Math 1172	Name:
Midterm 1	OSU username (name.nn):
Autumn 2013	Lecturer:
	Recitation Instructor: Yanli Wang
Form A	Recitation Time:

Instructions

•) SHOW ALL WORK!!! Incorrect answers with work shown may receive partial credit, but unsubstantiated answers may receive NO credit.

•) Give EXACT answers unless asked to do otherwise.

•) You do not need to simplify numerical answers such as $\frac{5}{\sqrt{8}} - \frac{5}{\sqrt{12}}$.

•) Calculators are permitted EXCEPT those calculators that have computer algebra systems (CAS) or ability to communicate with others. Furthermore, all memory must be cleared and all apps must be removed. PDA's, laptops, and cell phones are prohibited. Do not have these devices out!

•) The exam duration is 55 minutes.

•) The exam consists of 5 problems starting on page 2 and ending on page 6. Make sure your exam is not missing any pages before you start. Page 7 contains formulas. Page 8 may be used for extra work space.

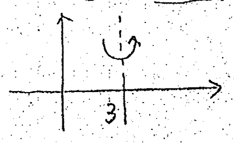
Problem Number	Maximum Point Value	Score
1	18	
2	22	
3	20	
4	20	
5	20	
Total	100	

Problem 1

[18 pts] True or False. Give a brief explanation or example to justify your answer.

True

a) [3 pts] Given a solid generated by revolving a region about the line $x = 3$, if we are using the shell method to compute its volume, then we would integrate with respect to x.



	dx	dy
	washer	shell
	shell	washer

~~True~~

False

b) [3 pts] Given a spring that obeys Hooke's Law, the work required to stretch the spring from equilibrium to 1 cm is the same as the work required to stretch the spring from 1 cm to 2 cm.

~~The same distance that is stretched.~~

$W_1 = \int_0^1 kx\,dx = \frac{1}{2}kx^2 \Big|_0^1 = \frac{1}{2}k$

$W_2 = \int_1^2 kx\,dx = \frac{1}{2}kx^2 \Big|_1^2 = \frac{1}{2}k(4-1) = \frac{3}{2}k$

$W_1 \neq W_2$

c) [3 pts] $\displaystyle\int \sin^6\theta \cos^3\theta \, d\theta = \int (u^6 - u^8)\, du$ where $\boxed{u = \sin\theta d\theta}$ × $u = \sin\theta$

$u = \sin\theta \quad du = \cos\theta\, d\theta$.

$\int \sin^6\theta\cos^3\theta\, d\theta = \int \sin^6\theta \cos^2\theta \cos\theta\, d\theta = \int \sin^6\theta(1-\sin^2\theta)\boxed{\cos\theta\, d\theta} = \int u^6(1-u^2)du$

$= \int u^6 - u^8\, du$.

False

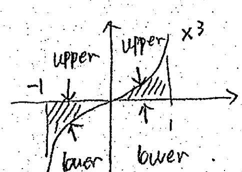
d) [3 pts] $\displaystyle\int \frac{1}{\cos x}\, dx = \ln|\cos x| + C$

$\int \frac{1}{\cos x}dx = \int \sec x\, dx = \ln|\sec x + \tan x| + C$.

☆ Or you can check the right hand side by taking derivative!

False

e) [3 pts] $\displaystyle\int_{-1}^{1} x^3\, dx$ represents the area of the region bounded by the graph of $y = x^3$ and the x-axis over the interval $[-1, 1]$.



$A = \int_{-1}^{0}(0 - x^3)dx + \int_0^1 (x^3 - 0)dx$

$= -\int_{-1}^{0} x^3\, dx + \int_0^1 x^3\, dx$.

upper and lower function change from $[-1, 0]$ to $[0, 1]$

True

f) [3 pts] $\displaystyle\int_0^{\pi/4} \sec x\, dx$ is the length of the curve $y = \ln(\sec x)$ on the interval $[0, \frac{\pi}{4}]$.

Arc length $L = \int_0^{\pi/4} \sqrt{1 + (y')^2}\, dx$

$\boxed{y' = \frac{1}{\sec x}\cdot(\sec x)' = \frac{1}{\sec x}\sec x \tan x = \tan x}$

$= \int_0^{\pi/4}\sqrt{1 + \tan^2 x}\, dx$

$= \int_0^{\pi/4} \sec x\, dx$ ↙ trig identity

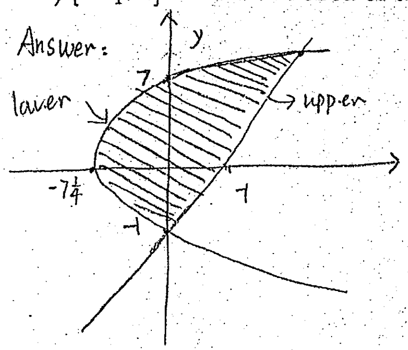
Problem 2

[22 pts] Areas and Volumes. Give the **exact values**. You do not need to simplify your final answer.

