Sliding Mode Multivariable Extremum Seeking Control with Application to Wind Farm Power Optimization

Yasser Bin Salamah* and Umit Ozguner*

***Abstract*— In this paper, a sliding mode based Extremum Seeking (ES) control scheme is proposed to solve a class of multivariable optimization problems. This approach recasts the problem of multivariable ES control into a sequence of single variable ES control. Our approach is suitable for non-separable problems, differentiating itself from previous work in this area. We determine the stability and convergence conditions from the unidimensional case and derive a sufficient condition for the multivariable scheme to converge to the vicinity of the optimal points. We investigate the application of the proposed scheme to optimize energy production in wind farms. Simulations are provided to illustrate the theoretical results and demonstrate its potential use.**

I. Introduction

Conventional control methods address stabilization, regulation, and/or fixed set-point tracking for a given dynamic system. However, many applications demand optimal set points, unknown in advance, or that must vary over time, such as Anti-lock Braking System control facing unpredictable changes in road conditions. In Extremum-Seeking (ES) control, the objective of the controller is to steer the system output to follow a non-predetermined optimal operating point. Fig. 1 illustrates a typical structure of an ES problem. Several design approaches were developed during the last two decades [1], [2], [3], [4], [5]. Prior research has proposed and analyzed Sliding-Mode based ES control, mainly for single-variable case. Korovin and Utkin [2] introduce the use of sliding mode extremum seeking control for static optimization. The main idea is to select a control law such that the system output tracks a monotonic decreasing (increasing) function in time towards the minimum (maximum). Furthermore, Özgüner and his co-worker generalized the method in the presence of dynamics [6]. It was successfully applied to a variety of applications such as: Anti-lock Braking System (ABS) [7], source seeking [8], and Maximum Power Point Tracking (MPPT) [9].

Multivariable ES literature currently features a variety of approaches [10]. Krstic and his co-worker propose a multivariable extension to the periodic perturbations approach by the use of multi-perturbation signals, provided they have different frequencies [11], [12]. In their recent work, a Newton-based ES approach for multivariable problem was proposed [13]. The Newton-based ES provides an estimate of the inverse of the Hessian of the objective function and makes the convergence rate independent of the unknown Hessian. The use of nonlinear programming was proposed by Teel and Popovic in [3]. The authors introduce a periodic sampling time to design a discrete-time extremum seeking controller. More recently, an extension to the single variable sliding mode ES control for multivariable applications was introduced [14], [15]. The authors propose the use of multiple sliding surfaces and different control parameters to search, simultaneously, for the set of optimal operating points. This approach works well when there is weak coupling between the decision variables in the multivariable cost function.

* The authors are with The Ohio State University Electrical and Computer Engineering Department, Columbus, OH, 43210 USA, email: (binsalamah.1@osu.edu),(ozguner.1@osu.edu)

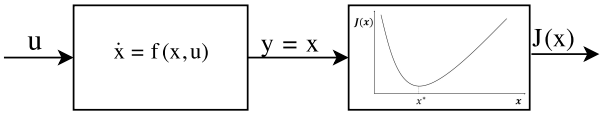

Fig. 1. Typical structure of Single Input Single Output Extremum Seeking problem.

In this paper, we develop and analyze a new control scheme for a Sliding Mode Multivariable ES control. The key idea is to recast the problem of n-dimensional ES control into an n-sequence of one-dimensional ES control. In contrast to current sliding-mode methods, our approach sheds the uncoupled-variable requirement. It also counters the high cost and low efficiency typical of decentralized ES controllers. However, even if the problem is reduced to a one-dimensional ES problem, from a stability point of view, restrictions must be placed on the algorithm to guarantee the stability of the overall ES system. The reason is that the sliding mode's reaching time must be achieved to ensure convergence towards the minimum. To begin our paper, Sliding Mode ES control is applied to a static mapping to focus on the forging of the algorithm itself and the convergence analysis. Next, the developed algorithm engages dynamical systems. Then, application of the proposed scheme for wind farm optimization is provided for illustration.

