REGULAR POLYGONS AND TANGIBLE INCOMPLETENESS

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This development was inspired by the talk I recently gave at Gent, [Fr21].

ABSTRACT. We investigate the structure of nonempty finite sequences of pairwise disjoint regular polygons in the plane, and obtain some new kinds of Π^0_2 Incompleteness at the important levels of PA, ATR_0, Π^1_2-TI_0, and Π^1_1-CA_0. All statements use the i,...,2i and j,...,2j conclusion first presented in [Fr01].

ATR_0 represents the threshold of Impredicativity associated with Γ_0, Π^1_2-TI_0 corresponds to Kruskal's Theorem with no labels, and Π^1_1-CA_0 corresponds to EKT (extended Kruskal's theorem), and the Graph Minor Theorem for finite tree width.

We work entirely in the Euclidean plane $\Re^2$. Regular polygons are defined as in Euclidean geometry, where, up to similarity, there is exactly one with n sides, n ≥ 3. We can think of regular polygons as sets of points in $\Re^2$ or as sets of sides, or as sets of endpoints, or as an integer (number of sides) together with a vertex and the center. As is well known, none of this makes any significant mathematical difference. Also, since we are mostly interested in combinatorial Incompleteness, an objection can be made that real numbers are inappropriate. So can instead replace real numbers by algebraic real numbers. Or we can stick with real numbers and use the decision procedure for the field of real numbers.

DEFINITION 1.1. We use P for regular polygons unless otherwise indicated. P is surrounded by P' if and only if P' surrounds P if and only if P lies in the interior of P'. We use X for nonempty finite lists of pairwise disjoint regular polygons. #(P) is the number of sides of P. Obviously every #(P) ≥ 3.

We now introduce four simple and natural notions of embeddings of X into X'. We begin with the most basic, which just asks that it preserve "surrounds".

DEFINITION 1.2. Let X,X' be given. An embedding from X into X' is a one-one function h from the terms of X into the terms of Y such that for all P_1,P_2 in X, P_1 surrounds P_2 if and only if $h(P_1)$ surrounds $h(P_2)$. X is embeddable in X' if and only if there is an embedding from X into X'. An embedding/# from X into X' is an embedding from X into X' that preserves # (the number of sides).

DEFINITION 1.4. Let X,X' be given. An embedding/inf from X into X' is an embedding from X into X' such that for all P_1,P_2,P_3 in X, P_1 is surround least among P in X that surrounds P_2,P_3 if and only if $h(P_1)$ is surround least among P in Y that surrounds $h(P_2),h(P_3)$. h is an embedding/#,inf from X into X' if and only if h is an embedding/# and an embedding/inf from X into X'.

DEFINITION 1.5. Let X be given. A gap of X consists of two terms P1,P2 in X such that P1 is surrounded by P2. We say that P is in the gap P1,P2 if and only if P1 is surrounded by P and P is surrounded by P2. A gap in X may have nothing in it from X.

DEFINITION 1.6. Let X,X' be given. h is a gap embedding from X into X' if and only if h is an embedding/#,inf from X into X' such that for all gaps P_1,P_2 in X', P_1,P_2 from X, all P in X' lying in that gap have at least as many sides as P_2.

POLYGON EMBED. POLYEMBED. Let k ≥ 1. In every sufficiently long $P_1,...P_n$ of pairwise disjoint regular polygons, there exist $k \leq i < j \leq n/2$ and an embedding from $P_i,...,P_{2i}$ into $P_j,...,P_{2j}$.

POLYGON EMBED/#. POLYEMBED/#. Let k ≥ 1. In every sufficiently long $P_1,...P_n$ of pairwise disjoint regular k-gons, there exist 1 ≤ i < j ≤ n/2 and an embedding/# from $P_i,...,P_{2i}$ into $P_j,...,P_{2j}$.

POLYGON EMBED/inf. POLYEMBED/inf. Let k ≥ 1. In every sufficiently long $P_1,...P_n$ of pairwise disjoint regular polygons, there exist k ≤ i < j ≤ n/2 and an embedding/inf from $P_i,...,P_{2i}$ into $P_j,...,P_{2j}$.

POLYGON EMBED/#,inf. Let k ≥ 1. In every sufficiently long $P_1,...P_n$ of pairwise disjoint regular ≤k-gons, there exist 1 ≤ i < j ≤ n/2 and an embedding/#,inf from $P_i,...,P_{2i}$ into $P_j,...,P_{2j}$.

POLYGON GAP EMBED. POLYGAPEMBED. Let k ≥ 1. In every sufficiently long $P_1,...P_n$ of pairwise disjoint regular ≤k-gons, there exist k ≤ i < j ≤ n/2 and a gap embedding from $P_i,...,P_{2i}$ into $P_j,...,P_{2j}$.

THEOREM 1. POLYEMBED, POLYEMBED/#, POLYEMBED/inf, POLYEMBED/#/inf, POLYGAPEMBED are provably equivalent, over EFA, to the following, respectively.
1. 1-Con(PA).
2. No primitive recursive descending sequences through the Γ_n, n ≥ 0. We can fix k = 2 and add a parameter for 1 ≤ k' ≤ i < j ≤ n/2 to get an exact match with Γ_0.
3. 1-Con(Π^1_2-TI_0).
4. 1-Con(Π^1_2-TI_0).
5. 1-Con(Π^1_1-CA_0).

For 2, we are relying on [SMW20].

Details will be presented later.

Note how simple we have already gone beyond the level of Predicative Analysis with POLYEMBED/#. It requires no explicit use of trees, let alone inf preservation.

REFERENCES

[Fr01] H. Friedman, Long Finite Sequences, Journal of Combinatorial Theory, Series A, Volume 95, Issue 1, July 2001, Pages 102-144.

[Fr21] H. Friedman, Line Figures and Incompleteness, 7 pages, Tangible Incompleteness Series, Lecture 2, Zoom, University of Gent, Gent, Belgium. April 21, 2021. This is the April 22, 2021 text. Downloadable Lecture Notes #71, https://u.osu.edu/friedman.8/foundational-adventures/downloadable-lecture-notes-2/

[SMW20] D. Schmidt, J. Van der Meeren, A. Weiermann: Calculating Maximal Order Types for Finite Rooted Unstructured Labeled Trees. p. 253-264 in: The Legacy of Kurt Schütte. Editors: Reinhard Kahle, Michael Rathjen. 2020. Springer International Publishing DOI: 10.1007/978-3-030-49424-7.