INTEGER AND WORD SEQUENCES: TANGIBLE INCOMPLETENESS

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ABSTRACT. In my previous paper on Long Finite Sequences, we prove and investigate lower bounds in connection with the fact that in any sufficiently long sequence $x_1,...,x_n$ from 1,...,k, there exists 1 <= i < j <= n/2 such that $x_i,...,x_{2i}$ is a subsequence of $x_j,...,x_{2j}$. We strengthen this statement in two different ways. One is to use my gap condition introduced for trees, here for $x_i,...,x_{2i}$ and $x_j,...,x_{2j}$, yielding an equivalence with 1-Con(PA). The other is to use sufficiently long sequences $w_1,...,w_n$ of finite words over a finite alphabet where $w_i,...,w_{2i}$ is required to be equivalent to a subsequence of $w_j,...,w_{2j}$ under certain natural equivalence relations on sequences of words. The resulting statements are equivalent to 1-Con(PA) and various impredicative systems.

We investigated a very simple statement about long sequences drawn from a finite alphabet in [Fr01].

BLOCK SUBSEQUENCE THEOREM. In any sufficiently long finite sequence $x_1,...,x_n$ from {1,...,k}, there exists $1 \leq i < j \leq n/2$ such that $(x_i,...,x_{2i})$ is a subsequence of $(x_j,...,x_{2j})$.

Before relaxing the {1,...,k} restriction, we strengthen the subsequence theorem in the following way.

DEFINITION 1. $x_1,...,x_r \in \mathfrak{R}$ is a gap subsequence of $y_1,...,y_s \in Z$ if and only if $x_1,...,x_r$ is a subsequence of $y_1,...,y_s$ placed in positions $a_1 < ... < a_r$, without change, where for all $1 < j < r$, the y's in positions $a_j+1,\ldots,a_{j+1}-1$ are numerically at least y_{j+1}.

This is precisely my "gap condition" for Extended Kruskal Theorem where the trees have no splitting and are therefore sequences.

It has been shown in [SS85] and reworked in ???? (A. Weiermann) that my EKT for sequences corresponds to $\in_0$ and 1-Con(PA). The following is the appropriate finite form in the style of [Fr01].

GAP SUBSEQUENCE THEOREM. In any sufficiently long finite sequence $x_1,\ldots,x_n$ from $\{1,\ldots,k\}$, there exists $1 \leq i < j \leq n/2$ such that $(x_i,\ldots,x_{2i})$ is a gap subsequence of $(x_j,\ldots,x_{2j})$.

THEOREM 1. The Gap Subsequence Theorem is provably equivalent to 1-Con(PA) over EFA.

It is not clear to us how to naturally get past predicativity using only finite sequences from a finite alphabet.

We now use finite words from a finite alphabet. Of course, words are sequences, but we like a different name than sequences as we are working with sequences of words.

DEFINITION 2. $\{1,\ldots,k\}^*$ is the set of all finite words in k letters $1,\ldots,k$. These are allowed to be empty. Let $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ be finite sequences of finite words from $\{1,\ldots,k\}$. We say that $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ are extension equivalent if and only if for all $1 \leq i,j \leq n$, x_i extends x_j if and only if y_i extends y_j. We say that $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are extension/last equivalent if and only if they are extension equivalent and for all $1 \leq i \leq n$, x_i and y_i have the same last letter (or both are empty). We say that $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are extension/last(1) equivalent if and only if they are extension equivalent and for all $1 \leq i \leq n$, x_i has last letter 1 $\leftrightarrow$ y_i has the last letter 1.

WORD SUBSEQUENCE THEOREM/1. In any sufficiently long finite sequence $\alpha 1,\ldots,\alpha n$ from $\{1,\ldots,k\}^*$, there exists $1 \leq i < j \leq n/2$ such that $(\alpha_i,\ldots,\alpha_{2i})$ is extension equivalent to a subsequence of $(\alpha_j,\ldots,\alpha_{2j})$.

The use of words here of unlimited finite length may look scary, but does not cause any problems. But it doesn't look like a Π^0_2 sentence. We can fix this by bounding the length of the words to as follows.

WORD SUBSEQUENCE THEOREM/1'. In any sufficiently long finite sequence $\alpha_1,...,\alpha_n$ from {1,...,k}* of lengths at most 1,...,n, respectively, there exists $1 \leq i < j \leq n/2$ such that $(\alpha_i,...,\alpha_{2i})$ is extension equivalent to a subsequence of $(\alpha_j,...,\alpha_{2j})$.

Then this is explicitly Π^0_2 in light of it being true for n (with k fixed in advance), it is obviously true when we raise n.

Below we also have a /1 and a /1' version with the same properties. We only write the /1 version.

THEOREM 2. Both forms of the Word Subsequence Theorem are provably equivalent to 1-Con(PA).

WORD/LAST(1) SUBSEQUENCE THEOREM/1. In any sufficiently long finite sequence $\alpha_1,...,\alpha_n$ from {1,...,k}*, there exists $1 \leq i < j \leq n/2$ such that $(\alpha_i,...,\alpha_{2i})$ is extension/last(1) equivalent to a subsequence of $(\alpha_j,...,\alpha_{2j})$.

THEOREM 3. Word/Last Subsequence Theorem is provably equivalent, over EFA, to 1-Con(ATR_0).

WORD/LAST SUBSEQUENCE THEOREM/1. In any sufficiently long finite sequence $\alpha_1,...,\alpha_n$ from {1,...,k}*, there exists $1 \leq i < j \leq n/2$ such that $(\alpha_i,...,\alpha_{2i})$ is extension/last equivalent to a subsequence of $(\alpha_j,...,\alpha_{2j})$.

THEOREM 4. Word/Last Subsequence Theorem/1 is provably equivalent, over EFA, to no infinite elementary recursive descending sequence through Γ_ω.

For Theorem 4, we make use of one of the ordinal calculations in [SMW20].

For the level of Kruskal's Theorem, we need the following.

DEFINITION 3. The inf of two finite sequences is taken to be the longest common initial segment. Let $x_1,...,x_n$ and $y_1,...,y_n$ be finite sequences of finite words from {1,...,k}. We say that $(x_1,...,x_n)$ and $(y_1,...,y_n)$ are inf equivalent if and only if for all $1 \leq i,j,r \leq n$, xi $= \inf(x_j,x_r)$ if and only if $y_i = \inf(y_j,y_r)$.

WORD/INF SUBSEQUENCE THEOREM/1. In any sufficiently long finite sequence $\alpha_1,...,\alpha_n$ from {1,...,k}*, there exists $1 \leq i < j \leq n/2$ such that $(\alpha_i,...,\alpha_{2i})$ is inf equivalent to a subsequence of $(\alpha_j,...,\alpha_{2j})$.

THEOREM 5. Word/Inf Subsequence Theorem is provably equivalent, over EFA, to 1-Con($\Pi^1{}_2$-TI_0).

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