The remainder of the paper is organized as follows: The problem statements and assumptions are outlined for a static map in section 2. Section 3 specifies the proposed controller design, followed by stability and convergence analyses in section 4. Sliding mode ES for dynamic systems is investigated in section 5, and section 6 presents an example to illustrate the proposed controller.

II. Problem Formulation

Consider a Multi-input and Single-output (MISO) static map:

$$y(t) = J(\theta(t)) \tag{1}$$

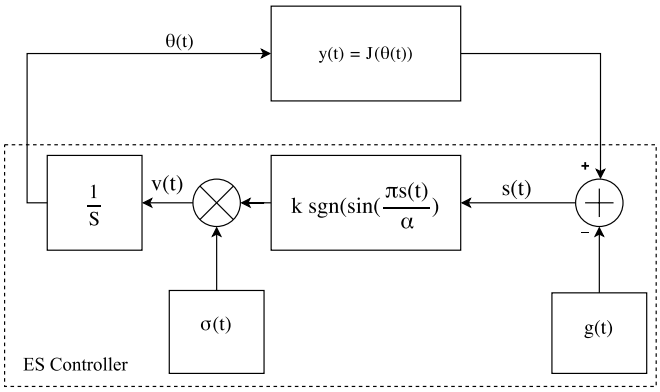

Fig. 2. Block diagram of the proposed extremum seeking control.

where $\theta(t) = [\theta_1(t)\ \theta_2(t)\ \dots\ \theta_n(t)]^T$ are decision variables and the function $J : \mathbb{R}^n \to \mathbb{R}$ is a multivariable cost function.

Assumption 2.1: The function $J : \mathbb{R}^n \to \mathbb{R}$ is differentiable and has a unique minimum at $\theta^\star = [\theta_1^\star\ \theta_2^\star\ \dots\ \theta_n^\star]^T$ and $\frac{\partial J}{\partial \theta_i} \neq 0$ with $\theta_i \neq \theta_i^\star$ for all $i \in \mathscr{I} = \{1, 2, \dots, n\}$.

Assumption 2.2: The decision variables $\theta(t)$ is designed to satisfy:

$$\dot{\theta}(t) = v(t) \tag{2}$$

where $v(t) \in \mathbb{R}^n$ is a control input to be specified in the next section.

With only the measurement of the performance function $J(\theta)$ being available, the problem is to design a control input $v(t)$ that generates $\theta(t)$ which minimizes the multi-variable cost function $J(\theta(t))$ without prior knowledge of the function or its gradient.

III. Controller Design

A. An Extremum Seeking Controller:

The proposed Multivariable Extremum Seeking Controller is shown in Fig. 2. In this approach, we transform the multivariable problem into a sequence of single dimensional ES problems by introducing a periodic search function $\sigma(t)$. The design of the control input aims to let the performance function track a monotonically decreasing function and converge to the vicinity of the minimum. Let $g(t)$ be a monotonically decreasing function of time with

$$\dot{g}(t) = -\rho, \quad \rho > 0. \tag{3}$$

Define a sliding mode manifold as

$$s(t) = J(\theta(t)) - g(t) \tag{4}$$

together with the search signal proposed in [6] the variable structure control is given by

$$v(t) = -\sigma(t)\ k\ \mathrm{sgn}(\sin\left(\pi \frac{s(t)}{\alpha}\right)) \tag{5}$$

α, k are positive design parameters, and $\sigma(t)$ is a search function to be defined in the next section.

B. Periodic Coordinate Search Function ($\sigma(t)$):

In this section we illustrate an example of the periodic coordinate search function. We start with some definitions:

Definition 1: A function $\sigma(t)$ is said to be periodic if there exists a real number $p > 0$ such that $\sigma(t+p) = \sigma(t)$. Any number p in which the equality holds is called a period.

Definition 2: Let $T = [T_l, T_u]$ be a given time interval. By a partition P of T, we mean a finite set of points $t_1, t_2, .., t_n$, where $T_l = t_1 \leq t_2 \leq t_3 \leq \dots \leq t_n = T_u$. We write $\Delta t_i = t_{i+1} - t_i$, $\forall i \in \mathscr{I}$.

The search function is designed to change the direction of the search periodically. Using the above definitions, let $\sigma(t)$ be a periodic function with period p. Define a time interval $T = [T_l, T_u]$ with length equal to p and let the set of points $t_1, t_2, .., t_n$ be given by a partition P. Let $\{e_1, e_2, \dots, e_n\}$ be the set of natural basis in $\mathbb{R}^n$. Then, the Periodic Coordinate Search Function is given by:

$$\sigma(t) = e_i \quad \forall t \in \Delta t_i \forall\ i\ \in \mathscr{I} \tag{6}$$

For simplicity, we choose $\Delta t_i = \frac{p}{n} \forall i \in \mathscr{I}$. By using this function and for any time t, the controller will search in one direction for an interval given by Δt before switching to search along another direction. This example of the search function is considered in the rest of this work, unless otherwise stated.

IV. Stability and Convergence Analysis

This section is organized as follows: First, we show the sliding mode existence condition for any given search direction and we discuss the convergence of the one-dimensional search. After that, we shall investigate the minimum value of Δt that guarantees the convergence of the the multivariable extremum seeking controller to the minimum point in a finite time.

Theorem 1: For the extremum seeking control input (5) with the switching manifold (4) and search function (6) with the i_{th} directional search being active, the sliding mode existence condition, of keeping s(t) at constant, is satisfied outside a region characterized by

$$\left| \frac{\partial J}{\partial \theta_i} \right| < \frac{\rho}{k} \quad \forall i \in \mathscr{I} \tag{7}$$

and kept on manifold $s(t) = n\alpha$ with:

- $n = 2K$ when $\frac{\partial J}{\partial \theta_i} > 0$.
- $n = 2K+1$ when $\frac{\partial J}{\partial \theta_i} < 0$.

where $n, K \in \mathbb{N}$.

Proof: The detailed proof of this theorem needs much space. However, the sketch of the proof is provided as follows:

Define a Lyapunov Candidate Function (LCF) as:

$$V(t) = \frac{1}{2} s^2(t).$$

The derivative of the function $V(t)$ is

$$\begin{aligned} \dot{V}(t) &= s(t)\dot{s}(t) \\ &= s(t) \left[-\sum_{i=1}^{n} \frac{\partial J(\theta(t))}{\partial \theta_i} \quad \dot{\theta}_i + \rho \right] \end{aligned} \tag{8}$$

With only the i_{th} search direction being active, $\dot{V}(t)$ can rewritten as

$$\dot{V}(t) = s(t)\left[-\frac{\partial J(\theta(t))}{\partial \theta_i} e_i \; k \; \text{sgn}(\sin\left(\frac{\pi s(t)}{\alpha}\right)) + \rho\right]$$

Note that in the neighborhood of $s = n\alpha$:

- For $n = 2K$; $\text{sgn}(\sin(\frac{\pi s}{\alpha})) = \text{sgn}(s - \alpha n)$
- For $n = 2K+1$; $\text{sgn}(\sin(\frac{\pi s}{\alpha})) = -\text{sgn}(s - \alpha n)$

thus, under the condition (7), we have

$$s(t)\dot{s}(t) < 0$$

That is, $\dot{V}(t)$ is strictly negative irrespective of the sign of $\frac{\partial J}{\partial \theta_i}$ and a sliding manifold will be reached in finite time and kept on manifold $s(t) = n\alpha$ until the controller changes the search direction. ■

Proposition 1: For any $i \in \mathscr{I}$ let $t_{r_i} < \infty$ be a finite sliding mode's reaching time for the i_{th} direction and, condition (7) be satisfied, then for any $t \in (t_{r_i}, t_{i+1}]$ we have $J(\theta(t)) < J(\theta(t_{r_i}))$.

