TANGIBLE INCOMPLETENESS FOR GIFTED YOUTH: EMULATION THEORY

Harvey M. Friedman
Distinguished University Professor
of Mathematics, Philosophy, Computer Science
Emeritus
Ohio State University
Columbus, Ohio
https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/
https://www.youtube.com/channel/UCdRdeExwKiWndBl4YOxBTEQ
July 16, 2020
DRAFT

ABSTRACT. We adapt the developments in the Tangible Incompleteness Interim Report [Fr20], for Gifted High School Student Summer Programs. This adaptation, which is of independent research interest, is called Emulation Theory. As Emulation Theory is so new, rich, and elementary, there are many open problems accessible to students that have yet to be attempted by professional mathematicians.

1. INTRODUCTION

Starting around 1966, as a graduate student at MIT, I clearly formulated my Tangible Incompleteness program, aimed at finding interesting, fundamental, elementary, tangible statements that are not provable or refutable using the standard ZFC axioms for mathematics. It is now 2020, and I have finally worked them up sufficiently for presentation to the general mathematical community, along with fully detailed proofs of the associated claims. A full statement of the current state of the art in this Tangible Incompleteness can be found in the extended abstract [Fr20].

The main statements in Tangible Incompleteness are independent of ZFC in the sense that they are neither provable nor refutable from the usual ZFC axioms for mathematics. They are provable in certain well studied extensions of ZFC by so called large cardinal hypotheses.

Tangible Incompleteness is aimed at a variety of audiences: mathematicians, philosophers, computer scientists, and general scientific thinkers. The theory comes in a few closely related settings, and one of these settings is particularly suitable for Gifted Mathematical Youth. This setting is based on "emulators" of finite sequences of tuples and we call this development Emulation Theory.

For the professional mathematical community perhaps the most immediately attractive formulation of our Tangible Incompleteness is based on squares in order invariant sets. In particular, maximal sets $A^2 \subseteq E$ where E is an order invariant set in $Q[-n,n]^k$. See [Fr20], sections 2.3, 2.4. The invariance takes the form of what we call negative stability (section 2.4). There are also some sharper notions of stability, culminating in full stability (section 2.4). In our student setting, negative stability and full stability are equivalent.

Emulation Theory deftly avoids the use of order invariant sets, as they are a wee bit too abstract at that tender level. Certainly many can handle the modest abstraction, but avoiding it will definitely create more comfort and enthusiasm.

Here is the Tangible Incompleteness from [Fr20], using emulators and negative stability. It is independent of ZFC (neither provable nor refutable), and asserts the following. See sections 2.1 - 2.3 here for definitions.

NEGATIVELY STABLE MAXIMAL EMULATOR. NSME. Every $E \subseteq Q[-n,n]^k$ has a negatively stable maximal emulator.

See the more general NSM on page 13 of [Fr20]. We have proved this using an extension of ZFC with a large cardinal hypothesis. Staying within the usual ZFC axioms for mathematics, we have also proved this for all $Q[-n,n]^2$ and for all $Q[-1,1]^k$. We don't think of any of these results, even in ZFC, as generally suitable for Mathematically Gifted Youth.

How much of NSME be proved for the students? For the students, we concentrate on $Q[-1,1]^2$ and small m:

LITTLE STUDENT THEOREM. Every $E \subseteq Q[-1,1]^2$, $|E| \leq 2$, has a negatively stable maximal emulator.

MAIN STUDENT THEOREM. Every $E \subseteq Q[-1,1]^2$, $|E| \leq 3$, has a negatively stable maximal emulator.

In $Q[-1,1]^2$ negative stability of $S \subseteq Q[-1,1]^2$ has a particularly simple formulation:

$(0,0) \in S \leftrightarrow (1,1) \in S$
For all $p < 0$, $(0,p) \in S \leftrightarrow (1,p) \in S$.
For all $p < 0$, $(p,0) \in S \leftrightarrow (p,1) \in S$.

The students need not come into contact with general negative stability - just the special case above. Negative stability of $S \subseteq Q[-1,1]^2$ can be taken to mean these three simple conditions above for the students.

Actually, the Main Student Theorem takes the following sharper form:

MAIN STUDENT THEOREM. Every $E \subseteq Q[-1,1]^2$, $|E| \leq 3$, has an algorithmic negatively stable maximal emulator.

If we add "algorithmic" in this way to the full NSME, then this algorithmic NSME is refutable. I know little about where the boundary is.

Here the student doesn't have to know what an algorithmic set is, but needs to recognize one when presented as such.

So this raises the open question of whether the Main Student Theorem holds for every $E \subseteq Q[-n,n]^2$. Or even for every $E \subseteq Q[-1,1]^2$, $|E| \leq 4$? This has not been investigated.

Note that the Main Student Chapter 3 is divided into two parts. First some substantial theory with exercises, and then some constructions. We recommend that the theory section be covered but with perhaps only selected proofs. So the students at least understand clearly what the theory says. However, we recommend that the entire construction section be covered in detail with proofs (many proofs to be worked out by the students). If these constructions are well understood then the students can engage in various exciting explorations into unknown territories, such as those suggested in Chapter 4.

A short course can go through the theory section 3.1 with only the easiest proofs and exercises, and only the core construction sections 3.2.1, 3.2.2, culminating with the Little Student Theorem.

2. INSTRUCTORS CHAPTER: r-EMULATION THEORY IN $Q[-n,n]^k$ WITH NEGATIVE AND FULL STABILITY

Here we discuss the development in [Fr20] in a professional way appropriate for the Instructors. The student level material is to be found in Chapter 3. Chapter 3 is divided into two parts. The first part contains background material, some theory, and some exercises. The second part consists of Constructions. The theory should be at least covered in terms of what it asserts, and the amount of time spent on proofs is as circumstances warrant. However, the core development consists of sections 3.2.1 - 3.2.4, and these four sections need to be covered in detail. It culminates with a complete proof of

MAIN STUDENT THEOREM. Every E ⊆ $Q[-1,1]^2$, |E| ≤ 3, has an algorithmic negatively stable maximal emulator.

as discussed in the Introduction. A short course culminates with a complete proof of

By way of comparison, we have proved

NEGATIVELY STABLE MAXIMAL EMULATORS (dim 2). NSME (dim 2). Every E ⊆ $Q[-n,n]^2$ has a negatively stable maximal emulator.

FULLY STABLE MAXIMAL EMULATORS (dim 2). FSME (dim2). Every E ⊆ $Q[-n,n]^2$ has a fully stable maximal emulator.

NEGATIVELY STABLE MAXIMAL EMULATORS. NSME. Every finite E ⊆ $Q[-n,n]^k$ has a negatively stable maximal emulator.

FULLY STABLE MAXIMAL EMULATORS. FSME. Every finite E ⊆ $Q[-n,n]^k$ has a fully stable maximal emulator.

where the latter two statements are proved only by going far beyond the usual ZFC axioms for mathematics into the large cardinal realm. We know that the latter two are independent of the usual ZFC axioms for mathematics (neither provable nor refutable).

For Q[-1,1]2, negative stability can be written in very simple form, and the students work with that simple form. We know that "algorithmic" cannot be obtained in NMSE. Also with all four of the above statements, only finite sets E need to be considered.

2.1. PRELIMINARIES

DEFINITION 2.1.1. N, Q are, respectively, the set of nonnegative integers and the set of all rationals. We use i,j,k,n,m,r,s,t with or without subscripts for positive integers, unless otherwise indicated. We use p,q with or without subscripts for rational numbers, unless otherwise indicated. We use f,g,h,F,G,H with or without subscripts for functions, unless otherwise indicated. We use A,B,C,D,E,K,S,T,U,V,W with or without subscripts for sets unless otherwise indicated. Q[-n,n] is the set of all rational numbers in the closed interval [-n,n]. for finite sequences x,y, we write xy for the concatenation of x and y.

DEFINITION 2.1.2. $x,y \in Q^k$ are order equivalent if and only if for all $1 \le i,j \le k$, $x_i < x_j \leftrightarrow y_i < y_j$.

This Definition must be understood by the students for dimensions k = 1,2,3,4, with plenty of examples and discussion.

STUDENT DEFINITION 1. $x,y \in Q^4$ are order equivalent if and only if for all $1 \le i,j \le 4$, $x_i < x_j \leftrightarrow y_i < y_j$.

2.2. MAXIMAL EMULATORS/r-EMULATORS IN $Q[-n,n]^k$

DEFINITION 2.2.1. S is an emulator of $E \subseteq Q[-n,n]^k$ if and only if $S \subseteq Q[-n,n]^k$ and for all $x,y \in S$ there exists $z,w \in E$ such that xy is order equivalent to zw.

Note that "xy is order equivalent to zw" is much much stronger in an essential way than saying

"x is order equivalent to z and y is order equivalent to w"

DEFINITION 2.2.2. S is a maximal emulator of $E \subseteq Q[-n,n]^k$ if and only if S is an emulator of $E \subseteq Q[-n,n]^k$ which is not a proper subset of any emulator of $E \subseteq Q[-n,n]^k$.

So with a maximal emulator, we cannot enlarge it and remain an emulator. Note that this is the same as saying that we cannot enlarge it by a single k-tuple and remain an emulator.

These Definitions are essential for the students just for the case $Q[-1,1]^2$. In this case, note that order equivalence is used only for elements of $Q[-1,1]^4$.

STUDENT DEFINITION 1. S is an emulator of $E \subseteq Q[-1,1]^2$ if and only if $S \subseteq Q[-1,1]^2$ and for all $x,y \in S$ there exists $z,w \in E$ such that the 4-tuple xy is order equivalent to the 4-tuple zw.

STUDENT DEFINITION 2. S is a maximal emulator of $E \subseteq Q[-1,1]^2$ if and only if S is an emulator of $E \subseteq Q[-1,1]^2$ which stops being an emulator if we add any new element of $Q[-1,1]^2$ to S.

THEOREM 2.2.1. Every $E \subseteq Q[-n,n]^k$ has a maximal emulator. In fact, every $E \subseteq Q[-n,n]^k$ has a maximal emulator containing any given emulator.

THEOREM 2.2.2. Every $E \subseteq Q[-n,n]^k$ has an algorithmic maximal emulator. In fact, $E \subseteq Q[-n,n]^k$ has an algorithmic maximal emulator containing any given finite emulator.

Theorem 2.2.1 needs to be covered for $Q[-1,1]^2$. I.e.,

STUDENT THEOREM 1. Every $E \subseteq Q[-1,1]^2$ has a maximal emulator. In fact, every $E \subseteq Q[-1,1]^2$ has a maximal emulator containing any given emulator.

STUDENT THEOREM 2. Every $E \subseteq Q[-1,1]^2$ has an algorithmic maximal emulator. In fact, $E \subseteq Q[-1,1]^2$ has an algorithmic maximal emulator containing any given finite emulator.

There is no simplification coming from using $Q[-1,1]^2$ rather than the more general $Q[-n,n]^k$. We just prefer to keep the student's attention focused on $Q[-1,1]^2$.

We present the proofs as deterministic sequential algorithms. Bringing in Zorn's Lemma would be optional. A proper treatment of Student Theorem 2 naturally involves quantifier elimination for (Q,<), which is a nice meaty entry point into logic that is quite mathematically oriented.

In the professional development of Emulation Theory, we proceed more generally and more powerfully. The students are not presented with r-emulators.

