CSE 2321: Homework 13 Solutions

1. (a) Two vertices \( u, v \) of a graph are \textit{adjacent} if there exists an edge \((u, v)\).

\begin{center}
\begin{tikzpicture}
    \node (u) at (0,0) {$u$};
    \node (v) at (1,0) {$v$};
    \draw (u) -- (v);
\end{tikzpicture}
\end{center}

(b) A \textit{self-loop} is an edge \((u, u)\).

\begin{center}
\begin{tikzpicture}
    \node (u) at (0,0) {$u$};
    \draw (u) to [loop above] (u);
\end{tikzpicture}
\end{center}

(c) A \textit{multi-graph} can have multiple edges between the same two vertices and self-loops.

\begin{center}
\begin{tikzpicture}
    \node (u) at (0,0) {$u$};
    \node (v) at (1,0) {$v$};
    \node (w) at (2,0) {$w$};
    \node (x) at (3,0) {$x$};
    \node (y) at (4,0) {$y$};
    \node (z) at (5,0) {$z$};
    \draw (u) -- (v);
    \draw (v) -- (w);
    \draw (u) -- (x);
    \draw (y) -- (z);
\end{tikzpicture}
\end{center}

(d) If \((u, v)\) is an edge in a graph, \((u, v)\) is \textit{incident} to vertices \( u \) and \( v \).

\begin{center}
\begin{tikzpicture}
    \node (u) at (0,0) {$u$};
    \node (v) at (1,0) {$v$};
    \draw (u) -- (v);
\end{tikzpicture}
\end{center}

(e) An \textit{acyclic} graph has no simple cycles.

\begin{center}
\begin{tikzpicture}
    \node (u) at (0,0) {$u$};
    \node (v) at (1,0) {$v$};
    \node (w) at (2,0) {$w$};
    \node (x) at (3,0) {$x$};
    \node (y) at (4,0) {$y$};
    \node (z) at (5,0) {$z$};
    \draw (u) -- (v);
    \draw (v) -- (w);
    \draw (w) -- (x);
    \draw (x) -- (y);
    \draw (y) -- (z);
\end{tikzpicture}
\end{center}

(f) A \textit{connected} graph is a graph in which every vertex is reachable from all other vertices.

\begin{center}
\begin{tikzpicture}
    \node (u) at (0,0) {$u$};
    \node (v) at (1,0) {$v$};
    \node (w) at (2,0) {$w$};
    \node (x) at (3,0) {$x$};
    \node (y) at (4,0) {$y$};
    \node (z) at (5,0) {$z$};
    \draw (u) -- (v);
    \draw (v) -- (w);
    \draw (w) -- (x);
    \draw (x) -- (y);
    \draw (y) -- (z);
\end{tikzpicture}
\end{center}
(g) An *Eulerian cycle* is a cycle through the graph which visits every edge once. 
   e.g. \((w, x, y, z, w)\)

(h) An *Eulerian path* is a path through the graph which visits every edge once. 
   e.g. \((w, x, y, z, w, y)\)

(i) A *Hamiltonian cycle* is a cycle through the graph which visits every vertex once 
   (with the exception of the starting vertex). 
   e.g. \((w, x, y, z, w)\)
(j) A *Hamiltonian path* is a path through the graph which visits every vertex once.  
e.g. \((w, x, y, z)\)

\[ \text{Diagram showing a Hamiltonian path} \]

(k) A *bridge* is an edge that is not contained in any cycle.

\[ \text{Diagram showing a bridge} \]

(l) A *complete graph* is a graph in which every pair of vertices is adjacent.  
Example for 7 vertices:
(m) A bipartite graph is a graph in which the vertices can be partitioned into two sets $V_1$ and $V_2$ such that for every edge $(u, v)$, $u$ is in one of the sets and $v$ is in the other.

Example for 7 vertices:

2. By Euler’s theorem, the sum of degrees of each vertex is double the number of edges in a graph. Therefore, a graph with 242 degrees total has 121 edges.

3. (a) $(0, 0, 1, 1, 1, 1, 4, 4)$ is not a degree sequence. There are 4 degree vertices. The minimum number of vertices (outside of themselves) that these two must be connected to is 6. Therefore, all 8 vertices are connected. However, the degree sequence has 2 vertices of degree zero, contradicting this.

(b) $(0, 2, 2, 2, 2, 4, 4, 4)$ is a degree sequence. Consider the following graph: