Transfer of Mathematical Knowledge: The Portability of Generic Instantiations

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ABSTRACT—***Mathematical concepts are often difficult to acquire. This difficulty is evidenced by failure of knowledge to transfer to novel analogous situations. One approach to this challenge is to present the learner with a concrete instantiation of the to-be-learned concept. Concrete instantiations communicate more information than their abstract, generic counterparts and, in doing so, they may facilitate initial learning. However, this article argues that extraneous information in concrete instantiations may distract the learner from the relevant mathematical structure and, as a result, hinder transfer. At the same time, generic instantiations, such as traditional mathematical notation, can be learned by both children and adults and can, in turn, allow for transfer, suggesting that generic instantiations result in a portable knowledge representation.***

KEYWORDS—*mathematics; child development; communication*

Mathematical concepts are often difficult to acquire. One approach to this challenge is to present such concepts to the learner via concrete instantiations such as physical manipulatives or contextualized examples. As noted by other authors in this issue (e.g., Martin, 2009; Uttal & McNeil, 2009), there is a widespread belief in the education community that concrete instantiations benefit the learning of abstract concepts. Some researchers (e.g., Martin, 2009; Sarama & Clements, 2009; Wearne & Hiebert, 1988) have demonstrated successful use of concrete instantiations for children's acquisition of some mathematical concepts. At the same time, the evidence of the effectiveness of concrete instantiations is not unequivocal (see Sarama & Clements, 2004; Sowell, 1989; Uttal, O'Doherty, Newland, Hand, & DeLoache, 2009). Concrete instantiations communicate more extraneous information than do their abstract counterparts, and as a result, they may interfere with learning and/or transfer. In what follows, we present this argument in greater detail.

Acquiring mathematical concepts is different from acquiring everyday concepts. Everyday concepts, such as "chair" or "cat," are grounded in perceptual similarity and are acquired with little effort through encounters with instances of a concept (Kloos & Sloutsky, 2008). Mathematical concepts, however, have precise definitions based on their relational structure, and superficial similarity of instances is irrelevant. Therefore, instances of the same concept can be vastly dissimilar, sharing little or no observable similarities. For example, the same mathematical function can describe the metabolism of medication in the body or the temperature of a cooling cup of coffee. Because superficial features can vary widely, it is often difficult to spontaneously recognize instances of the same concept. As a result, the acquisition of such concepts is often difficult for both children and adults and typically requires some supervision (e.g., Kloos & Sloutsky, 2008), which may take a form of explicit instruction that begins with an initial instantiation.

What is the best way to instantiate a concept to promote recognition of novel isomorphs and successful transfer? One view is that concrete instantiations may have an advantage over abstract instantiations. An instantiation is concrete to the extent that it communicates more extraneous, concept-irrelevant information than an abstract, generic counterpart. For example, consider the increase in information in the following possible symbols for the concept "person": a black dot, a stick figure, a detailed drawing, a real person. Added information can be perceptual—what we can actually sense—or conceptual—what other information we may know about the representation. By this interpretation, concrete instantiations include contextualized examples as well as

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physical objects. It is possible that some concrete instantiations may facilitate initial learning by tapping prior knowledge that communicates relevant structural aspects of the to-be-learned concept. For example, children are familiar with equal sharing of items prior to formally knowing the operation of division. In addition, concrete instantiations may better facilitate learning by being more engaging than abstract instantiations.