DEFINITION 2.2.3. S is an r-emulator of $E \subseteq Q[-n,n]^k$ if and only if $S \subseteq Q[-n,n]^k$ and for all $x_1,\ldots,x_r \in S$ there exists $y_1,\ldots,y_r \in E$ such that $x_1\ldots x_r$ is order equivalent to $y_1\ldots y_r$.

Note that S is an emulator of $E \subseteq Q[-n,n]^k$ if and only if S is a 2-emulator of $E \subseteq Q[-n,n]^k$.

DEFINITION 2.2.4. S is a maximal r-emulator of $E \subseteq Q[-n,n]^k$ if and only if S is an r-emulator of $E \subseteq Q[-n,n]^k$ which is not a proper subset of any r-emulator of $E \subseteq Q[-n,n]^k$.

So with a maximal r-emulator, we cannot enlarge it and remain an r-emulator. Note that this is the same as saying that we cannot enlarge it by a single k-tuple and remain an r-emulator.

2.3. STABLE SETS IN $Q[-n,n]^k$

DEFINITION 2.3.1. $S \subseteq Q[-n,n]^k$ is stable if and only if for all order equivalent $x,y \in \{0,\ldots,n\}^k$, $x \in S \leftrightarrow y \in S$.

For the students, stable takes on the following simple form:

STUDENT DEFINITION 3. $S \subseteq Q[-1,1]^2$ is stable if and only if $(0,0) \in S \leftrightarrow (1,1) \in S$.

Stable is a lot weaker than negatively stable, and so we can consider the weaker results for stable:

STABLE MAXIMAL EMULATOR. SME. Every $E \subseteq Q[-n,n]^k$ has a stable maximal emulator.

We have proved SME well within ZFC, although we don't regard this as generally suitable for Mathematically Gifted Youth. However for Q[-1,1]2, we have

STUDENT THEOREM 3. Every $E \subseteq Q[-1,1]^2$ has an algorithmic stable maximal emulator.

which has a nice interesting simple proof entirely suitable for Gifted Youth.

2.4. NEGATIVELY STABLE SETS IN $Q[-n,n]^k$

DEFINITION 2.4.1. $S \subseteq Q[-n,n]^k$ is negatively stable if and only if for all order equivalent $x,y \in \{0,...,n\}^k$, if $x',y' \in \{0,...,n\}^k$ is obtained from x,y by replacing zero or more x_i,y_i by p,p < 0, then $x' \in S \leftrightarrow y' \in S$.

EXAMPLE OF SUCH NEGATIVE PARAMETERIZATION: Use $Q[-8,8]^6$, (1,4,2,4,8,6), (2,4,3,4,6,5). Replace third coordinates by -3/2, and fourth coordinates by -7, obtaining (1,4,-3/2,-7,8,6), (2,4,-3/2,-7,6,5).

We do not require that the students work with Definition 2.4.1. However, we do require the students to understand and work with negative stability for $S \subseteq Q[-1,1]^2$. This takes on a particularly simple form, which is the way it is defined for the students.

STUDENT DEFINITION 5. $S \subseteq Q[-1,1]^2$ is negatively stable if and only if the following holds:

i. $(0,0) \in S \leftrightarrow (1,1) \in S$.

ii. For all $p < 0$, $(p,0) \in S \leftrightarrow (p,1) \in S$.

iii. For all $p < 0$, $(0,p) \in S \leftrightarrow (1,p) \in S$.

According to [Fr20], section 2.8, this Student Definition 5 is equivalent to professional Definition 2.4.1 for $Q[-1,1]^2$.

2.5. NEGATIVELY STABLE MAXIMAL EMULATORS (r-EMULATORS) IN $Q[-n,n]^k$

We now have one of the principal forms of Tangible Incompleteness:

NEGATIVELY STABLE MAXIMAL EMULATORS. NSME. Every $E \subseteq Q[-n,n]^k$ has a negatively stable maximal emulator.

NOTE: NSME is equivalent to NSME for finite $E \subseteq Q[-n,n]^k$.

See NSM in section 2.6 of [Fr20]. NSME is an equivalent choice for Tangible Incompleteness that is alternative to the NSMS of [Fr20], section 2.4. In the Introduction to [Fr20], see the header Metamathematical Properties of NSMS, which are shared by NSME.

In particular, NSME is independent of the usual ZFC axioms for mathematics. I.e., it is neither provable nor refutable in ZFC.

THEOREM 2.5.1. Every $E \subseteq Q[-1,1]^2$ has a negatively stable maximal r-emulator.

We have proved Theorem 2.5.1 well within ZFC but our proof uses transfinite recursion of uncountable length. We have not shown that that is necessary. The main result that the students will be taught is the following.

MAIN STUDENT THEOREM. Every $E \subseteq Q[-1,1]^2$, $|E| \leq 3$, has an algorithmic negatively stable maximal emulator.

A main challenge for the students is to handle $|E| \leq 4$. Presumably they would find algorithmic negatively stable maximal emulators, and therefore this would be a new result, not covered by my sledgehammers.

2.6. FULLY STABLE SETS IN $Q[-n,n]^k$

Full Stability is the strongest of a series of five stability notions discussed in section 2.6 of [Fr20]. The weakest is stable and the next weakest is negatively stable.

DEFINITION 2.6.1. $S \subseteq Q[-n,n]^k$ is fully stable if and only if for all order equivalent $x,y \in \{0,...,n\}^k$, if $x',y' \in \{0,...,n\}^k$ is obtained from x,y by replacing zero or more x_i,y_i by p,p < min(xy), then $x' \in S \leftrightarrow y' \in S$.

According to [Fr20], section 2.8, full stability for S ⊆ Q[-1,1]2 is the same as negative stability for $S \subseteq Q[-1,1]^2$ and is the same as Student Definition 5.

2.7. FULLY STABLE MAXIMAL r-EMULATORS IN $Q[-n,n]^k$

We now come to the strongest result in Emulation Theory, not intended for the students:

FULLY STABLE MAXIMAL r-EMULATORS. FSME. Every $E \subseteq Q[-n,n]^k$ has a fully stable maximal r-emulator.

NOTE: NSME is equivalent to NSME for finite $E \subseteq Q[-n,n]^k$.

See NSM in section 2.6 of [Fr20]. NSME is an equivalent choice for Tangible Incompleteness that is alternative to the NSMS of section 2.4. In the Introduction to [Fr20], see the header Metamathematical Properties of NSMS, which are shared by NSME.

In particular, NSME is independent of the usual ZFC axioms for mathematics. I.e., it is neither provable nor refutable in ZFC.

2.8. TOPICS FOR FURTHER STUDY

The development in Chapter 3 for the students is generally only elementary combinatorics and involved only the rationals numbers with its usual ordering. However, various ancillary topics of great mathematical importance arise during the development. There is no time to go into these in real detail, but the basics can be reasonably explained.

The student sees these fascinating topics in action and may be motivated to look into them further at a later time. Here is a list.

1. Section 3.1.1. There is some background material that discusses characterizations of some fundamental countable linear ordering up to isomorphisms. Also algorithmic is defined informally.
2. Section 3.1.2. The general notion of order equivalence in every dimension, and the notion of equivalence relation and equivalence classes and cosets. Also counting of those equivalence classes, as there is a substantial literature on those counts, with nontrivial results in the literature including asymptotics.
3. Sections 3.1.3, 3.1.4. Algorithms. This and later algorithms are used only in the positive sense - often that there is an algorithmic subset of Q[-1,1]2 with certain properties. Theorem 3.1.4.4 calls for pseudo code which is used for the important Theorem 3.1.4.5.
4. Section 3.1.5. Every one of the five categories of subsets of $Q[-1,1]^2$ have substantial advanced theory surrounding them.
5. Theorem 3.1.8.8. Here we use the compactness theorem for first order predicate calculus with equality.
6. In material to be filled in later, in section 3.2, showing that the Lemmas are best possible. I.e., finding the smallest finite, showing cannot replace order theoretic by finite, cannot replace semi linear by order theoretic, cannot replace semi algebraic by semi linear, cannot replace algorithmic by semi algebraic. This uses some advanced theory (see 4 above).
7. In material to be filled out later, all sorts of counting problems, starting already with sections 3.1.2, 3.1.3. I.e., counting the number of subsets of Q[-1,1]2 of cardinality 2 or 3, up to order isomorphism/switching with various properties like having a negatively stable maximal emulator within a certain of the five categories. Various combinatorial counting techniques come into play as well as computer methods.

3. MAIN STUDENT CHAPTER: EMULATION THEORY IN $Q[-1,1]^2$ WITH NEGATIVE STABILITY

3.1. PRELIMINARIES AND EMULATION THEORY

3.1.1. PRELIMINARIES

We often use ∧ for "and" and ∨ for "or" and → for "if then" and ↔ for "if and only if".

Q is the set of all rational numbers.
Z is the set of all integers.
N is the set of all nonnegative integers.
< is the usual comparison between rationals.
Q[-1,1] is the set of all rationals p such that -1 ≤ p ≤ 1.

We will use a,b,c,d,e,p,q,r,s,t,u,v,w for rational numbers in Q[-1,1] unless indicated otherwise. We use n,m,i,j,k for positive integers unless otherwise indicated.

We will always use the term "pair" for ordered pair. Thus (1,1/3) is not the same as (1/3,1). The unordered pair of 1 and 1/3 is {1,1/3} = {1/3,1}.

We use A,B,C,D,E,S,T,U,V for subsets of Q[-1,1]2 unless indicated otherwise. We use ⊆ for subset.

We use |E| for the cardinality of E, or the number of elements of E. All sets that we encounter here are of cardinality 0,1,2,3,..., or of cardinality ω.

A set of tuples of rationals is ALGORITHMIC if and only if there is a computer algorithm which tells us whether or not a given tuple of rationals is in the set. We will use this important concept without getting into its well known mathematical treatments.

THEOREM 3.1.1. Every countable dense linear ordering (D,<) with a left and right endpoints, both different, is isomorphic to (Q[-1,1],<).

THEOREM 3.1.2. Every countable linear ordering (D,<) where every element has an immediate successor and an immediate predecessor, and where there are only finitely many elements strictly between any two elements, is isomorphic to (Z,<).

THEOREM 3.1.3. Every countable linear ordering (D,<) with a least element, where every element has an immediate successor, and where there are finitely many elements less than any given element, is isomorphic to (N,<).

Prove these Theorems.

3.1.2. ORDER EQUIVALENCE AMONG TUPLES

A^k is the set of all k-tuples of elements of A. They are written $(x_1,...,x_k)$, where $x_1,...,x_k \in A$. Usually, we focus on small k, say k = 1,2,3,4.

DEFINITION 3.1.2.1. Let $x,y \in Q^k$. x,y are order equivalent if and only if for all $1 \leq i,j \leq k$,
$x_i < x_j \leftrightarrow y_i < y_j$.

THEOREM 3.1.2.1. Order equivalence is an equivalence relation on Q^k. I.e., for all $x,y,z \in Q^k$,
i. x,x are order equivalent.
ii. x,y are order equivalent if and only if y,x are order equivalent.
iii. If x,y are order equivalent and y,z are order equivalent, then x,z are order equivalent.

Equivalence relations on a nonempty set classify the elements of the set into one or more "equivalence classes", and these equivalence classes are pairwise disjoint. Normally, the number of equivalence classes is generally much smaller than the number of elements.