Proof: According to Theorem 1, sliding mode will be reached in finite time. To prove that $J(\theta(t))$ is decreasing in sliding mode, it is suffices to show show that $\theta_i(t)$ converges to $\theta_i^\star$ asymptotically. Without loss of generality, suppose that at the time $t = t_{r_i}$ the sliding surface s(t)=0 is reached. In sliding mode, the equivalent control v_{eq} can be obtained by solving the equation $\dot{s}(t) = 0$ for $v(t)$. That is,

$$\dot{s}(t) = \frac{\partial J(\theta(t))}{\partial \theta_i} v(t) + \rho = 0, \tag{9}$$

which implies,

$$v_{eq} = -\frac{\rho}{\frac{\partial J(\theta(t))}{\partial \theta_i}}. \tag{10}$$

Furthermore, note that under the Assumption 2.1 we have

$$(\theta_i(t) - \theta_i^\star)\frac{\partial J(\theta(t))}{\partial \theta_i} > 0, \; \forall \theta_i(t) \neq \theta_i^\star. \tag{11}$$

Let $\hat{\theta}_i(t) = \theta_i(t) - \theta_i^\star$, the time derivative of the function $\hat{\theta}_i(t)$ is

$$\dot{\hat{\theta}}_i(t) = v_{eq}(t). \tag{12}$$

From (10) and (11), it follows that

$$\hat{\theta}_i(t)\dot{\hat{\theta}}_i(t) < 0. \tag{13}$$

Using this result and the fact that by the search direction (6), we have

$$\theta_j(t_r) = \theta_j(t), \; \forall j \neq i$$

therefore,

$$J(\theta_1(t), ..., \theta_n(t)) < J(\theta_1(t_{r_1}), ..., \theta_n(t_{r_1})).$$

That is, J is decreasing in the i_{th} direction towards a neighborhood of the minimum, which is characterized by the region (7) while the i_{th} search is enabled. ■

Note that, after entering the region (7), it is possible that either the system output stays inside that region or goes through it. In the latter case, another sliding mode will happen and the system will enter the region again on the sliding mode.

The sliding mode's reaching time will be used in the analysis. Therefore, we devote our attention in the next paragraph to find an expression for the reaching time.

Without loss of generality, let $t_0 = 0$. By Theorem 1, the nearest switching surface will be reached in a finite time. Suppose that at $t = t_{r_1}$, the surface $s(t_{r_1}) = n\alpha$ is reached. Then, for any $s(0)$ we have

$$|s(0) - s(t_{r1})| = |s(0) - n\alpha| \leq \alpha. \tag{14}$$

Therefore, the reaching time for the first direction can be found as follows:

$$\begin{aligned} |t_{r1}| &= \frac{|s(0) - s(t_{r_1})|}{|\dot{s}(t)|} \\ &= \frac{|s(0) - n\alpha|}{|\frac{dJ}{d\theta_1} k \; \text{sgn}(\sin\left(\frac{\pi s(t)}{\alpha}\right)) + \rho|} \\ &\leq \frac{\alpha}{\rho}. \end{aligned} \tag{15}$$

Now, let the value of Δt_i satisfy:

$$\Delta t_i > \frac{\alpha}{\rho} \tag{16}$$

and consider the initial time for the overall search $t_0 = t_1 = 0$, with $\sigma(t) = [1 \; 0 \; ... \; 0]^T$ for all $t \in [t_1, t_2]$. The sliding mode will then be enabled with $t_{r_1} < t_2$ for a period given by

$$\Delta\hat{t}_1 = t_2 - t_{r_1}$$

in the 1st direction. By Proposition 1 we have

$$J(\theta_1(t_2), ..., \theta_n(t_2)) < J(\theta_1(0), ..., \theta_n(0)).$$

That is, $J(\theta(t))$ will track the monotonically decreasing function towards the minimum in the 1st direction before switching to another direction. The same is true for other directions in which we can conclude that (12) is a sufficient condition for the controller to converge towards the vicinity of the minimum for all directions. With this, we conclude our analysis with the following theorem:

Theorem 2: For the Multi-input Single-output static map (1) with control law (5), if the switching function is designed such that $\Delta t_i > \frac{\alpha}{\rho}$, then $J(\theta(t))$ will stably converge towards a vicinity of the minimum characterized by (7).