While some concrete instantiations may assist initial learning, successful learning does not necessarily result in successful transfer (Gick & Holyoak, 1980, 1983; Goswami, 1991; Novick, 1988; Reed, Dempster, & Ettinger, 1985; Reed, Ernst, & Banerji, 1974; Simon & Reed, 1976). There are a number of reasons to be skeptical of the effectiveness of concrete instantiations for transfer that stem from the notion that concrete instantiations convey extraneous information. First, transfer is influenced by superficial similarities across domains, with transfer to similar instances being more likely than transfer to dissimilar instances (Holyoak & Koh, 1987; Holyoak & Thagard, 1997; Ross, 1987, 1989). Extraneous information of the learning domain may limit transfer only to isomorphs with common superficial features. In addition, transfer may require alignment of the common structure between the learned and novel domains (Gentner, 1983, 1989; Holyoak & Thagard, 1989, 1997), and there is evidence suggesting that extraneous information may hinder structural recognition and alignment. Superficial features of an instantiation may also compete with relational structure for attention (Goldstone & Sakamoto, 2003), possibly making the detection of relations more difficult than it would be in an abstract, generic format. Another potential difficulty for transfer is that irrelevant information may be misinterpreted as part of the relevant structure (Bassok & Olseth, 1995; Bassok, Wu, & Olseth, 1995). Also, even for simple relations such as the relation of monotonic increase, relational structure common to two situations is less likely to be noticed when the situations are represented in a concrete, perceptually rich manner than when represented in a more generic form (Gentner & Medina, 1998; Markman & Gentner, 1993). Finally, concrete objects make poor symbols: Both children and adults tend to reason about them as objects themselves, not as signs denoting other entities (DeLoache, 2000; Schwartz, 1995; Uttal, Liu, & DeLoache, 1999; Uttal, Scudder, & DeLoache, 1997). If concreteness hinders transfer of simple relations involved in symbol use, it is likely to create an obstacle to transfer of more complex relations such as mathematical concepts.

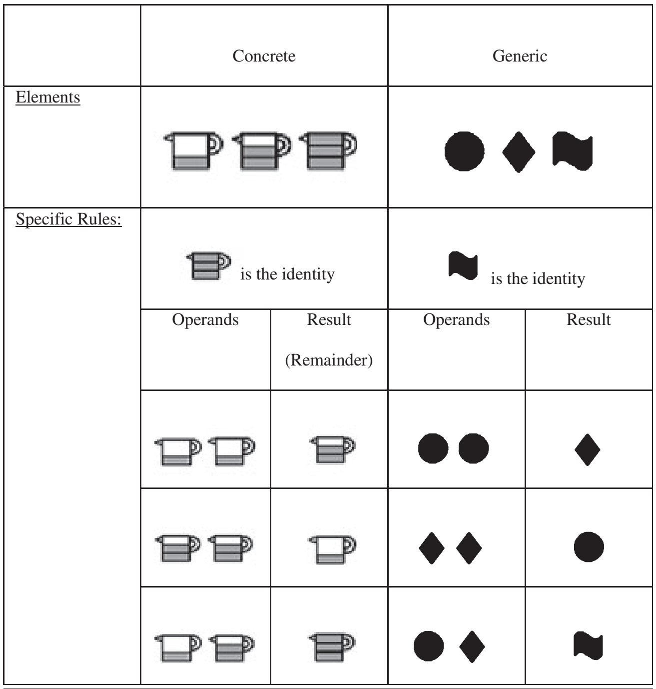
In a series of experiments, we investigated the effect of concreteness on initial learning and analogical transfer of a simple mathematical concept of a commutative mathematical group of order 3. This concept is a set of three elements, or equivalence classes, and an associated operation that has the properties of associativity and commutativity. In addition, the group has an identity element and inverses for each element (see Table 1 for properties). We instantiated this structure with different degrees of concreteness, where levels of concreteness were varied across participants. Training was presented via computer and consisted of explicit presentation of the group rules using the elements of the given instantiation, questions with feedback, and examples. After training, participants received a multiple-choice test of novel questions.