GEOGRAPHY: "born in the same country" is an equivalence relation on people. It classifies people by their countries of origin. The number of countries is far smaller than the number of people.

Let's see how this order equivalence definition works for small k.

Any two elements of Q^1 are order equivalent. So there is exactly one kind of element of Q^1 according to order equivalence. In general, it is customary to write A^1 as simply A.

Let $x \in Q^2$. There are three possibilities.
a. $x_1 = x_2$.
b. $x_1 < x_2$.
c. $x_1 > x_2$.

Two elements of Q^2 are order equivalent if and only if they lie in the same one of these three categories.

Let $x \in Q^3$. Given $x = (x_1,x_2,x_3)$, there are 13 possibilities. First of all, there are the above three possibilities for x_1,x_2. Under a, x_3 can go in one of three positions - equal to $x_1 = x_2$, below $x_1 = x_2$, or above $x_1 = x_2$. Under b, x_3 can go in one of five positions - below x_1, at x_1, strictly between x_1,x_2, at x_2, and above x_2. Under c, x_3 can go in one of five positions - below x_3, at x_3, strictly between x_3,x_1, at x_1, and above x_1. So there are 3+5+5 = 13 possibilities. So x,y are order equivalent if and only if they are in the same one of these 13 categories.

Let $x \in Q^4$. Given $x = (x_1,x_2,x_3,x_4)$, there are 75 possibilities. Verify this by considering all 13 possibilities of the x_1,x_2,x_3, and for each one of the 13, considering all of the varying number of possibilities for the placement of x_4. You can save yourself a lot of work if you realize how to organize the 13 possibilities of the x_1,x_2,x_3 into three groups, where the varying number of possibilities for the placement of x4 is the same for any x1,x2,x3 in that same one of the three groups. Then it all adds up to 75 (verify).

In our main development, we will not need to consider any tuples of higher length than 4.

THEOREM 3.1.2.2. The number of equivalence classes under order equivalence from Q^k is the same as the number of equivalence classes under order equivalence from A^k, where $A \subseteq Q$ has at least k elements. In particular, the number of equivalence classes under order equivalent from Q^k is the same as the number of equivalence classes under order equivalence from $\{1,2,...,k\}^k$. It is greater than the number of equivalence classes under order equivalence from $\{1,2,...,i\}^k$, no matter what integer $0 \leq i \leq k-1$ we choose.

Proof: Prove this. QED

The number of equivalence classes under order equivalence on Q^k (or $\{1,...,k\}^k$) is written ot(k), for "order types of dimension k". Thus we have seen that ot(1) = 1, ot(2) = 3, ot(3) = 13, ot(4) = 75.

The name preferential arrangement is used in the literature. E.g., see https://www.jstor.org/stable/2312725?seq=1 The preferential arrangements of 1,2,...,k are in nice one-one correspondence with the equivalence classes of k-tuples from {1,...,k} under order equivalence, which has ot(k) equivalence classes.

ot(1) = 1, ot(2) = 3, ot(3) = 13, ot(4) = 75, ot(5) = 541, ot(6) = 4,683, ot(7) = 47,293, ot(8) = 545,835, ot(9) = 7,087,261, ot(10) = 102,247,563, ot(11) = 1,622,632,573, ot(12) = 28,091,567,595, ot(13) = 526,858,348,381, ot(14) = 10,641,342,970,443.

The above is from O.A. Gross, Preferential Arrangements, American Mathematical Monthly, 1962.

3.1.3. EMULATORS

DEFINITION 3.1.3.1. Let x,y be two finite sequences. xy is the concatenation of x with y, and is obtained by taking x and continuing with y. Thus the length of xy is the sum of the lengths of s and y.

DEFINITION 3.1.3.2. An emulator of $E \subseteq Q[-1,1]^2$ is an $S \subseteq Q[-1,1]^2$ such that any pair of elements of S "look like" some pair of elements of E. I.e., for all $x,y \in S$ there exists $z,w \in E$ such that xy is order equivalent to zw. Note that we are allowing E to be finite or infinite, or even ∅.

Bear in mind that here that the $x,y \in S$ and the $z,w \in E$ are each a pair of rationals, (a,b). So we are using xy and zw which are both 4-tuples of rational numbers. Hence we are using order equivalence on $Q[-1,1]^4$. We will not generally need to consider order equivalence for longer tuples. This is because we will be focusing on the space Q[-1,1]2, which is rich enough to challenge you.

We don't have to worry about large E ⊆ Q[-1,1]2 because of the following.

THEOREM 3.1.3.1. Every $E \subseteq Q[-1,1]^2$ has the same emulators as some $E' \subseteq E$ of cardinality ≤ 150.

Proof: We find $E' \subseteq E$ such that every xy, $x,y \in E$, is order equivalent to some zw, $z,w \in E'$. (Why would such an E' have the same emulators as E) We can construct E' with at most ot(4) pairs of elements of E thrown into E'. (How?) So $|E'| \leq 150 = 2ot(4)$. QED

OPEN PROBLEM A. What is the least n such that we can replace 150 with n in Theorem 3.1.3.1?

EXERCISES. Prove the following. Assume each $E \subseteq Q[-1,1]^2$. I.e., we are working in $Q[-1,1]^2$.
1. Every subset of E is an emulator of E.

2. If S is an emulator of E and T is an emulator of S then T is an emulator of E.
3. Every emulator of ∅ is ∅.
4. If |E| ≤ 1 then every emulator of E has at most one element.
5. There exists E with |E| = 2 with an emulator of cardinality 2 but no greater.
6. There exists E with }E} = 2 with an infinite emulator.
7. Any E with |E| = 2 that has an emulator of cardinality 3 has an infinite emulator.
8. There exists E with |E| = 3 whose emulators all have at most 3 elements.
9. Any E with |E| = 3 that has an emulator of cardinality 4 has an infinite emulator.
10. Every E with |E| = 4 has an infinite emulator.
11. There is a finite set E such that every S is an emulator of E.
12. We can strengthen "infinite emulator" in 6,7,9,10 to "algorithmic infinite emulator".

OPEN PROBLEM B. What is the smallest cardinality m of such an E in 9? What is the relationship between the n in Open Problem A and this m here?

We now define an equivalence relation (on subsets of $Q[-1,1]^2$.

E is related to S if and only if E is an emulator of S and S is an emulator of E.

OPEN PROBLEM. How many equivalence classes does this equivalence relation have?

OPEN PROBLEM. What are the relationships between the three Open Problems above in this section?

NOTE: Since Emulation Theory is so new, coming into clear and concise form only in 2020, Open Problem really means, at the moment, that either I have not found the time to look seriously at it, or I have looked at it with some seriousness but see some significant obstacles.

3.1.4. ORDER INVARIANT, ORDER THEORETIC, SEMI LINEAR, ALGORITHMIC SUBSETS OF $Q[-1,1]^2$

An arbitrary subset of $Q]-1,1]^2$ is generally a rather complicated, unruly, wild, impenetrable mathematical object. Essentially nothing mathematically interesting can be said about them in general. Over time mathematicians have gotten very interested in certain naturally defined categories of subsets of Q[-1,1]2, and have uncovered interesting properties that they have. They have developed techniques for showing that certain sets are not present in certain categories.

These explorations are normally conducted for subsets of X^k, where X is a mathematical domain with natural operations defined on it. Four very common spaces are the Z^k, Q^k, $\mathfrak{R}^k$, C^k, where $Z,Q,\mathfrak{R},C$ are the integers, rationals, reals, and complex numbers, incorporating addition for the first two, and both addition and multiplication for the last two. Here we use our $Q[-1,1]^2$ which is very much like Q^2 for present purposes. It turns out that the "nice" subsets of $Q[-1,1]^2$ as the intersections of the "nice" subsets of Q^2 with $Q[-1,1]^2$.

IMPORTANT CATEGORIES OF SUBSETS OF $Q[-1.1]^2$

ORDER INVARIANT. $S \subseteq Q[-1,1]^2$ is order invariant if and only if for all order equivalent $x,y \in Q[-1,1]^2$, $x \in S \leftrightarrow y \in S$.

Order Invariance is extremely restrictive.

THEOREM 3.1.4.1. The order invariant $S \subseteq Q[-1,1]^2$ are the following.
i. $\emptyset$.
ii. $Q[-1,1]^2$.
iii. $\{(p,q) \in Q[-1,1]^2: p = q\}$.
iv. $\{(p,q) \in Q[-1,1]^2: p < q\}$.
v. $\{(p,q) \in Q[-1,1]^2: q < p\}$.
vi. $\{(p,q) \in Q[-1,1]^2: p \leq q\}$.
vii. $\{(p,q) \in Q[-1,1]^2: q \leq p\}$.
viii. $\{(p,q) \in Q[-1,1]^2: p \neq q\}$.

Proof: Prove this. QED

There is another way to look at order invariant $S \subseteq Q[-1,1]^2$. Consider the statements $p < q$, $q < p$. We can build up more statements by using the five common logical connections "not", "and", "or", "if then", "if and only if". They are written as ¬, ∧, ∨, →, ↔. For example,

$$(q \wedge (p \rightarrow (\neg q))) \leftrightarrow (p \vee (q \rightarrow (\neg p)))$$

just a nasty uninteresting mess for illustrative purposes only. So we consider what we call the $S \subseteq Q[-1,1]^2$ given by propositional combinations of $p < q$ and $q < p$.

THEOREM 3.1.4.2. Write ∧,→,↔ in terms of ¬,∨. Write ∨,→,↔ in terms of ¬,∧. Write ∧,∨,↔ in terms of ¬,→. None of ∧,∨,→ be written in terms of ¬,↔.

Proof: Prove this. Precise statements of these results and proofs are standard in math logic courses. QED

THEOREM 3.1.4.3. Let $S \subseteq Q[-1,1]^2$. The following are equivalent.
i. S is order invariant.
ii. S is the set of all $(p,q) \in Q[-1,1]^2$ satisfying a propositional combination of clauses $p < q$, $q < p$.

Proof: Prove this. QED

ORDER THEORETIC. $S \subseteq Q[-1,1]^2$ is order theoretic if and only if S is the set of all $(p,q) \in Q[-1,1]^2$ satisfying a propositional combination of clauses $p < q$, $q < p$, $c < p$, $p < c$, $c < q$, $q < c$, where c is various constants from $Q[-1,1]$.

For instance, we get vertical and horizontal line segments in $Q[-1,1]^2$ is $\{(p,q) \in Q[-1,1]^2: p = c\}$, $\{(p,q) \in Q[-1,1]^2: q = c\}$. They are order theoretic.

THEOREM 3.1.4.4. For order theoretic, we can allow the constants c to be from Q.

Proof: Prove this. QED

The next level is semi linear.

SEMI LINEAR. $S \subseteq Q[-1,1]^2$ is semi linear if and only if S is the set of all $(p,q) \in Q[-1,1]^2$ satisfying a propositional combination of linear inequalities in variables p,q with coefficients (including constant coefficients) from Q[-1,1].

THEOREM 3.1.4.5. For semi linear, we can allow the coefficients c to be from Q.

Proof: Prove this. QED

Next is semi algebraic.

SEMI ALGEBRAIC. $S \subseteq Q[-1,1]^2$ is semi algebraic if and only if S is the set of all $(p,q) \in Q[-1,1]^2$ satisfying a propositional combination of polynomial inequalities in variables p,q with coefficients (including constant coefficients) from Q[-1,1].

