maximal". This results in 2.10C. Note that these are explicitly $\Pi^0_3$ by unraveling the >>.

THEOREM 2.8.1. All ten in 2.10C are explicitly $\Pi^0_3$ and provably equivalent to Con(SRP) over EFA.

As the final step towards Tangibility, we replace r >> k by r > (8k)!, obtaining 2.10D.

THEOREM 2.8.2. All ten statements in 2.10D are explicitly $\Pi^0_1$ and provably equivalent to Con(SRP) over EFA.

## 2.9. RELEVANT FORMAL SYSTEMS

EFA. Exponential function arithmetic. Based on 0, successor, addition, multiplication, exponentiation and bounded induction. Same as $I\Sigma_0$(exp), [HP93], p. 37, 405.

$RCA_0$. Recursive comprehension axiom naught. Our base theory for Reverse Mathematics. [Si99,09].

$WKL_0$. Weak Konig's Lemma axiom naught. Our second level system for Reverse Mathematics. [Si99,09].

$ACA_0$. Arithmetic comprehension axiom naught. Our third level system for Reverse Mathematics. [Si99,09].

ACA'. Arithmetic comprehension axiom prime. Extension of ACA0 with "for all n and $x \subseteq \omega$, the n-th Turing jump of x exists".

ZF(C). Zermelo Frankel set theory (with the axiom of choice). ZFC is the usual currently accepted foundation for mathematics. [Ka94].

SRP. ZFC + $(\exists\lambda)(\lambda$ has the k-SRP), as a scheme in k. [Fr01]. Here $\lambda$ has the k-SRP if and only if $\lambda$ is a limit ordinal such that for every partition of the unordered k-tuples from $\lambda$ into two pieces, there is a stationary subset of $\lambda$ all of whose unordered k-tuples lie in the same piece.

$SRP^+$. ZFC + $(\forall k)(\exists\lambda)(\lambda$ has the k-SRP). [Fr01].

## 2.10. FORTY STATEMENTS

### A. TEN LEAD STATEMENTS
### IN $Q[-n,n]^k$
### infinite maximal objects
### implicitly $\Pi^0_1$ via the Gödel Completeness Theorem

INVARIANT MAXIMAL SQUARES. IMS. Every order invariant subset of every $Q[-n,n]^k$ has a completely USH/N invariant maximal square.

INVARIANT MAXIMAL CUBES. IMC. Every order invariant subset of every $Q[-n,n]^k$ has, for all r, a completely USH/N invariant maximal r-cube.

INVARIANT MAXIMAL LEGS. IML. Every order invariant subset of every $Q[-n,n]^k$ has a completely USH/N invariant maximal leg.

INVARIANT MAXIMAL LEGS*. IML*. Every order invariant subset of every $Q[-n,n]^k$ has, for all r, a completely USH/N invariant maximal r-leg.

INVARIANT MAXIMAL EMULATORS. IME. Every subset of every $Q[-n,n]^k$ has a completely USH/N invariant maximal emulator.

INVARIANT MAXIMAL DUPLICATORS. IMD. Every subset of every $Q[-n,n]^k$ has a completely USH/N invariant maximal duplicator.

INVARIANT MAXIMAL MULTI EMULATORS. IMME. Every subset of every $Q[-n,n]^k$ has, for all r, a completely USH/N invariant maximal r-emulator.

INVARIANT MAXIMAL MULTI DUPLICATORS. IMMD. Every subset of every $Q[-n,n]^k$ has, for all r, a completely USH/N invariant maximal r-duplicator.

INVARIANT MAXIMAL CLIQUES. IMCL. Every order invariant graph on every $Q[-n,n]^k$ has a completely USH/N invariant maximal clique.

INVARIANT MAXIMAL HYPERCLIQUES. IMH. Every order invariant r-graph on every $Q[-n,n]^k$ has a completely USH/N invariant maximal r-clique.

**B. TEN DERIVED STATEMENTS**
**IN $Q[-n,n]^k$**
**finite maximal objects**
**explicitly $\Pi^0_2$**
**explicitly $\Pi^0_1$ by a priori bounding**

INVARIANT N MAXIMAL SQUARES. INMS. Every order invariant subset of every $Q[-n,n]^k$ has a completely USH/N invariant finite N maximal square.

INVARIANT N MAXIMAL CUBES. INMC. Every order invariant subset of every $Q[-n,n]^k$ has, for all r, a completely USH/N invariant finite N maximal r-cube.

INVARIANT MAXIMAL LEGS. IML. Every order invariant subset of every $Q[-n,n]^k$ has a completely USH/N invariant finite N maximal leg.

INVARIANT MAXIMAL LEGS*. IML*. Every order invariant subset of every $Q[-n,n]^k$ has, for all r, a completely USH/N invariant finite N maximal r-leg.

INVARIANT N MAXIMAL EMULATORS. INME. Every subset of every $[-n,n]^k$ has a completely USH/N invariant finite N maximal emulator.

INVARIANT N MAXIMAL DUPLICATORS. INMD. Every subset of every $Q[-n,n]^k$ has a completely USH/N invariant finite N maximal duplicator.

INVARIANT N MAXIMAL MULTI EMULATORS. INMME. Every subset of every $Q[-n,n]^k$ has, for all r, a completely USH/N invariant finite N maximal r=emulator.

INVARIANT N MAXIMAL MULTI DUPLICATORS. INMMD. Every subset of every $Q[-n,n]^k$ has, for all r, a completely USH/N invariant finite N maximal r-duplicator.