In our first experiment, undergraduate students were trained and tested with either an abstract, generic instantiation or a perceptually rich, concrete instantiation (Sloutsky, Kaminski, & Heckler, 2005). The generic instantiation was described as a written language involving three symbols in which combinations of two or more symbols yield a predictable resulting symbol. Statements were expressed as *symbol 1*, *symbol 2* → *resulting symbol*. The concrete condition presented an artificial phenomenon involving images of three colorful, three-dimensional shapes. Participants in this condition watched movies of two or more shapes coming into contact, then disappearing, and a resulting shape appearing. In both conditions, the resulting symbol or shape was specified by the mathematical structure. After training and testing of one instantiation, participants were trained and tested with the other instantiation. We found that participants successfully learned both instantiations, with no difference in average test score on the generic instantiation no matter which instantiation they learned first. However, there was a marked difference in average test score on the concrete instantiation, with participants who were initially trained with the generic instantiation scoring higher on the concrete test than participants who were initially trained with the concrete instantiation. In other words, learning the concrete instantiation resulted in no improved learning of the generic instantiation. On the other hand, learning the generic instantiation resulted in better performance on the concrete instantiation, suggesting that participants were able to transfer their knowledge from the generic to the concrete instantiation. In a second experiment, we considered

Table 1
Principles of a Commutative Mathematical Group

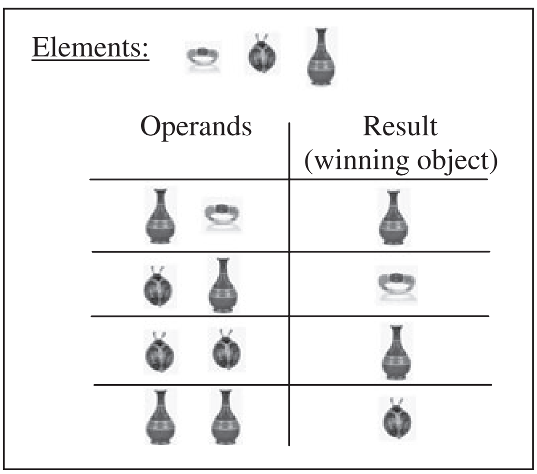
A commutative group of order 3 is a closed set of three elements and a binary operation (denoted +) with the following properties:	
Associative	For any elements x, y, z: $((x + y) + z) = (x + (y + z))$
Commutative	For any elements x, y: $x + y = y + x$
Identity	There is an element, **I**, such that for any element, x: $x + \mathbf{I} = x$
Inverses	For any element, x, there exists another element, y, such that $x + y = \mathbf{I}$

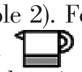
Table 2
Concrete and Generic Instantiations of Commutative Group



the effects of perceptual richness on initial learning. Participants learned an instantiation of a group that had different levels of concreteness: (a) generic black symbols; (b) colorful, patterned symbols; (c) classes of colorful, patterned symbols; or (d) classes of real objects. While all participants learned the rules, those who learned with the generic symbols scored significantly higher than did the other participants, with no differences across these three conditions. Therefore, the mere addition of color lowered learning.

Table 3
Stimuli for Transfer Domain



The results of these experiments demonstrated that perceptual richness that is irrelevant to the to-be-learned concept hindered both learning and transfer. However, not all concreteness is irrelevant perceptual richness. As noted above, some concreteness may help to communicate relevant structure by tapping prior knowledge or by presenting perceptual information that is correlated with structure. This "relevant concreteness" would most likely facilitate learning and possibly, in turn, facilitate transfer. To investigate the effects of such relevant concreteness, we instantiated the concept of a mathematical group in a context involving familiar objects that might facilitate learning of the group rules (Kaminski, Sloutsky, & Heckler, 2005). In this case, the elements of the group were three measuring cups (see Table 2). Instead of learning arbitrary rules of symbol combinations, participants were told that they needed to determine a leftover amount of liquid when different measuring cups were combined (see Table 2). For example, combining  and  resulted in  leftover. We compared learning this instantiation with learning a generic instantiation and found that with minimal training (i.e., explicit statement of the rules and one example), both instantiations were successfully learned, but the relevantly concrete instantiation did have an advantage over the generic (81% vs. 63% correct, chance = 38%).