THEOREM 3.1.4.6. For semi algebraic, we can allow the coefficients to be from Q.

Proof: Prove this. QED

Now we go all the way up to algorithmic.

ALGORITHMIC $S \subseteq Q[-1,1]^2$. There is an algorithm for determining whether a given element of $Q[-1,1]^2$ is or is not an element of S.

There is an enormous jump from semi algebraic $S \subseteq Q[-1,1]^2$ to algorithmic $S \subseteq Q[-1,1]^2$. The usual intermediate steps are treated in terms of another subject, computational complexity, in theoretical computer science. There are interesting computational complexity issues that arise in our Emulation Theory. Perhaps the most famous category except for algorithmic is "polynomial time computable", but we won't work with this here.

THEOREM 3.1.4.7. Every infinite order theoretic $S \subseteq Q[-1,1]^2$ contains a vertical line segment, a horizontal line segment, or a line segment in the line y = x. $\{(x,y) \in Q[-1,1]2: y = -x\}$ is semi linear but not order theoretic.

Proof: First claim falls out as part of advanced theory. Prove the second claim. QED

THEOREM 3.1.4.8. Let S ⊆ Qn, T ⊆ Qm be semi linear. Then
i. Qn\S ⊆ Qn is semi linear.
ii. S ∪ T ⊆ Qn is semi linear if n = m.
iii. S[T] ⊆ Qn-m is semi linear if n > m.
iv. Every semi linear subset of Q is a finite union of intervals whose endpoints are from Q ∪ {∞,-∞}.

Proof: Part of advanced theory. QED

THEOREM 3.1.4.9. There is an algorithmic subset of $Q[-1,1]^2$ which is not a subset or a superset of any infinite semi linear subset of $Q[-1,1]^2$.

Proof: Prove this. QED

3.1.5. MAXIMAL EMULATORS

DEFINITION 3.1.5.1. S is a maximal emulator of E ⊆ $Q[-1,1]^2$ if and only if S is an emulator of E ⊆ $Q[-1,1]^2$ and no proper superset of S is an emulator of E ⊆ $Q[-1,1]^2$.

THEOREM 3.1.5.1. S is a maximal emulator of E ⊆ $Q[-1,1]^2$ if and only if S is an emulator of E ⊆ $Q[-1,1]^2$ where no new element can be added to S and S remain an emulator of E ⊆ $Q[-1,1]^2$.

Proof: Prove this. QED

It is convenient to write ∪. for disjoint union. I.e., when we write A ∪. B we mean A ∪ B with the understanding that A,B are disjoint (i.e., A ∩ B = ∅).

THEOREM 3.1.5.2. Every E ⊆ $Q[-1,1]^2$ has a maximal emulator.

Proof: List the elements of $Q[-1,1]^2$ without repetition, $x_1,x_2,...$. Create the maximal emulator S in stages, $\emptyset = S_0 \subseteq S_1 \subseteq ...$, taking $S = \cup_i S_i$. Suppose S_i, $i \geq 0$, has been constructed. Take $S_{i+1} = S_i \cup \{x_1\}$ if this is an emulator of E; S_i otherwise. Prove that S is an emulator of E. Prove that S is a maximal emulator of E by assuming that some $S \cup. \{x_i\}$ is an emulator of E, and obtaining a contradiction. QED

There is an important generalization.

THEOREM 3.1.5.3. For all $E \subseteq Q[-1,1]^2$, every emulator of E is contained in a maximal emulator of $E \subseteq Q[-1,1]^2$.

Proof: Let X be an emulator of $E \subseteq Q[-1,1]^2$. List the elements of $Q[-1,1]^2$ without repetition, $x_1,x_2,...$. Create the maximal emulator $S \supseteq X$ in stages, $X = S_0 \subseteq S_1 \subseteq ...$, taking $S = \cup_i S_i$. Suppose S_i has been constructed, $i \geq 0$. Take $S_{i+1} = S_i \cup \{x_i\}$ if this is an emulator of E; S_i otherwise. Prove that S is a maximal emulator of E by assuming that some $S \cup. \{x_i\}$ is an emulator of E, and obtaining a contradiction. QED

THEOREM 3.1.5.4. There is an algorithm that determines, for two finite sets $S,E \subseteq Q[-1,1]^2$, whether S is an emulator of $E \subseteq Q[-1,1]^2$.

Proof: Write pseudo code for this problem. QED

THEOREM 3.1.5.5. Every $E \subseteq Q[-1,1]^2$ has an algorithmic maximal emulator.

Proof: By Theorem 3.1.3.1 we can assume that $|E| \leq 150$. Follow the proof of Theorem 3.1.5.2, and see that the construction there can be carried out algorithmically using Theorem 3.1.5.4. QED

THEOREM 3.1.5.6. E = {(-1,1),(0,1/2)} has no order theoretic maximal emulator.

Proof: Let S be an order theoretic maximal emulator of E. Prove that S is infinite. By Theorem 3.1.4.6, S contains a vertical line segment, a horizontal line segment, or a line segment in the line y = x. Derive a contradiction. QED

Again there is an important generalization.

THEOREM 3.1.5.7. For all E ⊆ $Q[-1,1]^2$, every finite emulator of E ⊆ $Q[-1,1]^2$ is contained in an algorithmic maximal emulator of E.

Proof: Follow the proof of Theorem 3.1.5.3 and use Theorem 3.1.5.4. QED

OPEN PROBLEM. Does every E ⊆ Q[-1,1]2 have a semi linear maximal emulator?

THEOREM 3.1.5.8. If S is a maximal emulator of E ⊆ $Q[-1.1]^2$ then S is a maximal emulator of S. There exists an emulator S of some E ⊆ $Q[-1,1]^2$ such that S is a maximal emulator of S but S is not a maximal emulator of E.

Proof: Prove this. QED

Determine which of the following statements is true for all E,S ⊆ $Q[-1,1]^2$:

If S is a (maximal) emulator of E and S' is a (maximal) emulator of S then S' is a (maximal) emulator of E.

Here there are 8 statements according to whether you choose "maximal" or not, with three independent choices. Prove or give counterexamples to all 8 statements.

Below all sets are subsets of $Q[-1,1]^2$.

OPEN PROBLEM. How many different maximal emulators can a set E have?
OPEN PROBLEM. How many different maximal emulators of E containing E can a set E have?
OPEN PROBLEM. What can the set of cardinalities of the maximal emulators of a set be?
OPEN PROBLEM. What can the set of cardinalities of the maximal emulators containing E can a set E have?
OPEN PROBLEM. What are the cardinalities of the sets S such that S is a maximal emulator of S?

3.1.6. STABLE AND NEGATIVELY STABLE SUBSETS OF $Q[-1,1]^2$

DEFINITION 3.1.6.1. $S \subseteq Q[-1,1]^2$ is stable if and only if the following holds.
$(0,0) \in S$ if and only if $(1,1) \in S$.

DEFINITION 3.1.6.2. $S \subseteq Q[-1,1]^2$ is negatively stable if and only if the following holds.
1. $(0,0) \in S$ if and only if $(1,1) \in S$.
2. For all $p < 0$, $(0,p) \in S \leftrightarrow (1,p) \in S$.
3. For all $p < 0$, $(p,0) \in S \leftrightarrow (p,1) \in S$.

Typical cases of clauses 2,3 are

$(0,-1/2) \in S \leftrightarrow (1,-1/2) \in S$.
$(-1/2,0) \in S \leftrightarrow (-1/2,1) \in S$.

These definitions of stable and negatively stable are obtained from taking the official definitions of stable and negatively stable for $S \subseteq Q[-n,n]^k$ from the professional manuscript, and specializing them to $S \subseteq Q[-1,1]^2$ and simplifying them (without changing their meaning). Our two dimensions (ordered pairs) is much simpler than dimension k, and also -1,1 is much simpler than -n,n. See chapter 4.

Order invariance is much stronger than negative stability.

THEOREM 3.1.6.1. Let S ⊆ $Q[-1,1]^2$ be order invariant. Then S is negatively stable.

THEOREM 3.1.6.2. There are continuumly many negatively stable S ⊆ $Q[-1,1]^2$. There are exactly 8 order invariant S ⊆ $Q[-1,1]^2$. They are all negatively stable.

Proof: Prove the first claim. The second claim is by Theorem 3.1.5.1. Prove the third claim using Theorem 3.1.5.1 and also more theoretically. QED

Here is a sometimes useful sufficient condition for negative stability.

THEOREM 3.1.6.3. Suppose S ⊆ $Q[-1,1]^2$, where fld(S\{(0,1)} is disjoint from {0,1}. Then S is negatively stable.

Proof: Prove this. QED

3.1.7. STABLE, NEGATIVELY STABLE MAXIMAL EMULATORS

If we just want stability, things are pretty straightforward in $Q[-1,1]^2$:

THEOREM 3.1.7.1. Every E ⊆ $Q[-1,1]^2$ has an algorithmic stable maximal emulator.

Proof: Let E ⊆ $Q[-1,1]^2$.
case 1. There is no (p,p) ∈ E. By Theorem 3.1.5.5, let S be any algorithmic maximal emulator of E. Then (0,0),(1,1) ∉ S. Hence S is stable.
case 2. There is exactly one (p,p) in E. Obviously {(1/2,1/2)} is an emulator. By Theorem 3.1.4.6, let S be an algorithmic maximal emulator of E that includes

(1/2,1/2). Then (0,0),(1,1) $\notin$ S (why?), and so S is stable.
case 3. There are at least two (p,p) in E. Then {(0,0),(1,1)} is an emulator, and so by Theorem 3.1.4.6, let S be an algorithmic maximal emulator of E that includes (0,0),(1,1). Then S is stable. QED

Can we improve Theorem 3.1.7.1 by sharpening "algorithmic"? By Theorem 3.1.5.6, we cannot even find an order theoretic maximal emulator for |E| = 2, even without "stable".

OPEN PROBLEM. Does every E $\subseteq$ Q$[-1,1]^2$ have a semi linear stable maximal emulator? Or semi algebraic?

Can we improve Theorem 3.1.7.1 to negatively stable maximal emulator or even negatively stable order invariant emulator?

THEOREM 3.1.7.2. {(0,0)} $\subseteq$ Q$[-1,1]^2$ does not have an order invariant maximal emulator. All of its maximal emulators have cardinality 1.

Proof: Prove this. QED

But we do have this:

THEOREM 3.1.7.3. (Friedman) Every E $\subseteq$ Q$[-1,1]^2$ has a negatively stable maximal emulator.

The only proof that we have of Theorem 3.1.7.3 uses some advanced machinery involving uncountable length transfinite recursion, and certainly does not establish that the negatively stable maximal emulator can be made algorithmic.

OPEN PROBLEM. Does every E $\subseteq$ Q$[-1,1]^2$ have an algorithmic negatively stable maximal emulator?

THEOREM. Not every E $\subseteq$ Q$[-1,1]^2$, |E| = 2, has a semi linear negatively stable maximal emulator.

Proof: We prove this in section 3.2.2 borrowing from some advanced theory. QED

It is with this Open Problem that you have the clearest path to new advances in Emulation Theory. Let me explain.

In section 3.2, we are going to prove the following.

MAIN STUDENT THEOREM. Every $E \subseteq Q[-1,1]^2$, $|E| \leq 3$, has an algorithmic negatively stable maximal emulator.