V. Multivariable ES for System with Dynamic

The extension of the developed approach in previous sections from static maps to systems with dynamics will be of a great interest. Let's consider a Multi-input Single-output dynamical system:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= z(t) \end{aligned} \tag{17}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}$ is the output and $z(t)$ is a cost function described by the equation:

$$z(t) = J(x_1(t), x_2(t), ..., x_n(t)) \tag{18}$$

where each function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $J : \mathbb{R}^n \to \mathbb{R}$ is smooth, continuous, and differentiable with respect to its arguments. Then, for the extension of the developed approach from static maps to systems with dynamics, the following assumptions about system (17) are required:

Assumption 5.1: There exists a smooth control law:

$$u(t) = \alpha(x(t), \theta(t))$$

to stabilize system (17), with $\theta(t)$ designed to satisfy

$$\dot{\theta}(t) = v(t)$$

and $v(t)$ being a variable structure control input given by the control input (5).
With this, the closed loop system

$$\dot{x} = f(x(t), \alpha(x(t), \theta(t)))$$

has equilibrium, which is a function of $\theta(t)$.

Assumption 5.2: There exists a smooth function $l : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(x(t), \alpha(x, \theta(t))) = 0,$$

if, and only if,

$$x(t) = l(\theta(t)).$$

Assumption 5.3: There exists $\theta^*(t) \in \mathbb{R}^n$ such that:

$$\frac{\partial}{\partial x}(J \circ l)(\theta^*(t)) = 0$$

and

$$\frac{\partial^2}{\partial x(t)}(J \circ l)(\theta^*(t)) > 0.$$

This assumption ensures that the function $z(t)$ has a minimum at $\theta(t) = \theta^\star(t)$.

Assumption 5.4: The dynamical system (17) is much faster than the controller dynamic. That is,

$$\left|\frac{d}{dt}x(t)\right| \gg \left|\frac{d}{dt}\theta(t)\right|.$$

This is to ensure a time scale separation between the stabilization control and the optimizer. It follows that the stability and the convergence analysis of the ES control with dynamics is analogous to the static map case. Fig. 3 illustrates the control scheme in the presence of dynamics.

VI. Illustrative Example: Wind Farm Power Capture Maximization

To illustrate the application of the proposed controller, we consider a problem of maximizing the total power production of a wind farm. The wind farm model will be briefly overviewed in the next section. Then, using the developed scheme, we will run simulations to validate the results and assess the performance of the scheme.

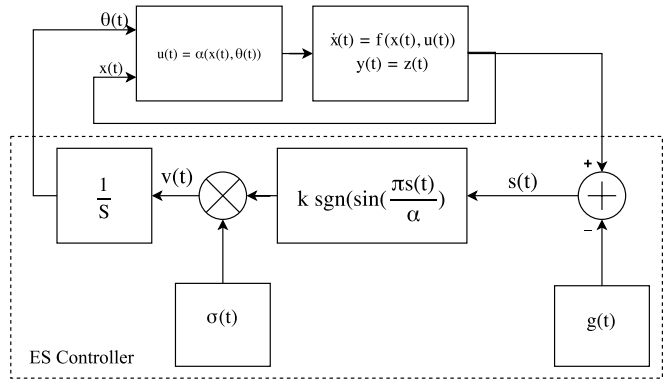

Fig. 3. ESC for system with Dynamic.

A. System Model and Problem Formulation

We consider a wind farm consisting of n wind turbines. Let θ_i be the control parameter of turbine i. Here, θ_i is the Axial Induction Factor (AIF) of turbine i, which takes values in $\left[0, \frac{1}{2}\right]$. The power produced by the wind turbine i is given by

$$J_i(\theta) = \frac{1}{2}\rho_{air} A_i C_p(\theta_i) V_i(\theta)^3, \tag{19}$$

where ρ_{air} is the density of the air, A_i is the area swept by blades of turbine i, $C_p(\theta_i)$ is the power efficiency coefficient, which is given by

