A Quantitative Approach to Quadratic Embedding of Graphs

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1. Notations and Motivations
1.1 Notations

Graph \( G = (V, E) \)
- finite
- unoriented
- simple, no loops
- connected

Graph distance \( d(x, y) \)
the length of a shortest walk from \( x \) to \( y \)

Distance matrix
\[
D = \begin{bmatrix} d(x, y) \end{bmatrix}_{x,y \in V}
\]

Q-matrix
\[
Q = Q_q = \begin{bmatrix} q^{d(x, y)} \end{bmatrix}_{x,y \in V}
\]

where \( q \) is a parameter, in fact,
\(-1 \leq q \leq 1 \) is an interesting case
### Definition

A real matrix $D = \left[ d_{x,y} \right]_{x,y \in V}$ with

$$d_{x,y} = d_{y,x} \ , \ d_{x,x} = 0$$

is called a **Euclidean distance matrix** if there exist points $\{ P_x \ ; \ x \in V \}$ in a Euclidean space such that

$$\| P_x - P_y \| ^2 = d_{x,y} \ , \ x, y \in V.$$  

The map: $x \rightarrow P_x$ is a quadratic embedding.

---

**Schoenberg (1935-37)**

(i) $D$ is a Euclidean distance matrix.

(ii) $D$ is conditionally negative definite (CND).

(iii) for any $o \in V$, the matrix $A = \left[ A_{xy} \right]_{x,y \in V \setminus \{o\}}$

$$A_{xy} = \frac{1}{2} \left( D_{ox} + D_{oy} - D_{xy} \right)$$

is positive definite, i.e.,

$$\langle f, Af \rangle \geq 0 \ \text{for all} \ f \in C \left( V \setminus \{o\} \right)$$
1.2 Motivation: Quadratic Embedding

Definition. For a graph $G = (V, E)$ a map $\psi : V \rightarrow \mathbb{R}^N$ is called a quadratic embedding if

$$\| \psi(x) - \psi(y) \|^2 = d(x,y), \quad x, y \in V.$$

In that case $G$ is called of QE class.

First examples

Bozejko's examples of non-QE graphs

Checked by elementary geometry
1.3 Quantitative Approach

For a (finite connected) graph $G = (V, E)$,

$G$ is of QE class,

i.e., admits a quadratic embedding $\psi : V \to \mathbb{R}^N$

such that $\|\psi(x) - \psi(y)\|^2 = d(x, y)$, $x, y \in V$.

$\iff D$ is a Euclidean distance matrix

$\iff D$ is CND (conditionally negative definite)

Definition [Obata-Zakiyyah (2018)]

\[
QEC(G) = \max \left\{ \langle f, Df \rangle : \begin{array}{c}
\langle f, f \rangle = 1, \\
\langle 1, f \rangle = 0
\end{array} \right\}
\]

quadratic embedding constant (QEC)

$\iff QEC(G) \leq 0$

QEC is a new invariant of (finite) connected graphs.

Project:
Classify graphs of QE/non-QE class by QE constants.
Remark (Schoenberg 1937)

If a graph $G = (V, E)$ admits a quadratic embedding:

$\psi : V \rightarrow \mathbb{R}^N$, i.e.,

$\| \psi(x) - \psi(y) \|^2 = d(x, y), \quad x, y \in V,$

then for any $2 \leq p < \infty$ there exists a map

$\phi : V \rightarrow \mathbb{R}^N$ such that

$\| \phi(x) - \phi(y) \|^p = d(x, y), \quad x, y \in V.$
1.4 Q-matrices

Bozejko Heidelberg Lecture (1987)

The Q-matrix $Q_q = \left[ q^d(x, y) \right]_{x,y \in V}$ is positive definite for $0 \leq q \leq 1$ if and only if $G$ admits a quadratic embedding.

Question:
For which $q$ the Q-matrix $Q_q$ is positive definite?

An application of Q-matrices

Spectral analysis of growing graphs [Hora-Obata book (2007)]

$\mathcal{A}(G_v)$: adjacency algebra of growing graph $G_v$

$\langle a \rangle_q = \langle Q_q e_o, a e_o \rangle, \quad a \in \mathcal{A}(G_v)$

If $\langle \cdot \rangle_q$ is a state, the spectral distribution of the adjacency matrix $\mu = \mu_{v,q}$ is defined by

$$
\langle A^m \rangle_q = \int x^m \mu(dx), \quad m = 1, 2, \ldots
$$

Study the asymptotics of $\mu_{v,q}$ as $v \to \infty$, $q \to 0$

[A variant of central limit theorem (CLT)]
2. Calculating QEC


2.1 A Basic Formula

For a finite connected graph \( G = (V, E) \) we have

\[
\text{QEC}(G) = \max \left\{ \langle f, Df \rangle : \langle f, f \rangle = 1, \langle 1, f \rangle = 0 \right\}
\]

Introduce Lagrange multipliers:

\[
\mathcal{F}(f, \lambda, \mu) = \langle f, Df \rangle - \lambda \langle f, f \rangle - 1 - \mu \langle 1, f \rangle
\]

\[
S = S(D) = \{ (f, \lambda, \mu) : \frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial \lambda} = \frac{\partial F}{\partial \mu} = 0 \}
\]

where \( f = [x_1, x_2, \ldots, x_n]^T \), \( n = |V| \)

**Theorem [Obata-Zakiyyah (2018)]**

\[
\text{QEC}(G) = \max \{ \lambda ; (f, \lambda, \mu) \in S(D) \}
\]

**Proof:** QEC(G) is attained by some \((f, \lambda, \mu) \in S(D)\).

On the other hand, if \((f, \lambda, \mu) \in S(D)\),

\[
0 = \frac{\partial F}{\partial x_i} = 2 \langle e_i, Df \rangle - 2\lambda \langle e_i, f \rangle - \mu \langle 1, e_i \rangle
\]

\[
= \langle e_i, 2(D-\lambda)f - \mu 1 \rangle
\]

Hence \( Df = \lambda f + \frac{1}{2} \mu f \) and \( \langle f, Df \rangle = \lambda \).
### 2.2. Complete Graphs etc.

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<td><strong>Complete graphs</strong></td>
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<tr>
<td>$K_n$</td>
<td>$n \geq 2$</td>
</tr>
<tr>
<td><strong>Complete bipartite graphs</strong></td>
<td></td>
</tr>
</tbody>
</table>
| $K_{m,n}$ | \[
\frac{2}{m+n} \left\{ (m-1)(n-1) - 1 \right\} \]
| QEC $\leq 0$ | (i) $m=1$ or $n=1$
| | (ii) $m=n=2$
| **Complete tripartite graphs** | |
| $K_{1,1,n}$ | $n \geq 1$ |
| | $\frac{n-4}{n+2}$ |
| | QEC $\leq 0$ |
| | $\iff n \leq 4$ |
### Graphs

<table>
<thead>
<tr>
<th>Graph Description</th>
<th>QEC Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_n \setminus {e}$</td>
<td>$\frac{2}{n}$</td>
</tr>
<tr>
<td>$K_n \setminus {e_1, \ldots, e_r}$ disjoint</td>
<td>0</td>
</tr>
<tr>
<td>$K_n \setminus P_3$</td>
<td>$\frac{n-10}{n + 2 + \sqrt{2(n-1)(n-2)}}$</td>
</tr>
<tr>
<td>$K_n \setminus K_r$</td>
<td>$r - 2 - \frac{r(r-1)}{n}$</td>
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</tbody>
</table>

$K_n \setminus \{\text{edges}\}$
2.3 Relation to Distance Spectra

The distance spectrum of a graph $G = (V, E)$ is the list of eigenvalues of $D$:

$$\lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n, \quad n = |V|$$

Peron-Frobenius theory

**Theorem**

$$\lambda_1 > \text{QEC}(G) \geq \lambda_2$$

**Proof:** By min-max theorem says:

$$\lambda_k = \min_{\text{codim } U = k-1} \max_{f \in U \setminus \{0\}} \frac{\langle f, Df \rangle}{\langle f, f \rangle}$$

On the other hand,

$$\text{QEC}(G) = \max_{f \in C(V) \setminus \{0\}} \frac{\langle f, Df \rangle}{\langle f, f \rangle}$$

$$\langle 1, f \rangle = 0$$

$$\rightarrow U : \text{codim } U = 1$$

Hence we have $\lambda_2 \leq \text{QEC}(G)$.

$\text{QEC}(G) < \lambda_1$ is by Peron-Frobenius theory.
When does the equality hold?

\[ \lambda_1 > QEC(G) \geq \lambda_2 \geq \cdots \geq \lambda_n \]

A graph \( G = (V, E) \) is called transmission regular if \( D \) has constant row sums, i.e.,

\[ \delta = \sum_{y \in V} d(x,y) \text{ is independent of } x. \]

In that case, \( \lambda_1 = \delta \) and \( D1 = \delta 1 \). Then we have

**Theorem** If \( G \) is transmission regular, we have

\[ \lambda_2 (G) = QEC(G). \]

In particular, for a distance-regular graph \( G \) we have \( \lambda_2 (G) = QEC(G) \).

The distance-regular graphs with \( \lambda_2 (G) \leq 0 \) are classified.

J. H. Koolen and S.V. Shpectorov:
Distance-regular graphs the distance matrix of which has only one positive eigenvalues, Europ. J. Comb. 15 (1994), 269-275.
Cycles $C_n$

- Obviously, $C_n$ is transmission regular.
- Eigenvalues of $D$ are known (easy to compute).

(1) $C_{2n+1}: \text{ev}(D) = \{ \lambda_0, \lambda_1(2), \ldots, \lambda_n(2) \}$

$$
\lambda_0 = n(n+1), \quad \lambda_k = -\left(4\cos^2 \frac{k\pi}{2n+1}\right)^{-1}, \quad 1 \leq k \leq n.
$$

(2) $C_{2n}: \text{ev}(D) = \{ \lambda_0, \lambda_1, \ldots, \lambda_{2n-1} \}$

$$
\lambda_0 = n^2, \quad \lambda_2 = \ldots = \lambda_{2n-2} = 0,
\lambda_k = -\left(\sin^2 \frac{k\pi}{2n}\right)^{-1}, \quad k = 1, 3, \ldots, 2n-1.
$$

<table>
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<th>QEC</th>
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<tr>
<td>cycles on odd vertices</td>
<td>$C_{2n+1}$ $n \geq 1$</td>
</tr>
<tr>
<td>$-(4\cos^2 \frac{\pi}{2n+1})^{-1}$</td>
<td></td>
</tr>
<tr>
<td>cycles on even vertices</td>
<td>$C_{2n}$ $n \geq 2$</td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
2.4 QE Constants of Paths

\[ P_n \]

\[ \text{QEC}(P_n) = \max \left\{ \langle f, Df \rangle : \langle f, f \rangle = 1, \langle 1, f \rangle = 0 \right\} \]

**Theorem [Mlotkowski (2022)]**

\[ \text{QEC}(P_n) = \frac{-1}{1 + \cos \frac{\pi}{n}} \quad (n \geq 2) \]

- **Comparison with \( \lambda_2(P_n) \)**
  
  [Ruzieh–Powers (1990)] \( \lambda_2(P_n) \) is known.
  
  explicitly for even \( n \) and
  
  an implicit formula for odd \( n \).

\[ \lambda_2(P_n) = \text{QEC}(P_n) \quad \text{for even } n \]

\[ \lambda_2(P_n) < \text{QEC}(P_n) \quad \text{for odd } n \]

**Open Problem**

Characterize the graphs \( G \) with \( \lambda_2(G) = \text{QEC}(G) \).
3. Graph Operations for QE Graphs


3.1 Cartesian (direct) products

The Cartesian product of $G_1$ and $G_2$, denoted by $G_1 \times G_2$ is a graph on $V_1 \times V_2$ with

$$(x_1, y_1) \sim (x_2, y_2) \iff \begin{cases} x_1 = x_2 \text{ or } \ y_1 = y_2 \\ y_1 \sim y_2 \end{cases}$$

**Theorem [Obata-Zakiyyah (2018)]**

If both $G_1$ and $G_2$ are of QE class, so is the Cartesian product $G_1 \times G_2$. Moreover,

$$\text{QEC}(G_1 \times G_2) = 0,$$

whenever $|V_1| \geq 2$ and $|V_2| \geq 2$

**Proof by construction of quadratic embedding.**

$\alpha_i : G_i \rightarrow \mathbb{R}^{d_i}$

Then $\beta : G_1 \times G_2 \rightarrow \mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2}$ defined by

$$\beta(x, y) = \alpha_1(x) \oplus \alpha_2(y), \quad x \in V_1, \quad y \in V_2$$

is a quadratic embedding. The second half is directly by existence of $f$ with $\langle f, Df \rangle = 0$. 

3.2 Star Products (vertex concatenation)

For two rooted graphs

\[ G_1 = (V_1, E_1, o_1), \quad G_2 = (V_2, E_2, o_2) \]

the star product \( G_1 \star G_2 \) is defined as follows:

\[ p: G_1 \star G_2 \rightarrow R^d \]

Theorem [Obata-Zakiyyah (2018)]

If both \( G_1 \) and \( G_2 \) are of QE class, so is the star product \( G_1 \star G_2 \).

Proof by construction of quadratic embedding.

\[ \alpha_i: G_i \rightarrow R^{d_i} \text{ with } \alpha_i(o_i) = 0 \]

Then \( \beta: G_1 \star G_2 \rightarrow R^{d_1} \oplus R^{d_2} \) defined by

\[ \beta(x) = \begin{cases} \alpha_1(x) \oplus 0, & x \in V_1 \\ 0 \oplus \alpha_2(x), & x \in V_2 \end{cases} \]

is a quadratic embedding.
3.3 An Estimate for $QEC(G_1 \star G_2)$

$G_1 = (V_1, E_1)$, $V_1 = m$, $QEC(G_1) = Q_1$

$G_2 = (V_2, E_2)$, $V_2 = n$, $QEC(G_2) = Q_2$

**Theorem [Mlotkowski-Obata (2018)]**

$$\max \{ Q_1, Q_2 \} \leq QEC(G_1 \star G_2) \leq \frac{mn}{2(m+n-1)} \left\{ Q_1 + Q_2 + \sqrt{(Q_1 + Q_2)^2 - \frac{4(m+n-1)}{mn} q_1 q_2} \right\}$$

**Corollary**

1. If $Q_1 < 0$ and $Q_2 < 0$, then $QEC(G_1 \star G_2) < 0$.
2. If $Q_1 = 0$ and $Q_2 \leq 0$, then $QEC(G_1 \star G_2) = 0$.

- A general formula of $QEC(G_1 \star G_2)$ in terms of $Q_1$ and $Q_2$ is hopeless.

**Problem**

Find a formula of $QEC(G_1 \star G_2)$ for a family of graphs $G_1$ and $G_2$. For example, trees.
3.3 An Estimate for $\text{QEC}(G_1 \star G_2)$

For a complete graph $K_n$ we have $\text{QEC}(K_n) = -1 < 0$.

Hence, if $\text{QEC}(G) < 0$, we have

$$\text{QEC}(G) \leq \text{QEC}(G \star K_n) < 0$$

**Problem**

When does the strict inequality or the equality:

$$\text{QEC}(G) < \text{QEC}(G \star K_n)$$

$$\text{QEC}(G) = \text{QEC}(G \star K_n)$$

hold?
Theorem [Baskoro-Obata (2021)]

Assume that $QEC(G) < 0$.

If $\exists f$ such that

\[ QEC(G) = \langle f, Df \rangle , \langle 1, f \rangle = 0, \]
\[ \langle f, f \rangle = 1 \text{ and } f(o) \neq 0, \]

then, we have $QEC(G) < QEC(G \star K_n)$.

Example of $QEC(G) = QEC(G \star K_2)$

We see that if $QEC(G) = \langle f, Df \rangle$, then $f(o) = 0$.
Hence in the above Theorem $f(o) \neq 0$ is necessary.
3.4 Graph Joins

The graph join of $G_1$ and $G_2$, denoted by $G_1 + G_2$, is a graph on $V_1 \cup V_2$ with

$$E = E_1 \cup E_2 \cup \{ \{ x, y \}; x \in V_1, y \in V_2 \}$$

![Graph join of $G_1$ and $G_2$](image)

Distance matrix in terms of adjacency matrices:

$$D = \begin{bmatrix}
2J - 2I - A_1 & J \\
J & 2J - 2I - A_2
\end{bmatrix}$$

Theorem [Lou-Obata-Huang (2019)]

Assume that $G_i$ is $r_i$-regular on $n_i$ vertices.

$$\lambda_{\min}(G_i) = \min \text{ ev } (A_i)$$

$$\text{QEC}(G_1 + G_2) = -2 + \max \left\{ -\lambda_{\min}(G_1), -\lambda_{\min}(G_2), \frac{2n_1n_2 - r_1n_2 - r_2n_1}{n_1 + n_2} \right\}$$
### Graph Joins

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<th>Wheels</th>
<th>Complete Split Graphs</th>
<th>Friendship Graphs</th>
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<tr>
<td>$W_n = C_n + K_1$ for $n \geq 3$</td>
<td>$\overline{K}_m + K_n$ for $m \geq 1$ and $n \geq 1$</td>
<td>$F_n = nK_2 + K_1$ for $n \geq 1$</td>
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### QEC