CONJECTURE. Every $E \subseteq Q[-1,1]^2$, $|E| \leq 4$, has an algorithmic negatively stable maximal emulator.

3.1.8. ORDER ISOMORPHISM AND COORDINATE SWITCHING

DEFINITION 3.1.8.1. f is an order isomorphism if and only if there exists b > -1 such that $f:Q[-1,1] \rightarrow Q[-1,b]$ is strictly increasing and has range Q[-1,b]. For $S \subseteq Q[-1,1]^2$, fld(S) is the field of S which is the set of all coordinates of elements of S.

DEFINITION 3.1.8.2. f is an order isomorphism from S onto S' if and only if f is an order isomorphism which maps fld(S) onto fld(S'), where for all $p,q \in$ fld(S), $(p,q) \in S \leftrightarrow (f(p),f(q)) \in S'$. $S,S' \subseteq Q[-1,1]2$ are order isomorphic if and only if there is an order isomorphism from S onto S'.

THEOREM 3.1.8.1. Let $E,E' \subseteq Q[-1,1]^2$ be order isomorphic. Then E,E' have the same emulators. If S,S' are order isomorphic then S is an emulator of E if and only if S' is an emulator of E.

THEOREM 3.1.8.2. If $E,E' \subseteq Q[-1,1]^2$ have the same emulators then they have the same maximal emulators and the same negatively stable maximal emulators. There are $E \subseteq Q[-1,1]2$ and order isomorphic $S,S' \subseteq$

Q[-1,1]2 such that S is a negatively stable maximal emulator of E and S' is not a maximal emulator of E.

Proof: Prove this. QED

So order isomorphisms of emulators are not enough to guarantee that we preserve emulator maximality. Order isomorphisms of sets are not enough to preserve negative stability (prove). We will fix this by using "global order isomorphisms" below. But first we show that we get a limited kind of maximality.

THEOREM 3.1.8.3. Let S be a maximal emulator of E $\subseteq$ $Q[0,1]^2$ and f:Q[-1,1] → Q[-1,b] be an order isomorphism from S onto S'. Then there is no emulator S' ∪. (c,d) of E with c,d ≤ b.

Proof: Prove this. QED

DEFINITION 3.1.8.3. S,S' $\subseteq$ $Q[0,1]^2$ are globally isomorphic if and only if there is an order isomorphism f from S onto S' such that f(1) = 1.

THEOREM 3.1.8.4. Let E,S,S' $\subseteq$ $Q[-1,1]^2$. If S and S' are globally isomorphic then S is a maximal emulator of E if and only if S' is a maximal emulator of E.

Proof: Prove using Theorem 3.1.8.3. QED

DEFINITION 3.1.8.4. The coordinate switch of (a,b) ∈ Q^2 is (b,a). The coordinate switch of S $\subseteq$ $Q[-1,1]^2$ is the set of coordinate switches of its elements. We write csw((a,b)) = (b,a), and csw(S) = {csw(x): x ∈ S}.

THEOREM 3.1.8.5. Let E,S $\subseteq$ Q[-1,1]2. S is an emulator of E if and only if csw(S) is an emulator of csw(E). S is a maximal emulator if and only if csw(S) is a maximal emulator. S is stable if and only if csw(S) is stable. S is negatively stable if and only if csw(S) is negatively stable. S is order theoretic, semi linear, algorithmic if and only if csw(S) is order theoretic, semi linear, algorithmic, respectively.

Now for some important application of order isomorphisms and global order isomorphisms.

THEOREM 3.1.8.6. Suppose $E \subseteq Q[-1,1]^2$ has a finite maximal emulator S whose field omits 1. Then E has a negatively stable maximal emulator of the same cardinality.

Proof: Make a global isomorphism whose image has field omitting both 0 and 1. QED

THEOREM 3.1.8.7. Suppose that among the finite emulators of $E \subseteq Q[-1,1]^2$ there is one of largest finite size. Then one of these of largest finite size is a negatively stable maximal emulator of $E \subseteq Q[-1,1]^2$.

Proof: Let S be a finite emulator of $E \subseteq Q[-1,1]^2$ of largest finite size $n \geq 0$. Make an order isomorphism of S onto $S' \subseteq Q[-1,1]^2$ whose field excludes 0,1. Then E' is a negatively stable maximal emulator of E of cardinality n (why?). QED

Using a topic in mathematical logic, we can easily sharpen Theorem 3.1.8.6 as follows.

THEOREM 3.1.8.8. Suppose every emulator of $E \subseteq Q[-1,1]^2$ is finite. There is a finite negatively stable maximal emulator of E.

Proof: Let E be as given. We claim that there is a largest size among the finite emulators of E. Suppose this is false. Then there are arbitrarily large finite emulators of $E \subseteq Q[-1,1]^2$. We can convert this to the context where the classical compactness theorem for countable models in first order predicate calculus with equality applies. In this way we get an infinite emulator of E ⊆ Q[-1,1]2, contradicting the hypotheses. From the claim we are done by Theorem 3.1.8.7. QED

3.2. CONSTRUCTIONS

3.2.1. EMULATING ZERO OR ONE PAIR - BABY STUDENT THEOREM

BABY STUDENT THEOREM. Let E ⊆ $Q[-1,1]^2$, |E| ≤ 1. E has a |E| element negatively stable maximal emulator. If |E| = 0 then E has exactly one emulator, ∅. If |E| = 1 then E has infinitely many emulators, one being ∅ and the rest being maximal emulators of cardinality 1. Some but not all of these maximal emulators are negatively stable maximal emulators.

Proof: Prove this. QED

EXERCISE. Let |E| = {(p,q)}. Determine exactly what the maximal emulators of E are in terms of p,q. Determine exactly the stable maximal emulators of E. Determine the negatively stable maximal emulators of E.

3.2.2. EMULATING TWO PAIRS - LITTLE STUDENT THEOREM

In this section, we prove the following.

LITTLE STUDENT THEOREM. Every E ⊆ $Q[-1,1]^2$, |E| = 2, has an algorithmic negatively stable maximal emulator.

If the two distinct elements are not order equivalent, then we have a particularly easy situation.

EASY STUDENT THEOREM. Every E ⊆ $Q[-1,1]^2$, |E| = 2, whose two elements are not order equivalent, has a two element negatively stable maximal emulator. E may or may not have a one element negatively stable maximal emulator.

Proof: Prove this. QED

LEMMA 3.2.2.1. Let E = {(p,q),(r,s)} ⊆ Q[-1,1]2, |E| = 2, p ≤ q ∧ (p,q) <lex (r,s) ∧ (p,q),(r,s) are order equivalent. Exactly one of the following holds.
i. E = {(p,p),(s,s)} ∧ p < s.
ii. E = {(p,q),(p,s)} ∧ p < q < s.
iii. E = {(p,q),(r,q)} ∧ p < r < q.
iv. E = {(p,q),(q,s)} ∧ p < q < s.
v. E = {(p,q),(r,s)} ∧ p < r < q < s
vi. E = {(p,q),(r,s)} ∧ p < r < s < q
vii. E = {(p,q),(r,s)} ∧ p < q < r < s.

Proof: Let E,p,q,r,s be as given. Clearly either (p = q ∧ r = s) ∨ (p > q ∧ r > s). Also |{p,q,r,s}| = 2,3, or 4.
case 1. p = q ∧ r = s. Then clause i applies.
case 2. p < q ∧ r < s. If |{p,q,r,s}| = 2 then (p,q) = (r,s). Hence |{p,q,r,s}| is 3 or 4.
   case 2a. |{p,q,r,s}| = 3. Then {p,q},{r,s} have exactly one element in common. If it is p then we have (p,q),(p,s), q < s, and clause ii applies. If it is q then we have (p,q),(r,q) and so clause iii applies.
   case 2b. |{p,q,r,s}| = 4. There are 24 possibilities of the ordering of p,q,r,s, and because of p < q ∧ r < s ∧ (p,q) <lex (r,s), many possibilities are impossible. Note that p ≤ r. Because of cardinality 4, we have p < r. This means that p must be the least of the four. Here is the complete list.
p < q < r < s yes clause vii
p < q < s < r s < r no
p < r < q < s yes clause v
p < r < s < q yes clause vi
p < s < q < r s < r no
p < s < r < q s < r no

QED

LEMMA 3.2.2.2. Let E = {(p,p),(q,q)} where p < q. E has an order theoretic negatively stable maximal emulator. E has no finite maximal emulator.

Proof: Prove that {(a,a): -1 ≤ a ≤ 1} is an order theoretic negatively stable maximal emulator of E. Prove that E has no finite maximal emulator. QED

LEMMA 3.2.2.3. Let E = {(p,q),(p,r)} where p < q < r. E has an order theoretic negatively stable maximal emulator. E has no finite maximal emulator.

Proof: Prove that {(-1,a): -1 < a ≤ 1} is an order theoretic negatively stable maximal emulator of E. Prove that E has no finite maximal emulator. QED

LEMMA 3.2.2.4. Let E = {(p,r),(q,r)} where p < q < r. E has an order theoretic negatively stable maximal emulator. E has no finite maximal emulator.

Proof: Prove that {(a,-1/2): -1 ≤ a < -1/2} is an order theoretic negatively stable maximal emulator of E. Prove that E has no finite maximal emulator. QED

LEMMA 3.2.2.5. Let E = {(p,q),(q,r)} where p < q < r. E has a two element negatively stable maximal emulator. Every maximal emulator of E is of cardinality 2.

Proof: Prove that S = {(-1,-1/2),(-1/2,-1/3)} is a two element negatively stable maximal emulator. Prove that every maximal emulator of E is of cardinality 2. QED

LEMMA 3.2.2.6. Let E = {(p,q),(r,s)} where p < r < q < s. E has a semi linear negatively stable maximal emulator. No maximal emulator of E is order theoretic.

Proof: Prove that {(a,a+.4): -1 ≤ a < -.6} is a semi linear negatively stable maximal emulator of E. Prove that no maximal emulator of E is order theoretic using Theorem 3.1.4.7. QED

LEMMA 3.2.2.7. Let E = {(p,q),(r,s)} where p < r < s < q. E has a semi linear negatively stable maximal emulator. No maximal emulator of E is order theoretic.

Proof: Prove that {(-1+a,-.5-a): 0 ≤ a < .25} is a semi linear negatively stable maximal emulator of E. Prove that no maximal emulator of E is order theoretic using Theorem 3.1.4.7. QED

LEMMA 3.2.2.8. Let E = {(p,q),(r,s)}, p < q < r < s. E has an algorithmic negatively stable maximal emulator. No negatively stable maximal emulator of E is semi linear. No maximal emulator of E is semi linear with the sole exception of the stable maximal emulator {(-1,1)}.

Proof: By Theorem 3.1.5.7, let S be an algorithmic maximal emulator of E containing (0,1). Since 0,1 do not appear in S\{(0,1)}, clearly S is negatively stable (why?). Hence S is an algorithmic negatively stable maximal emulator of E. Prove that {(-1,1)} is a stable maximal emulator of E. Prove that no maximal emulator of E is order theoretic, other than {(-1,1)}, using Theorem 3.1.4.9. QED

LITTLE STUDENT THEOREM. Every E ⊆ $Q[-1,1]^2$, |E| ≤ 2, has an algorithmic negatively stable maximal emulator.

Proof: Let E be as given. If |E| ≤ 1 then use the Baby Student Theorem. Suppose |E| = 2. If the two elements of E are not order equivalent then apply Easy Student Theorem. Now suppose the two elements of E are order equivalent. In the First Case, assume (a,b) ∈ E → a ≤ b. Write E = {(p,q),(r,s}}, where (p,q) <lex (r,s), so that E is in the form given by Lemma 3.2.2.1. In the Second Case, assume (a,b) ∈ E → a > b. Write E = {(p,q),(r,s)}, where (q,p) <lex (s,r). Then the coordinate switch csw(E) of E is {(q,p),(s,r)} which is in the form given by Lemma 3.2.2.1.

So in the First Case, we have exactly one of i-vii. These correspond exactly to Lemmas 3.2.2.2 - 3.2.2.8. Hence there is an algorithmic negatively stable maximal emulator of E. In the Second Case, we have exactly one of i-vii for csw(E), and again there is an algorithmic negatively stable maximal emulator of csw(E). Hence there is an algorithmic negatively stable maximal emulator of E (why?). QED

We have seen in Lemma 3.2.2.8 that we cannot replace "algorithmic" by "semi linear" in the Little Student Theorem. However, what if we are just looking for stability rather than negative stability?

THEOREM 3.2.2.9. Every E ⊆ $Q[-1,1]^2$, |E| ≤ 2, has a semi linear stable maximal emulator. Not every E ⊆ $Q[-1,1]^2$, |E| = 2, has an order theoretic maximal emulator. Not every E ⊆ $Q[-1,1]^2$, |E| = 2, has a semi linear negatively stable maximal emulator.

Proof: We have semi linear (negative) stability in Lemmas 3.2.2.? - 3.2.2.7. In Lemma 3.2.2.8, use the stable maximal emulator {(-1,1)} as cited. For the second claim, use Lemma 3.2.2.6. QED

We now give a framework for stating more precise information including giving some quantitative information (counting). This depends on having a suitable equivalence relation(s) on the E ⊆ $Q[-1,1]^2$, |E| ≤ 2.

We have already encountered the equivalence relation of order isomorphism between E ⊆ $Q[-1,1]^2$, |E| ≤ 2, in section 3.1.8. The fundamental point was that order isomorphic E ⊆ $Q[-1,1]^2$ have the same emulators. Therefore they have the same:
maximal emulators
stable emulators
negatively stable emulators
stable maximal emulators
negatively stable maximal emulators

and we can also add the adjectives order theoretic, semi linear, and algorithmic. Why? But it is natural and convenient to incorporate coordinate switching.

DEFINITION 3.2.2.1. E,E' ⊆ $Q[-1,1]^2$ are order isomorphic/switching if and only if E is order isomorphic to E' or csw(E').

THEOREM 3.2.2.10. Order isomorphic/switching is an equivalence relation on the E ⊆ $Q[-1,1]^2$.

Proof: Prove this. QED

OPEN PROBLEM. How many equivalence classes are there under order isomorphic and order isomorphic/switching of E ⊆ Q[-1,1]2, |E| = 2? What are, or what can we say about the various cardinalities of the equivalence classes?

LITTLE STUDENT THEOREM*. Every E ⊆ $Q[-1,1]^2$, |E| ≤ 2, has a semi linear negatively stable maximal emulator, with exactly one exception up to order isomorphic/switching, where it has an algorithmic negatively stable maximal emulator. Every E ⊆ $Q[-1,1]^2$, |E| ≤ 2, has an order theoretic negatively stable maximal emulator, with exactly 3 exceptions up to order isomorphic/switching.

Proof: Note that the seven cases i-vii in Lemma 3.2.2.1 are exhaustive (with no repetitions) of the E

$\subseteq$ Q[-1,1]^2, |E| = 2, both elements of E order equivalent, up to order isomorphic/switching. Only vii gives rise to a merely algorithmic negatively stable maximal emulator rather than a semi linear negatively stable maximal emulator (Lemma 3.2.2.8). The E $\subseteq$ Q[-1,1]^2, |E| = 2, both elements of E not order equivalent, and the E $\subseteq$ Q[-1,1]^2, |E| ≤ 1, have finite, and therefore semi linear, negatively stable maximal emulators. For the second claim, Lemmas 3.2.2.6 - 3.2.2.8 correspond to those three exceptions, which are exactly the E whose field has cardinality 4. QED

NOTE: SOME COUNTING MATERIAL WILL BE ADDED LATER.

3.2.3. EMULATING THREE PAIRS: STUDENT STRAY THEOREM

In this section we prove the following.

STUDENT STRAY THEOREM. Every E $\subseteq$ Q[-1,1]^a, |E| = 3, where not all x,y ∈ E are order equivalent, has an algorithmic negatively stable maximal emulator.

The name "stray" arises as follows. If x ∈ E is not order equivalent to any other element of E then we can view x ∈ E as stray in E.

THEOREM 3.2.3.1. In E $\subseteq$ Q[-1,1]^2, |E| = 3, there are exactly zero, one, or three strays.

Prove: Prove this. QED

So we can restate the Student Stray Theorem as follows.

STUDENT STRAY THEOREM. Every E $\subseteq$ Q[-1,1]^2, |E| = 3 with at least one stray has an algorithmic negatively stable maximal emulator.

The hard case is where there is exactly one stray. The no stray case is handled in section 3.2.4. The easy case is where there are three strays.

EASY STUDENT STRAY THEOREM. Every $E \subseteq Q[-1,1]^2$, $|E| = 3$, where no two distinct elements are order equivalent, has a three element negatively stable maximal emulator.

EASY STUDENT STRAY THEOREM (restated). Every $E \subseteq Q[-1,1]^2$, $|E| = 3$ with three strays has a three element negatively stable maximal emulator.

Proof: Show that every emulator of E has at most three elements. Now apply Theorem 3.1.8.7. QED

DEFINITION 3.2.3.1. Let $A \subseteq Q[-1,1]$. $p \geq A$ if and only if for all $q \in A$, $p \geq q$. p is surrounded in A if and only if $p \in A$ and A contains an element less than p and A contains an element greater than p.

LEMMA 3.2.3.1. Let $S \subseteq Q[-1,1]^2$ be an emulator of $E \subseteq Q[-1,1]^2$, $|E| = 2$. Let (t,u) be not order equivalent to any element of E. Then S is a maximal emulator of $E \cup. \{(t,u)\}$; or there exists a maximal emulator $S \cup. \{(b,c)\}$ of $E \cup. \{(t,u)\}$ where (b,c),(t,u) are order equivalent. and i-iii holds:
i. b is not surrounded in fld(S).
ii. c is not surrounded in fld(S).
iii. If $\max(b,c) \geq fld(S)$ then there is a semi linear order isomorphism from $S \cup. \{(t,u)\}$ onto a maximal emulator of $E \cup. \{(t,u)\}$ contained in $Q[-1,=1/2]^2$..

Proof: Let S,E,t,u be as given. Then (t,u) is not order equivalent to any element of S (why?). Obviously S is an emulator of $E \cup. \{(t,u)\}$. If S is a maximal emulator of $E \cup. \{(t,u)\}$ then there are no claims left to verify. Assume S is not a maximal emulator of $E \cup. \{(t,u)\}$, and let $S \cup. \{(b,c)\}$ be an emulator of $E \cup. \{(t,u)\}$. Then (b,c) is order equivalent to an element of E ∪. {(t,u)}. If (b,c) is order equivalent to some element of S then S ∪. {(b,c)} is an emulator of E (the (t,u) playing no role in the emulation of E ∪. {(t,u)} by S ∪. {(b,c)}), contradicting the maximality of S. Hence (b,c) is not order equivalent to any element of E. Hence (b,c) is order equivalent to (t,u).

We now claim that S ∪ {(b,c)} is a maximal emulator of E ∪. {(t,u)}. To see this, if we can adjoin (b',c') then (b',c') cannot be order equivalent to (b,c), and so (b',c') is order equivalent to an element of S. But then S ∪. {(b',c')} is an emulator of E, contradicting the maximality of S.

Suppose b is surrounded in fld(S). Let p,b,q ∈ fld(S), p < b < q. Write (p,_),(b,c) and (b,_),(b,c) and (q,_),(b,c), where three quadruples are pairwise order inequivalent, and where there may be coordinate switching of the pairs with _ - all for use in the definition of S ∪. {(b,c)} being an emulator of E ∪. {(t,u)}. These must be order equivalent to (p',_),(t,u) and (t,_),(t,u) and (q',_),(t,u), respectively, all from E ∪. {(t,u)}, again with possible coordinate switching. Note that (p',),(t,_),(q',_) are different and not order equivalent to (t,u), and therefore must be from E. But this contradicts that |E| = 2.

Suppose c is surrounded in fld(S). Argue the same way as in the previous paragraph.

Now assume max(b,c) ≥ fld(S). Then max(t,u) ≥ fld(E). Let f:Q[-1,1] → Q[-1,-1/2] be a semi linear surjective order isomorphism. By Theorem 3.1.8.2, f[S ∪. {(b,c)}] = f[S] ∪. {(f(b),f(c)} = T is an emulator of E ∪. {(t,u)}. Obviously T is algorithmic negatively stable. It remains to use max(a,b) ≥ fld(S) ∧ max(t,u) ≥ fld(E) to see that T is a maximal emulator of E ∪. {(t,u)}. Note that (f(b),f(c)) has the largest possible max in T. Suppose T ∪. {(v,w)} is an emulator of E ∪. {(t,u)}. Now max(v,w) cannot be bigger than max(f(b),f(c)) because we would have an element of E ∪. {t,u)} with higher max than max(t,u), which is impossible. Hence max(v,w) ≤ max(f(b),f(c)) and so we apply Theorem 3.1.8.3 to obtain a contradiction. QED

LEMMA 3.2.3.2. Let E = {((p,p),(q,q)} ⊆ $Q[-1,1]^2$, p < q and t ≠ u from Q[-1,1]. There exists an order theoretic negatively stable maximal emulator of E ∪ {(t,u)}.

Proof: Let E,p,q,t,u be as given. By Lemma 3.2.2.3, S = {(a,a): -1 ≤ a ≤ 1} is an order theoretic negatively stable emulator of E. By Lemma 3.2.3.1, S is an order theoretic negatively stable maximal emulator of E ∪. {(t,u)} of E; or some S ∪. {(b,c)} is an order theoretic maximal emulator of E ∪. {(t,u)}, where (b,c),(t,u) are order equivalent, and i-iii there. If the former holds then we are done. So assume the latter. Since b,c are not surrounded in Q[-1,1], b,c ∈ {-1,1}. Since b ≠ c, we have max(b,c) = 1 ≥ max(fld(S)), and so by Lemma 3.2.3.1, there is an order theoretic negatively stable maximal emulator of E ∪. {(t,u)}. QED

LEMMA 3.2.3.3. Let E = {(p,q),(p,r)} ⊆ $Q[-1,1]^2$, p < q < r ∧ t,u ∈ Q[-1,1] ∧ t ≥ u. There exists an order theoretic negatively stable maximal emulator of E ∪ {(t,u)}.