INVARIANT N MAXIMAL CLIQUES. INMCL. Every order invariant graph on every $Q[-n,n]^k$ has a completely USH/N invariant finite N maximal clique.

INVARIANT N MAXIMAL HYPERCLIQUES. INMH. Every order invariant r-graph on every $Q[-n,n]^k$ has a completely USH/N invariant finite N maximal r-clique.

**C. TEN DERIVED STATEMENTS**
**IN INTEGER $[-kt,kt]^k$**
**explicitly $\Pi^0{}_3$ by >>**

INTEGER INVARIANT MAXIMAL SQUARES/>>. IIMS/>>. Let t >> k. Every order invariant subset of every $[-kt,kt]^k$ has a completely USH/tN invariant tN maximal square.

INTEGER INVARIANT MAXIMAL CUBES/>>. IIMC/>>. Let t >> k. Every order invariant subset of every $[-kt,kt]^k$ has, for all r, a completely USH/tN invariant tN maximal r-cube.

INTEGER INVARIANT MAXIMAL LEGS/>>. IML/>>. Let t >> k. Every order invariant subset of every $Q[-kt,kt]^k$ has a completely USH/tN invariant tN maximal leg.

INTEGER INVARIANT MAXIMAL LEGS*/>>. IML*/>>. Let t >> k. Every order invariant subset of every $Q[-kt,kt]^k$ has, for all r, a completely USH/tN invariant tN maximal r-leg.

INTEGER INVARIANT MAXIMAL EMULATORS/>>. IIME/>>. Let t >> k. Every subset of every $[-kt,kt]^k$ has a completely USH/tN invariant tN maximal emulator.

INTEGER INVARIANT MAXIMAL DUPLICATORS/>>. IIMD/>>. Let t >> k. Every subset of every $[-kt,kt]^k$ has a completely USH/tN invariant tN maximal duplicator.

INTEGER INVARIANT MAXIMAL MULTI EMULATORS/>>. IIMME/>>. Let t >> k. Every subset of every $[-kt,kt]^k$ has, for all r, a completely USH/tN invariant tN maximal r-emulator.

INTEGER INVARIANT MAXIMAL MULTI DUPLICATORS/>>. IIMMD/>>. Let t >> k. Every subset of every $[-kt,kt]^k$ has, for all r, a completely USH/tN invariant tN maximal r-duplicator.

INTEGER INVARIANT MAXIMAL CLIQUES/>>. IIMCL/>>. Let t >> k. Every order invariant graph on every $[-kt,kt]^k$ has a completely USH/tN invariant tN maximal clique.

INTEGER INVARIANT MAXIMAL HYPERCLIQUES/>>. IIMH/>>. Let t >> k. Every order invariant r-graph on every $[-kt,kt]^k$ has a completely USH/tN invariant tN maximal r-clique.

**D. TEN DERIVED STATEMENTS**
**IN INTEGER $[-kt,kt]^k$**
**explicitly $\Pi^0_1$**

INTEGER INVARIANT MAXIMAL SQUARES. IIMS. Let t > (8k)!. Every order invariant subset of every $[-kt,kt]^k$ has a completely USH/tN invariant tN maximal square.

INTEGER INVARIANT MAXIMAL CUBES. IIMC. Let t > (8k)!. Every order invariant subset of every $[-kt,kt]^k$ has, for all r, a completely USH/tN invariant tN maximal r-cube.

INTEGER INVARIANT MAXIMAL LEGS. IIML. Let t > (8k)!. Every order invariant subset of every $Q[-kt,kt]^k$ has a completely USH/tN invariant tN maximal leg.

INTEGER INVARIANT MAXIMAL LEGS*. IIML*. Let t > (8k)!. Every order invariant subset of every $Q[-kt,kt]^k$ has, for all r, a completely USH/tN invariant tN maximal r-leg.

INTEGER INVARIANT MAXIMAL EMULATORS. IIME. Let t > (8k)!. Every subset of every $[-kt,kt]^k$ has a completely USH/tN invariant tN maximal emulator.

INTEGER INVARIANT MAXIMAL DUPLICATORS. IIMD. Let t > (8k)!. Every subset of every $[-kt,kt]^k$ has a completely USH/tN invariant tN maximal duplicator.

INTEGER INVARIANT MAXIMAL MULTI EMULATORS. IIMME. Let t > (8k)!. Every subset of every $[-kt,kt]^k$ has, for all r, a completely USH/tN invariant tN maximal r-emulator.

INTEGER INVARIANT MAXIMAL MULTI DUPLICATORS. IIMMD. Let t > (8k)!. Every subset of every $[-kt,kt]^k$ has, for all r, a completely USH/tN invariant tN maximal r-duplicator.

INTEGER INVARIANT MAXIMAL CLIQUES. IIMCL. Let t > (8k)!. Every order invariant graph on every $[-kt,kt]^k$ has a completely USH/tN invariant tN maximal clique.

INTEGER INVARIANT MAXIMAL HYPERCLIQUES. IIMH. Let r > (8k)!. Every order invariant r-graph on every $[-kt,kt]^k$ has, for all r, a completely USH/tN invariant tN maximal r-clique.

THEOREM 2.10.1. All 40 statements are provable in SRP+. Each is not provable from any consistent set of theorems of SRP that includes $RCA_0$. All statements in A are provably equivalent to Con(SRP) over $WKL_0$, with the implication to Con(SRP) provable in $RCA_0$. All 32 statements are unprovable in ZFC, assuming ZFC is consistent.

THEOREM 2.10.2. All statements under B,C,D are provably equivalent to Con(SRP) over EFA.

## 2.11. REFERENCES

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