$y = x - 1$ $\quad (y-\frac{1}{2})^2 - 7 - \frac{1}{4}$

a) [10 pts] Find the area of the region bounded by the graphs $x = y+1$ and $x = y^2 - y - 7$.

Answer:



Find the area using the integral in y direction is easier.

Solve for the intercepts.

$y+1 = y^2 - y - 7$

$y^2 - 2y - 8 = 0$

$(y-6)(y+2) = 0 \qquad y = -2 \quad y = 6$

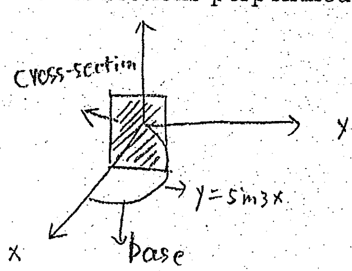
$$A = \int_{-2}^{6} (y+1) - (y^2 - y - 7)\, dy$$

$$= \int_{-2}^{6} -y^2 - 2y + 8\, dy$$

$$= -\frac{1}{3}y^3 - y^2 + 8y \Big|_{-2}^{6}$$

$$= 36.$$

b) [12 pts] Consider the region R bounded by the graph of $y = \sin(3x)$ and the x-axis over the interval $[0, \frac{\pi}{3}]$. Find the volume of the solid whose base is the region R and whose cross sections perpendicular to the x-axis are squares.



$V = \int_0^{\pi/3} A(x)\, dx$ $\qquad$ $A(x)$: area of cross-section.

$= \int_0^{\pi/3} (\sin 3x)^2 dx$ $\qquad$ $A(x) = (\sin 3x)^2$

$\downarrow$ half-angle formula.

$= \int_0^{\pi/3} \frac{1 - \cos 6x}{2}\, dx$

$= \frac{1}{2}x - \frac{1}{12}\sin 6x \Big|_0^{\pi/3}$

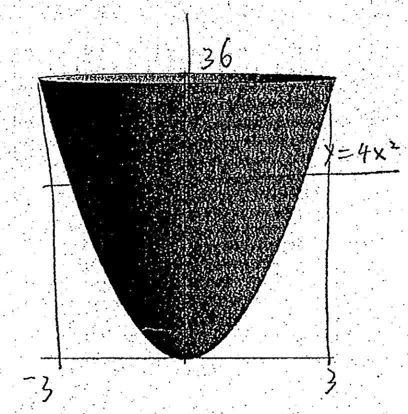
$= \frac{\pi}{6} - \frac{1}{12}\cancel{\sin 2\pi} - \frac{1}{2}\cdot 0 + \frac{1}{12}\cancel{\sin 0}$

$= \frac{\pi}{6}.$

Note: You can use reduction formula to find $\int (\sin 3x)^2 dx$. But before you use reduction formula, you need substitution $\underline{\underline{u = 3x}}$

Problem 3

[20 pts] A tank is shaped like a parabolic bowl. It is formed by revolving the graph of $y = 4x^2$ for $0 \le x \le 3$ (in *meters*) about the y-axis. The tank is filled with water to a height of 30 *meters*. How much work is required to pump all of the water to an exit pipe at the top of the tank? [Note: The density of water is 1000 kg/m^3.]



30 - water level

The height of the bowl:

$y = 4 \cdot 3^2 = 4 \cdot 9 = 36$

$y = 4x^2$

$x = \sqrt{\frac{y}{4}}$ solve for x

[Note: Most of the credit for this problem comes from setting up the correct integral.]

Draw a horizontal slice, the slice is a circle.