- $n$: even
  - $0$
- $n$: odd
  - $-4\sin^2\frac{\pi}{2n}$

- $m \geq 1$ and $n \geq 1$
- $m = 2$ or $n = 1$
- $m = 3$, $n = 2$ or $3$
- $m = 4$, $n = 2$

\[
\frac{mn - m - 2n}{m + n} \leq QEC 
\]

- $QEC \leq 0$
- $m = 2$ or $n = 1$
- $m = 3$, $n = 2$ or $3$
- $m = 4$, $n = 2$

\[
\frac{-3}{2n+1} 
\]
complete tri-partite graphs

\[ K_{\ell,m,n} = \overline{K}_{\ell} + K_{m,n} \]

QEC\((K_{\ell,m,n})\)

\[ = -2 + \frac{\ell m + mn + nl + \sqrt{\ell^2 m^2 + m^2 n^2 + n^2 l^2 - \ell mn (\ell + m + n)}}{\ell + m + n} \]

problem

\[ P_n + K_1 \]

?
Theorem [Baskoro-Obata (2021)]

1. \( \text{QEC}(G) = -1 \iff G = K_n, n \geq 2 \)
2. No graph with \(-1 < \text{QEC}(G) < -\frac{2}{3}\)
3. \( \text{QEC}(G) = -\frac{2}{3} \iff G = P_3 = K_2 \ast K_2 \)
4. \(-\frac{2}{3} < \text{QEC}(G) < -2 + \sqrt{2} \)
   \[ \iff G = K_n \ast K_2 \text{ with } n \geq 3 \text{ or } = K_3 \ast K_3 \]

Moreover,

\[
\text{QEC}(K_n \ast K_2) = \frac{-2}{2 + \sqrt{2}(1 - \frac{1}{n})}
\]

\[
\text{QEC}(K_3 \ast K_3) = -\frac{3}{5}
\]

Open problem

Characterize the graphs \( G \) with \( \text{QEC}(G) < -\frac{1}{2} \).
4. Exploring Non-QE Graphs


4.1 QE Constants of Small Graphs

All Small Connected Graphs

When working on a problem involving graphs recently, I needed a comprehensive visual list of all the (non-isomorphic) connected graphs on small numbers of nodes, and was surprised to find a dearth of such images on the web. So I made some. Below are images of the connected graphs from 2 to 7 nodes. In the upper left of each box is a cardinal number (starting from 1) which can be used as a unique identifier for each graph having $n$ nodes. The number in the upper right of each box is the number of edges and the string of numbers at the bottom of the box gives the degree of each node from node 0 to node $n-1$, in order.

These pictures were created by starting from Brendan McKay’s very useful data sets, processing these with showg (found here) into a more convenient format, sorting them within each $n$ by increasing number of edges, and then converting into images.
Graphs on $n \leq 5$ vertices

N. Obata and A. Y. Zakiyyah: Distance matrices and quadratic embedding of graphs, Electronic J. Graph Th. Appl. 6 (2018), 37-60.

Note: (1) $\lambda_1$ is the maximal real root of $5\lambda^3 - 2\lambda^2 - 4\lambda + 2 = 0.$
All graphs on $n \leq 4$ vertices are of QE class.

Among 21 connected graphs on 5 vertices two graphs are of non-QE class.

QEC = \(\frac{2}{5}\)

QEC = \(\frac{4}{11 + \sqrt{161}}\)

Note: (2) $\lambda_2^*$ is the maximal real root of $5\lambda^3 + 3\lambda^2 - 5\lambda + 1 = 0$. 
4.2 Primary Non-QE Graphs

**Lemma**

\[ G \hookrightarrow \tilde{G} : \text{isometrically embedded subgraph} \]

Then we have \( \text{QEC}(G) \leq \text{QEC}(\tilde{G}) \).

Therefore, if \( G \) is non-QE, so is \( \tilde{G} \).

Proof is immediate from

\[ \tilde{D} = \begin{bmatrix} D & * \\ * & * \end{bmatrix} \]

To classify non-QE graphs it is essential to find "minimal ones."

**Definition** A non-QE graph is called primary if it does not contain a non-QE graph as an isometrically embedded proper subgraph.