Proof: Let E,p,q,r,t,u be as given. By Lemma 3.2.2.4, S = {(-1,a): -1 < a ≤ 1} is an order theoretic negatively stable emulator of E. By Lemma 3.2.3.1, we can assume that some S ∪. {(b,c)} is an order theoretic maximal emulator of E ∪. {(t,u)}, (b,c),(t,u) order equivalent, with i-iii there. Since b,c are not surrounded in Q(-1,1], b,c ∈ {-1,1}. If b = -1 then S ∪. {(b,c)} is an order theoretic negatively stable maximal emulator of E ∪. {(t,u)}. if b = 1 then max(b,c) = 1 ≥ max(fld(S)), and so by Lemma 3.2.3.1, there is an order theoretic negatively stable maximal emulator of E ∪. {(t,u)}. QED

LEMMA 3.2.3.4. Let E = {(p,r),(q,r)} ⊆ $Q[-1,1]^2$, p < q < r ∧ t,u ∈ Q[-1,1] ∧ t ≥ u. There exists an order theoretic negatively stable maximal emulator of E ∪ {(t,u)}.

Proof: Let E,p,q,r,t,u be as given. By Lemma 3.2.2.5, S = {(a,-1/2): -1 < a < -1/2} is an order theoretic negatively stable emulator of E. By Lemma 3.2.3.1, we can assume that some S ∪. {(b,c)} is a maximal emulator of E ∪. {(t,u)}, (b,c),(t,u) order equivalent, with i-iii there. If b or c lies in Q{-1,-1/2) then b or c is surrounded in fld(S). Hence b,c ∈ Q[-1/2,1]. Hence max(b,c) ≥ max(fld(S)), and so there is an order theoretic negatively stable maximal emulator of E ∪. {(t,u)}. QED

LEMMA 3.2.3.5. Let E = {(p,q),(q,r)} ⊆ $Q[-1,1]^2$, p < q < r ∧ t,u ∈ Q[-1,1] ∧ t ≥ u. There exists a two or three element negatively stable maximal emulator of E ∪ {(t,u)}.

Proof: Let E,p,q,r,t,u be as given. By Lemma 3.2.2.6, S = {(-1,-1/2),(-1/2,-1/3)} is a two element negatively stable maximal emulator of E. By Lemma 3.2.3.1, we can assume that some S ∪. {(b,c)} is a maximal emulator of E ∪. {(t,u)}, (b,c),(t,u) order equivalent. Then if b,c ≤ -1/3 then there is a two element negatively stable maximal emulator of E ∪ {(t,u)}. Otherwise, max(b,c) ≥ max(fld(S)), in which case there is a three element negatively stable maximal emulator of E ∪ {(t,u). QED

LEMMA 3.2.3.6. Let E = {(p,q),(r,s)} ⊆ $Q[-1,1]^2$, p < r < q < s ∧ t ≥ u. There exists a semi linear negatively stable maximal emulator of E ∪ {(t,u)}.

Proof: Let E,p,q,r,s,t,u be as given. By Lemma 3.2.2.7, S = {(a,a+.4): -1 ≤ a < -.6} is a semi linear negatively stable maximal emulator of E. By Lemma 3.2.3.1, we can assume that some S ∪ {(b,c)} is a maximal emulator of E ∪. {(t,u)}, (b,c),(t,u) order equivalent. Now b,c are not in the interior of fld(S) and so max(b,c) ≥ S. Hence there is a semi linear negatively stable maximal emulator of E ∪. {(t,u)}. QED

LEMMA 3.2.3.7. Let E = {(p,q),(r,s)} ⊆ $Q[-1,1]^2$, p < r < s < q ∧ t ≥ u. There exists a semi linear negatively stable maximal emulator of E ∪ {(t,u)}.

Proof: Let E,p,q,r,s,t,u be as given. By Lemma 3.2.2.8, S = {(-1+a,-.5-a): 0 ≤ a < .25} is a semi linear negatively stable maximal emulator of E. By Lemma 3.2.3.1, we can assume that some S ∪ {(b,c)} is a maximal emulator of E ∪. {(t,u)}, (b,c),(t,u) order equivalent. Now b,c are not surrounded in fld(S) and so max(b,c) ≥ S. Hence there is a semi linear negatively stable maximal emulator of E ∪. {(t,u)}. QED

LEMMA 3.2.3.8. Let E = {(p,q),(r,s)}, p < q < r < s ∧ t ≥ u. There exists an algorithmic negatively stable maximal emulator of E ∪ {(t,u)}.

Proof: Let E,p,q,r,s,t,u be as given. By Lemma 3.2.2.9, let S be an algorithmic negatively stable maximal emulator of E, where (0,1) ∈ fld(S) and fld(S) has a negative element. By Lemma 3.2.3.1, we can assume that some S ∪ {(b,c)} is a maximal emulator of E ∪. {(t,u)}, (b,c),(t,u) order equivalent. By Lemma 3.2.3.1, b,c ≠ 0. Also if b = 1 then max(b,c) ≥ fld(S), and so there is a negatively stable maximal emulator of E ∪. {(t,u)}. So we can assume that b,c ∉ {0,1}. Hence S ∪ {(b,c)} is an algorithmic negatively stable emulator of E ∪ {(t,u)}. QED

STUDENT STRAY THEOREM. Every E ⊆ $Q[-1,1]^2$, |E| = 3, where not all x,y ∈ E are order equivalent, has an algorithmic negatively stable maximal emulator.

Proof: Let E = {(p,q),(r,s),(t,u)} ⊆ $Q[-1,1]^2$ be as given. If no two distinct elements of E are order equivalent then apply Easy Student Stray Theorem. Otherwise, we can assume that (p,q),(r,s) are order equivalent and not order equivalent to (t,u), using Theorem 3.2.3.1. By coordinate switching we can assume that Lemma 3.2.2.1 applies to p,q,r,s, with (p,q) not order equivalent to (t,u). Now apply Lemma 3.2.2.1 and the various constructions of S in section 3.2.1 to activate Lemmas 3.2.3.2 - 3.2.3.8. In this way we obtain an algorithmic stable maximal emulator of E. QED

We now want to extend Theorem 3.2.2.9 from |E| ≤ 2 to |E| = 3 with a stray.

OPEN PROBLEM. Does every E ⊆ $Q[-1,1]^2$, |E| = 3, where not all x,y ∈ E are order equivalent, has a semi linear stable maximal emulator? A semi linear maximal emulator?

3.2.4. EMULATING THREE PAIRS: MAIN STUDENT THEOREM

In this section, we prove the following.

MAIN STUDENT THEOREM. Every E ⊆ $Q[-1,1]^2$, |E| ≤ 3 has an algorithmic negatively stable maximal emulator.

In light of the Student Stray Theorem, we can assume that all x,y ∈ E are order equivalent.

The use of a quantity called weights makes certain arguments a bit simpler.

DEFINITION 3.2.4.1. Let $S \subseteq Q[-1,1]^2$. W(S), the weight of S, is the number of elements $(x,y) \in S^2$, $x \neq y$, up to order equivalence on 4-tuples and switching (x,y) with (y,x).

THEOREM 3.2.4.1. For $E \subseteq Q[-1,1]^2$, $|E| = 3$, $W(E) \leq 3$. If S is an emulator of $E \subseteq Q[-1,1]^2$ then $W(S) \leq W(E)$.

Proof: Obvious since every element of S^2 is an element of E up to order equivalence of 4-tuples. QED

Since the three elements of E are assumed to be order equivalent, they are all of the form (p,p), all of the form (p,q), $p < q$, or all of the form (p,q), $p > q$. By coordinate switching, we only need consider (p,p) and (p,q), $p < q$.

LEMMA 3.2.4.2. Let $E = \{(p,p),(q,q),(r,r)\}$, $|E| = 3$. E has an order theoretic negatively stable maximal emulator.

Proof: Let $S = \{(q,q): -1 \leq q \leq 1\}$. S is an order theoretic negatively stable maximal emulator. Prove this. QED

Throughout this section, we assume $E = \{(p,q),(r,s),(t,u)\}$, $p < q \wedge r < s \wedge t < u \wedge |E| = 3$. Other conditions on E,p,q,r,s,t,u are explicitly stated.

LEMMA 3.2.4.3. Suppose at least two elements of E have the same first term. There is an order theoretic negatively stable maximal emulator of E.

Proof: Note that S = {(-1,a): -1 < a ≤ 1} is an emulator of E. Prove this. Suppose S ∪. (b,c) is an emulator of E. Then b < c and we claim b = -1. Otherwise -1 < b < 1 and S ∪. {(b,c)| has the following five quadruples:

(-1,b/2),(b,c)
(-1,b),(b,c)
(-1,(b+c)/2),(b,c)
(-1,c),(b,c)
(-1,1),(b,c)

so that the weight of S is at least 5, more than the weight of E is at most 3. So b = -1. Hence (b,c) ∈ S, which is impossible. QED

So we have handled duplicate first coordinates. Now we want to handle duplicate second coordinates.

LEMMA 3.2.4.4. Let E = {(p,r),(q,r),(t,r)}, p < q < t < r. There is an order theoretic negatively stable maximal emulator of E.

Proof: S = {(a,-1/2): -1 ≤ a < -1/2} is an order theoretic negatively stable maximal emulator of E. QED

LEMMA 3.2.4.5. Let E = {(p,r),(q,r),(t,u)}, p < q < r ∧ t < r ∧ p,q,t distinct. There is an order theoretic negatively stable maximal emulator of E.

Proof: Let S = {(a,-1/2): -1 ≤ a < -1/2}. Let S ∪. {(b,c)} be an emulator of E. Suppose b < -1/2. Then (b,-1/2),(b,c) is not reflected in E. Hence b ≥ -1/2. Now (-1,-1/2),(-1/2,c) is not reflected in E, and so b > 1/2. Since (-1,-1/2),(b,c) is reflected in E. we have a contradiction. Hence S is an order theoretic negatively stable maximal emulator of E. QED

LEMMA 3.2.4.6. Let E = {(p,r),(q,r),(r,s)}, p < q < r. There is an order theoretic negatively stable maximal emulator of E.

Proof: Let S = {(a,-1/2): -1 ≤ a < -1/2}. Then S ∪ {(-1/2,1/2)} is an emulator of E. Let (S ∪ {(-1/2,1/2)}) ∪. {(b,c)} be an emulator of E. Suppose b ≤ -1/2. Then (b,-1/2),(b,c) is not reflected in E. Hence b > -1/2. Now (-1,-1/2),(b,c) is reflected in E. This is impossible. Hence S ∪ {(-1/2,1/2)} is an order theoretic negatively stable maximal emulator of E. QED

LEMMA 3.2.4.7. Let E = {(p,r),(q,r),(s,t)}, p < q < r < s. There is an algorithmic negatively stable maximal emulator of E.

Proof: Let S = (-1,1/2),(0,1/2),(2/3,1). Then S is an emulator of E. Let S' be an algorithmic maximal emulator of E containing E. Let p < 0. Then (-1,1/2),(p,1) is not reflected in E, and so (p,1) ∉ S'. Also (0,1/2),(p,0) is not reflected in E. So for no p < 0 is (p,1) or (p,0) in S'. Hence S' is an algorithmic negatively stable maximal emulator of E. QED

LEMMA 3.2.4.8. Let E = {(p,q),(r,s),(t,u)}, where some pair of first coordinates are equal or some pair of second coordinates are equal. There is an algorithmic negatively stable maximal emulator of E.