$$C_p(\theta_i) = \theta_i(1-\theta_i)^2, \tag{20}$$

and $V_i(\theta)$ is the wind speed at turbine i which is given by

$$V_i(\theta) = V_\infty\left(1 - \sqrt{\sum_{j<i}(\theta_j C[j,i])^2}\right), \tag{21}$$

where V_∞ is the free stream wind speed and $C \in R^{n\times n}$ is a matrix depending on the farm layout, which is based on the Park model [16]. Note that in a wind farm, it is difficult to accurately model the coupling between the wind turbines due to the interaction between the wind-turbine wake aerodynamics and the dynamics of the turbines. Therefore, it is assumed that we can only measure the total power production, which is given by

$$P_{Total}(\theta_1, \theta_2, ..., \theta_n) = \sum_{i=1}^{n} J_i(\theta_1, \theta_2, ..., \theta_n) \tag{22}$$

and the problem is to maximize the wind farm total produced power by manipulating the AIF, without knowing the explicit forms of the function P_{Total}.

B. Simulation Results

We consider three turbines, $n = 3$, as shown in Fig. 4, with identical diameter, $D = 77m$, located at $\{(0,0),(0,5D),(0,10D)\}$ and a fixed air density $\rho_{air} = 1.225kg/m^3$, with free stream wind speed equaling $8m/s$. Simulations are carried out using MATLAB. ES parameters are set as $\rho = 500$, $\alpha = 0.5$, $\Delta t = 0.1s$ and $k = 0.5$. Note that the optimal solution for individual turbines can be obtained from equations (19) and (20). That is, we have

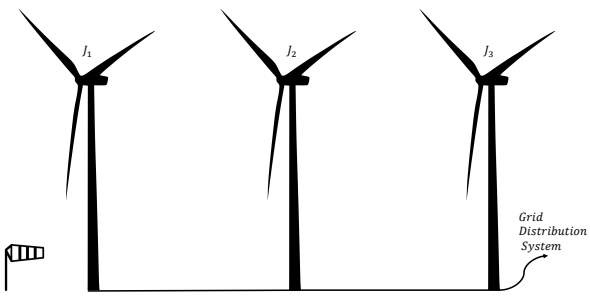

Fig. 4. Illustration of the Wind Farm layout considered for simulation.

$\frac{d}{d\theta}C_p = 0 \iff \theta = \frac{1}{3}$. Therefore, it is natural to set all $\theta_i = \frac{1}{3}$. On the other hand, if we consider the optimal solution for the total power generated by all three turbines, we have:

$$\begin{aligned} P_{Total} &= J_1 + J_2 + J_3 \\ &= \frac{1}{2}\rho A(C_p(\theta_1)V_\infty^3 + C_p(\theta_2)V_2(\theta_1)^3 \\ &+ C_p(\theta_3)V_3(\theta_1,\theta_2)^3). \end{aligned} \tag{23}$$

Optimizing the above equation using a numerical optimizer yields

$$\theta^* = (0.232, 0.208, 0.333).$$

The total power at $\theta_i = \frac{1}{3}$ is considered as a baseline to assess the benefit of using the proposed ES scheme. Fig. 5 shows that sliding mode is reached in a finite time after each switch. Fig. 6 demonstrates that starting initially with optimal setting for individual turbine $\theta(0) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, the proposed control steers the farm power output to a vicinity of the maximum power generation. Next, to examine the effect of the switching time Δt on the convergence of the total power, we run the simulation for different values of Δt. According to Theorem 2 above, it is sufficient to consider a switching time greater than $\frac{\alpha}{\rho} = 1 \times 10^{-3}$ sec. Fig. 7 depicts the simulation results for different values of Δt. It is evident that condition (11) is sufficient to guarantee convergence of the total power to a vicinity of the maximum.

VII. Conclusion

We proposed a Sliding Mode ES control scheme for multivariable optimization. The strategy of reducing the problem to a sequence of single variable ES has been considered to solve this class of problems. Stability and convergence analysis are discussed. The controller was successfully implemented to solve the ES problem for wind farm optimization. Simulations show that this method is independent from accurate turbine power modeling and wind measurement. Future research will include the development of ES controllers for constrained optimization and the adaptation of different search signals for complex applications.