**Problem** Classify the primary non-QE graphs.
Complete bipartite graphs $K_{m,n}$ ($m \geq n \geq 1$)

(1) $K_{3,2}$ is of non-QE class (Bozejko's example).

(2) $\text{QEC}(K_{m,n}) = \frac{2\{(m-1)(n-1) - 1\}}{m+n}$ \hspace{1cm} (m \geq n \geq 1)

(3) $K_{m,n}$ is non-QE if and only if $m \geq 3$ and $n \geq 2$.

In fact, $K_{3,2} \hookrightarrow K_{m,n}$ isometrically.

(4) Hence only $K_{3,2}$ is primary non-QE.
4.3 Graphs on Six Vertices

![Graphs on Six Vertices](http://www.cadaeic.net/graphpics.htm)
Sieve for primary non-QE graphs

(1) The star product of QE graphs is of QE class.
There are 51 graphs of this type including 6 trees.

(2) The cartesian product of QE graphs is of QE class.
There are 2 graphs of this type.

(3) A graph containing G5-10 or G5-17 as an isometrically embedded subgraph is of non-QE class.

\[ G_{5-10} = K_{3,2} \quad \text{and} \quad G_{5-17} \]

There are 24 graphs of this type, which exhaust non-primary non-QE graphs.

(4) Special series of graphs with known QE constants:

- \( K_n, P_n, C_n \)
- \( K_{m_1, m_2, \ldots, m_k} \) (multipartite graphs)
- \( G_1 + G_2 \) (graph joins of regular graphs)
- \( K_n \land K_{m,1} \)
(6) Direct construction of quadratic embeddings from known ones of smaller graphs

(7) Seven graphs are left. Their QE constants are explicitly calculated.

\[ \frac{-4 + \sqrt{19}}{3} > 0 \quad \lambda^* < 0 \quad \frac{-5 + \sqrt{19}}{3} < 0 \quad \lambda^* > 0 \]

\[ \frac{-3 + \sqrt{5}}{2} < 0 \quad \lambda^* > 0 \quad \frac{-3 + \sqrt{5}}{2} < 0 \]

\(\lambda^*\) is determined as a root of cubic or quartic equations.
Theorem [Obata (2022)]

Among 112 graphs on six vertices there are 3 primary non-QE graphs (see figures), 24 non-primary non-QE graphs and 85 QE graphs.
4.4. More on primary non-QE Graphs

Multipartite graphs \( K_{m_1, m_2, \ldots, m_k} \)

An explicit formula for \( \text{QEC}(K_{m_1, m_2, \ldots, m_k}) \) is recently obtained [N. Obata: arXiv:2206.05848]

**Theorem**

Among the complete multipartite graphs there are four primary non-QE graphs:

\[
\begin{align*}
\text{QEC}(K_{3,2}) &= 2/5 \\
\text{QEC}(K_{4,1,1,1}) &= 2/7 \\
\text{QEC}(K_{5,1,1}) &= 1/7 \\
\text{QEC}(K_{3,1,1,1,1}) &= 1/7
\end{align*}
\]

**Remark**

Recently, Mlotkowski found an infinite series of primary non-QE graphs.
Summary

(1) We discussed QE/Non-QE graphs by means of quadratic embedding constant:

\[ \text{QEC}(G) = \max \left\{ \langle f, Df \rangle : \langle f, f \rangle = 1, \langle 1, f \rangle = 0 \right\} \]

(2) The value of QEC(G) is obtained explicitly for some special series of graphs.

(3) Graph operations preserving QE are studied.

(4) Classification along with QEC(\(P_n\)) - just started

\[
\begin{align*}
\text{QEC}(P_2) & \quad \text{QEC}(P_3) & \quad \text{QEC}(P_4) \quad \cdots \\
-1 & \quad -1/2 & \quad -2/3 & \quad -2 + \sqrt{2} & \quad \cdots \\
\end{align*}
\]

(5) All primary/non-primary non-QE graphs on six vertices are classified.

\[
\begin{align*}
-1/2 & \quad 0 & \quad G_6-30 & \quad G_6-84 & \quad G_5-17 & \quad G_6-60 & \quad G_5-10 \\
\text{primary non-QE graphs} & \quad \Delta & \quad \Delta & \quad \Delta & \quad \Delta & \quad \Delta & \quad \Delta \\
& \quad \text{distribution?} \\
\end{align*}
\]
Thank you very much for your patience!

Any comments and suggestions are welcome!