Proof: Suppose some pair of first coordinates are equal. Apply Lemma 3.2.4.3. Now suppose some pair of second coordinates are equal. We can assume that the first coordinates are distinct. If all three second coordinates are equal then apply Lemma 3.2.4.4. Now suppose exactly two second coordinates are equal. If the first coordinate of the remaining pair is less than the common second coordinate, then apply Lemma 3.2.4.5. If the first coordinate of the remaining pair is the same as the common second coordinate, apply Lemma 3.2.4.6. If the first coordinate of the remaining pair is greater than the common second coordinate, apply Lemma 3.2.4.7. This covers all cases. QED

LEMMA 3.2.4.9. Let E = {(p,q),(q,r),(s,t)}. There is a two element negatively stable maximal emulator of E.

Proof: Let S = {(-1,1/2),(1/2,1). Let S ∪. {b,c)} be an emulator of S. Suppose b < 1/2. Then (1/2,1),(b,c) is reflected in E, and so c = 1/2. Hence (1,1/2),(b,1/2) is reflected in E, which is impossible. So b ≥ 1/2. Now if b = 1/2 then (1/2,1)),(1/2,c) is reflected in E, which is impossible. Finaly suppose b > 1/2. But then (-1,1/2),(b,c) is not reflected in E. Therefore S is a negatively stable maximal emulator of E. QED

LEMMA 3.2.4.10. Let E = {(p,q),(r,s),(t,u)}, where p,q,r,s,t,u are not entirely distinct. There is an algorithmic negatively stable maximal emulator of E.

Proof: Let E,p,q,r,s,t,u be as given. By Lemma 3.2.4.8, we can assume that p,r,t are distinct and q,s,u are distinct. Then the second coordinate of some pair in E is the same as the first coordinate of some other pair in E. This is covered by Lemma 3.2.4.9. QED

LEMMA 3.2.4.11. Let E = {(p,q),(r,s),(t,u)}, where p,q,r,s,t,u are distinct. There is an algorithmic negatively stable maximal emulator of E.

Proof: Let S be an algorithmic maximal emulator of E containing (0,1). No coordinate of any element of E\{(0,1)} lies in {0,1}. Hence S is negatively stable. QED

MAIN STUDENT THEOREM. Every E ⊆ $Q[-1,1]^2$, |E| ≤ 3, has an algorithmic negatively stable maximal emulator.

Proof: By Lemmas 3.2.4.10 and 3.2.4.11. QED

We cannot improve the Main Student Theorem always getting a semi linear negatively stable maximal emulator. See Lemma 3.2.2.8. However if we just want stability, can we?

OPEN PROBLEM. Does every $E \subseteq Q[-1,1]^2$, $|E| \leq 3$, have a semi linear stable maximal emulator? Have a semi linear maximal emulator?

4. SUPPLEMENTAL MATERIAL more will be added later

4.1. ATTACKING $E \subseteq Q[-1,1]^2$, $|E| = 4$

CONJECTURE. Every $E \subseteq Q[-1,1]^2$, $|E| = 4$, has an algorithmic negatively stable maximal emulator.

We already know that

THEOREM. (Friedman) Every $E \subseteq Q[-1,1]^2$ has a negatively stable maximal emulator.

but the only proof I have uses transfinite recursions of uncountable length, and does not yield an algorithmic negatively stable maximal emulator.

I recommend that the work be divided into the following groups. Some of these groups would need a major subdivision.

A. $E = \{(p,p),(q,q),(r,r),(s,s)\}$, $|E| = 4$
B. $E = \{(p,p),(q,q),(r,r),(s<t)\}$, $|E| = 4$.
C. $E = \{(p,p),(q,q),(r<s),(t<u)\}$, $|E| = 4$
D. $E = \{(p,p),(q,q),(r<s),(t>u)\}$, $|E| = 4$
E. $E = \{(p,p),(q<r),(s<t),(u<v)\}$, $|E| = 4$
F. $E = \{(p,p),(q<r),(s<t),(u>v)\}$, $|E| = 4$
G. $E = \{(p<q),(r<s),(t<u),(v<w)\}$, $|E| = 4$
H. $E = \{(p<q),(r<s),(t<u),(v>w)\}$, $|E| = 4$
I. $E = \{(p<q),(r<s),(t>u),(v>w)\}$, $|E| = 4$

Here are some specific E to work with, each consisting of four pairs. It is simpler to allow the elements of E to be outside $Q]-1.1]^2$, but we still insist that we look for emulators S ⊆ $Q[-1,1]^2$. E need only be presented up to isomorphism, so there is no reason to stay within $Q[-1,1]^2$.

1a. {(1,1),(2,2),(.5,1),(1.5,2.5)}
1b. {(1,1),(2,2),(.5,1),(2.5,1.5)}
1c. {(1,1),(2,2),(1,.5),(1.5,2.5)}
1d. {(1,1),(2,2),(2.5,1.5),(2.5,1.5)}

2a. {(1,2),(1,3),(1.5,3.5),(2,2.5)}
2b. {(1,2),(1,3),(1.5,3.5),(2.5,2)}
2c. {(1,2),(1,3),(3.5,1.5),(2,2.5)}
2d. {(1,2),(1,3),(3.5,1.5),(2.5,2)}

3a. {(1,2),(2,3),(0,1),(2.5,3)}
3b. {(1,2),(2,3),(0,1),(3,2.5)}
3c. {(1,2),(2,3),(1,0),(2.5,3)}
3d. {(1,2),(2,3),(1,0),(3,2.5)}

4a. {(1,2),(3,4),(2,4),(2.5,4.5)}
4b. {(1,2),(3,4),(2,4),(4.5,2.5)}
4c. {(1,2),(3,4),(4,2),(2.5,4.5)}
4d. {(1,2),(3,4),(4.2),(4.5,2.5)}

5a. {(1,3),(2,3)},(.5,2),(1.5,3.5)}
5b. {(1,3),(2,3)},(.5,2),(3.5,1.5)}
5c. {(1,3),(2,3)},(2,.5),(1.5,3.5)}
5d. {(1,3),(2,3)},(2,.5),(3.5,1.5)}

6a. {(1,3},(2,4),(.5,3),(.5,4.5)}
6b. {(1,3},(2,4),(.5,3),(4.5,.5)}
6c. {(1,3},(2,4),(3,.5),(.5,4.5)}
6d. {(1,3},(2,4),(3,.5),(4.5,.5)}

7a. {(1,4),(2,3)},(1.5,4.5),(2.5,3)}
7b. {(1,4),(2,3)},(1.5,4.5),(3,2.5)}
7c. {(1,4),(2,3)},(4.5,1.5),(2.5,3)}
7d. {(1,4),(2,3)},(4.5,1.5),(3,2.5)}

For each of these 28 examples, we ask the following question:

Is there a (algorithmic, semi linear, order theoretic, finite, blank) (negatively stable, stable, blank) maximal emulator S ⊆ $Q[-1,1]^2$ of E?

In section 3.1 we showed that for all E ⊆ $Q[-1,1]^2$ there is an algorithmic stable maximal emulator.

We stated earlier that I know that for all E ⊆ $Q[-1,1]^2$ there is a negatively stable maximal emulator. My proofs use advanced methods, and we don't know if in each (or any) case there is an algorithmic negatively stable maximal emulator.

4.2. LONGER INTERVALS, HIGHER DIMENSIONS AND UNPROVABILITY

We now consider the spaces $Q[-n,n]^2$ based on the longer intervals Q[-n,n], n ≥ 1. Stability now takes the following form.

S ⊆ $Q[-n,n]^2$ is stable if and only if for all 0 ≤ i < j ≤ n and 0 ≤ i' < j' ≤ n,

(i,j) ∈ S ↔ (i',j') ∈ S
(j,i) ∈ S if (j',i') ∈ S
(i,i) ∈ S ↔ (i',i') ∈ S

S ⊆ $Q[-n,n]^2$ is negatively stable if and only if for all 0 ≤ i < j ≤ n and 0 ≤ i' < j' ≤ n,

(i,j) ∈ S ↔ (i',j') ∈ S
(j,i) ∈ S if (j',i') ∈ S
(i,i) ∈ S ↔ (i',i') ∈ S
for all p < 0, (p,i) ∈ S ↔ (p,i') ∈ S
for all p < 0, (i,p) ∈ S ↔ (i',p) ∈ S

We now consider both longer intervals and higher dimensions, Q[-n,n]k.

$S \subseteq Q[-n,n]^k$ is stable if and only if for all order equivalent $x,y \in \{0,...,n\}^k$, $x \in S \leftrightarrow y \in S$.

$S \subseteq Q[-n,n]^k$ is negatively stable if and only if for all order equivalent $x,y \in \{0,...,n\}^k$, if $x',y' \in Q[-n,n]^k$ is obtained by replacing zero or more x_i,y_i by $p,p < 0$, then $x' \in S \leftrightarrow y' \in S$.

THEOREM 4.2.1. The definitions of stability and negative stability for S ⊆ Q[-1,1]2 are the same of those for Q[-n,n]k with k set to 2 and n set to 1.

We have shown that the following statement is independent (neither provable or refutable in) of ZFC.

EVERY $E \subseteq Q[-n,n]^k$ HAS A NEGATIVELY STABLE MAXIMAL EMULATOR.

CONJECTURE. The above statement is independent of ZFC with $k = n = 3$.

In the above Conjecture we quantifier over all E. We see from section 3.1 that we can instead look for all E of at most some standard finie aspo

4.3. r-EMULATORS, FULL STABILITY - ANOTHER REALM OF CHALLENGES

The key notion of emulators can be greatly intensified, creating a whole new realm of complications.

$S \subseteq Q[-n,n]^k$ is a k-emulator of $E \subseteq Q[-n,n]^k$ if and only if for all $x_1,...,x_r \in S$ there exists $y_1,...,y_r \in E$ such that the concatenation $x_1...x_r$ is order equivalent to the concatenation $y_1...y_r$.

It is obvious that emulators are 2-emulators.

EVERY $E \subseteq Q[-n,n]^k$ HAS A NEGATIVELY STABLE MAXIMAL r-EMULATORS.

We have shown that this statement is independent of ZFC.

We also use a stronger form of stability than negatively stable. We call it full stability.

$S \subseteq Q[-n,n]^k$ is fully stable if and only if for all order equivalent $x,y \in \{0,...,n\}^k$, if $x',y' \in Q[-n,n]^k$ is obtained by replacing zero or more x_i,y_i by $p,p < \min(xy)$, then $x' \in S \leftrightarrow y' \in S$.

4.4. CHALLENGE: SIX 4-TUPLES

Here is a randomly generated sequence of 24 elements of {0,...,7}. This forms a set of r-tuples of cardinality 6.

E = {(2,0,0,1),(1,2,7,1),(0,6,7,1),(1,5,6,1),(2,7,4,4),(7,2,4,3)}

As a special case of our general Emulator Theorem, using large cardinal hypotheses, that go far beyond ZFC, we know that

STATEMENT. $E \subseteq Q[-7,7]^4$ HAS A NEGATIVELY STABLE MAXIMAL EMULATOR $S \subseteq Q[-7,7]^4$.

We conjecture that the STATEMENT can be proved well within ZFC. Is it utterly hopeless for a human being to have any chance of proving such a STATEMENT without simply invoking the general theorem for any $E \subseteq Q[-n,n]^k$?

In any case, we also know that there is a specific finite $E \subseteq Q[-n,n]^k$ for which the STATEMENT is independent of ZFC. We conjecture that |E| = 15 is enough.

REFERENCES

[Fr20] H. Friedman, Tangible Incompleteness Interim Report, #110, Downloadable Manuscripts, https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/, July 15, 2020, 26 pages.