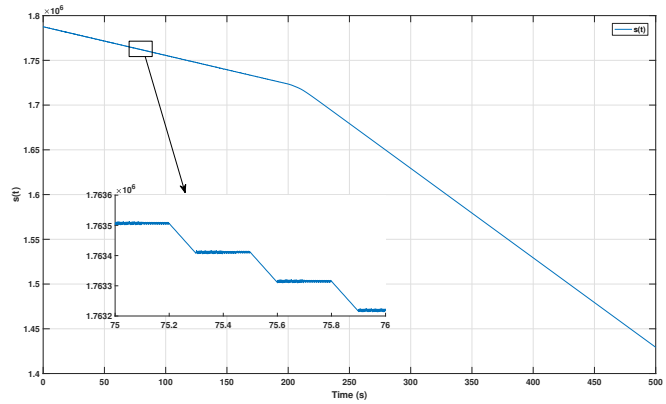

Fig. 5. Switching surfaces with sliding mode ESC.

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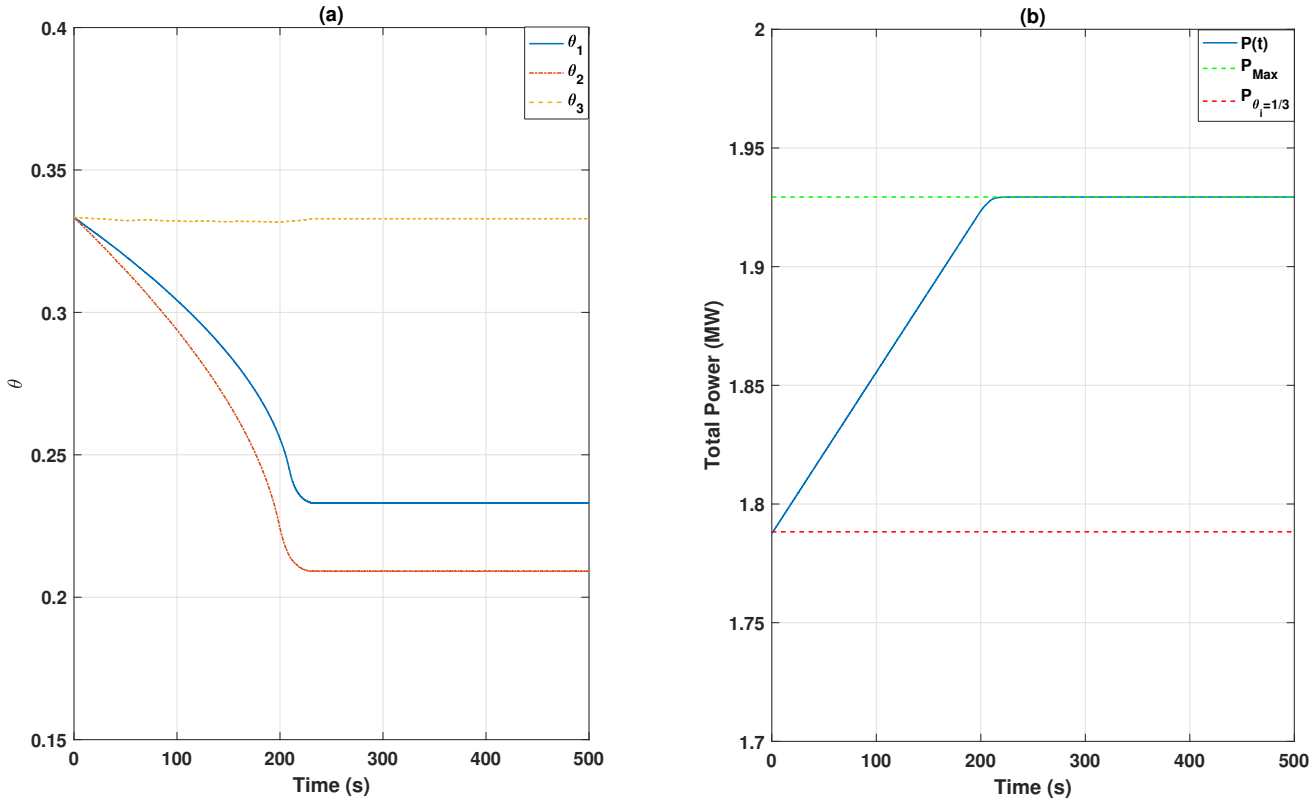

Fig. 6. Simulation results on Power maximization of three turbines.(a) Convergence of the axial induction factors θ_i. (b) The Evolution of the total power produced by the farm.