The area of the slice $A(y) = \pi r^2 = \pi x^2 = \pi \cdot \left(\sqrt{\frac{y}{4}}\right)^2 = \frac{\pi}{4} y$

the distance that the slice is lifted: $D(y) = 36 - y$

$$W = \int_0^{\overset{\text{water level}}{30}} \rho g A(y) D(y) \, dy$$

$$= \int_0^{30} \rho g \cdot \frac{\pi}{4} y \cdot (36 - y) \, dy$$

$$= \rho g \frac{\pi}{4} \int_0^{30} y(36 - y) \, dy$$

$$= \rho g \frac{\pi}{4} \int_0^{30} 36y - y^2 \, dy$$

$$= \rho g \frac{\pi}{4} \left(18y^2 - \frac{y^3}{3}\right) \Bigg|_0^{30}$$

$$= \rho g \frac{\pi}{4} \cdot \left(18 \cdot 30^2 - \frac{30^3}{3}\right) \text{ J}$$

$\boxed{\text{Don't forget the unit!}}$

Problem 4

[20 pts] Evaluate the following integrals. Show your work.

a) [10 pts] $\int x^8 \ln x \, dx$ Integration by parts LIATE

$u = \ln x \quad dv = x^8 dx$

$du = \frac{1}{x} dx \quad v = \int x^8 dx = \frac{1}{9}x^9$

$= uv - \int v\,du$

$= \ln x \cdot \frac{1}{9}x^9 - \int \frac{1}{9}x^9 \cdot \frac{1}{x} dx$

$= \frac{1}{9}x^9 \ln x - \frac{1}{9}\int x^8 dx$

$= \frac{1}{9}x^9 \ln x - \frac{1}{81}x^9 + \boxed{C}$

Don't forget !

b) [10 pts] $\int_0^3 \frac{w^2-3}{w+1} dw$ Long division or substitution

Method 1.

$$\begin{array}{r|l} & w-1 \quad \text{Quotient} \\ w+1 & w^2-3 \\ & w^2+w \\ \hline & -w-3 \\ & -w-1 \\ \hline & -2 \quad \text{remainder} \end{array}$$

$\int_0^3 \frac{w^2-3}{w+1} dw$

$= \int_0^3 w - 1 - \frac{2}{w+1} dw$

$= \frac{1}{2}w^2 - w - 2\ln|w+1| \Big|_0^3$

$= \frac{1}{2}\cdot 3^2 - 3 - 2\ln 4$

$= \frac{3}{2} - 2\ln 4$

Method 2 Let $u = w+1 \Rightarrow w = u-1$

$du = dw$

$\int \frac{w^2-3}{w+1} dw$

$= \int \frac{(u-1)^2-3}{u} du$

$= \int \frac{u^2-2u+1-3}{u} du$

$= \int \frac{u^2-2u-2}{u} du$

$= \int u - 2 - \frac{2}{u} du$

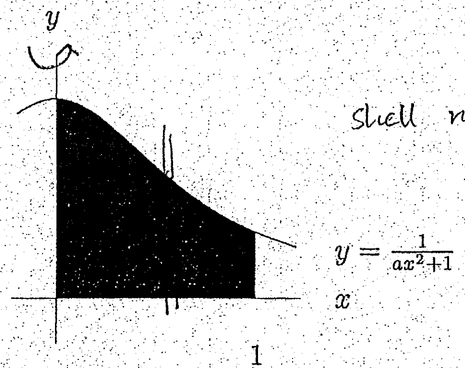
$= \frac{1}{2}u^2 - 2u - 2\ln u$

$= \frac{1}{2}(w+1)^2 - 2(w+1) - 2\ln|w+1| \Big|_0^3$

$= \frac{3}{2} - 2\ln 4$

Problem 5

[20 pts] Consider the region under the graph $y = \dfrac{1}{ax^2+1}$ over the interval $0 \le x \le 1$, where $a > 0$ is a constant.



Shell method will be better !

a) [12 pts] Set up an integral (or integrals) that represents the volume of the solid generated by revolving the region about the y-axis.

Shell :

$$V = \int_0^1 2\pi \underbrace{x}_{\text{radius}} \cdot \underbrace{\left(\frac{1}{ax^2+1} - 0\right)}_{\text{height}} dx$$

b) [8 pts] Evaluate your integral(s) in part a to find the volume of the solid.
Note: Your answer will possibly contain the constant a.

$$V = \int_0^1 2\pi x \frac{1}{ax^2+1} dx$$

$$= 2\pi \int_0^1 \frac{x}{ax^2+1} dx$$

Let $u = ax^2+1$ $\quad du = 2ax\,dx$ $\quad x\,dx = \frac{du}{2a}$

$$2\pi \int \frac{x}{ax^2+1} dx = 2\pi \int \frac{1}{u} \frac{1}{2a} du$$

$$= \frac{2\pi}{2a} \int \frac{1}{u} du = \frac{2\pi}{2a} \ln|u|$$

$$= \frac{2\pi}{2a} \ln|ax^2+1|$$

$$V = \frac{2\pi}{2a} \ln|ax^2+1| \Big|_0^1 = \frac{2\pi}{2a} \ln(a+1) = \boxed{\frac{\pi}{a} \ln(a+1)}$$