To test whether this advantage would exist for transfer, we gave participants slightly more detailed training (questions with feedback and examples) and then presented them with a novel instantiation of an isomorphic group. This novel domain was intentionally concrete and contextually rich, as are many real-world instantiations of mathematics. Participants were asked to figure out the rules of a children's game. In the game, children point to a series of objects, then the child who is "it" points to a final object. This child wins if he or she points to the correct object according to the rules (see Table 3). Participants were told that the rules of the game were like the rules of the system they had just learned (i.e., either the concrete or the generic instantiation). Then participants were shown a series of examples from which the rules could be deduced. The results revealed that with the slightly protracted training, there was no difference in learning scores across the two conditions (78% vs. 75%). However, there was a striking difference in transfer: Participants in the concrete condition had an average test score of approximately 54% correct, while the average score in the generic condition was approximately 79% (chance was 38%). We replicated these results in another experiment and further found that when asked to match analogous elements across learning and transfer domains, 100% of participants in the generic condition were able to do so, while only 25% of participants in the concrete condition made the correct match, a rate no better than that of chance guessing.

These reported results involved undergraduate college students. Perhaps concreteness is helpful, but only for younger learners. That is, if children under 12 years of age are in a concrete operational stage of development in which their thinking and problem solving are limited to concrete contexts, they may need a concrete instantiation to begin to grasp an abstract concept. In this case, we may see an advantage for the concrete. However, if the difficulty with concrete instantiation stems from the effect of extraneous information on attentional focus, then concreteness may be at least as detrimental for children's transfer as it is for adults', since children are less able to control their focus of attention (Dempster & Corkill, 1999; Napolitano & Sloutsky, 2004). To test this possibility, we taught 11-year-old children either the concrete or the generic instantiation and presented them with the transfer domain, as in our earlier experiments. The concrete instantiation did result in better learning than did the generic (82% vs. 66% correct). However, for the concrete learners, transfer scores were only marginally above the chance score of 38% (47% correct), whereas scores were significantly above chance for the generic learners (61% correct).

These findings make several important points. First, successful learning does not necessarily translate into successful transfer. Second, irrelevant perceptual richness of some concrete instantiations can potentially hinder both learning and transfer. Third, while some concreteness may give a leg-up in the learning process, this advantage comes at the cost of transfer because the learner has difficulty recognizing learned structure in novel contexts. Structure is learned but is bound to the initial learning scenario. Generic instantiations, on the other hand, do allow spontaneous recognition of learned structure and successful transfer. Finally, the advantage of generic instantiations is not limited to adults. Children can learn a generic instantiation and successfully transfer their knowledge.

We suggest that a mechanistic component that underlies the effect of concreteness on transfer is, in fact, attentional focus. Successful analogical transfer is dependent on the ability to focus on the common relation, while ignoring irrelevant information, whether it is a simple relation such as "bigger than" or a more complex relation, or set of relations, of a mathematical definition. Transfer fails if a common relation is undetected. Concrete instantiations present extraneous information that can compete for attention with the relevant relational structure. With development, the ability to ignore irrelevant information is likely strengthened, which may explain improved performance with age on simple symbolic tasks (e.g., DeLoache, 2000; Uttal et al., 2009). At the same time, our studies demonstrate that even adults can be distracted by irrelevant superficial information. However, it is possible that with the development of expertise, people learn (depending on the task demands) to deliberately shift attention between superficial and deep structural information.

If a primary goal of learning abstract concepts such as mathematical concepts is the ability to recognize novel instantiations and successfully transfer knowledge, then educational material should maximize the likelihood of attending to relational structure and minimize the likelihood of diverting attention primarily to the superficial. One way of achieving this is to present mathematical concepts via generic formats, such as traditional symbolic notation. Learning a generic instantiation can result in portable knowledge that can be spontaneously transferred to novel isomorphic domains. In addition, because generic instantiations may help learners focus attention on relevant structure, we would expect that learning generic instantiations may also benefit transfer to nonisomorphic tasks such as those involving problem solving, estimation, and comparison. We do not suggest that concrete instantiations should never be used in the teaching of mathematics. Rather, we suggest that if they are used, measures need to be taken to help the learner extract the relevant structure from the learning context and recognize it in novel contexts.

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