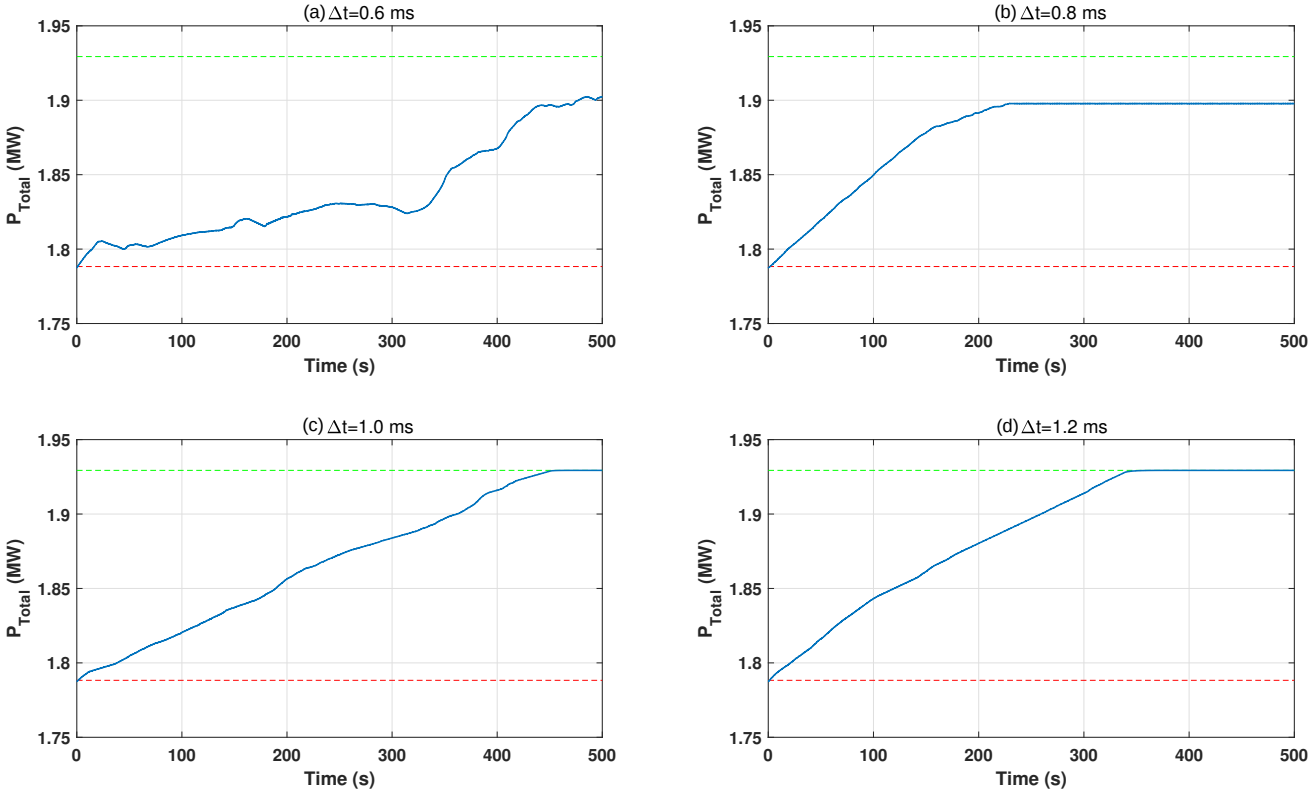

Fig. 7. Simulation results on Power maximization of three turbines with different switching time.