1. (20 pts) **Let R be the region bounded by the graphs of** $y = 2x$ **and** $y = (x-4)^2$.

(a) (2 pts) **Find the points of intersection of the line** $y = 2x$ **and the parabola** $y = (x-4)^2$.

SOLUTION :

$2x = (x-4)^2$

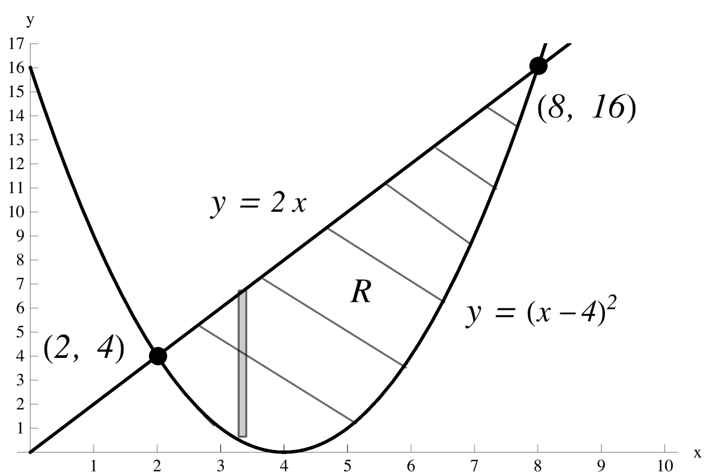
$2x = x^2 - 8x + 16$

$0 = x^2 - 10x + 16 \Longrightarrow (x-2)(x-8) = 0 \Longrightarrow x = 2,\ x = 8$

POINTS of intersection : (2, 4) and (8, 16)

(b) (2 pts) **Sketch the region R.**

SOLUTION :



(c) (10 pts) Find the area of region R.

sOLUTION :

$$A = \int_2^8 \left[2x - (x-4)^2\right] dx =$$

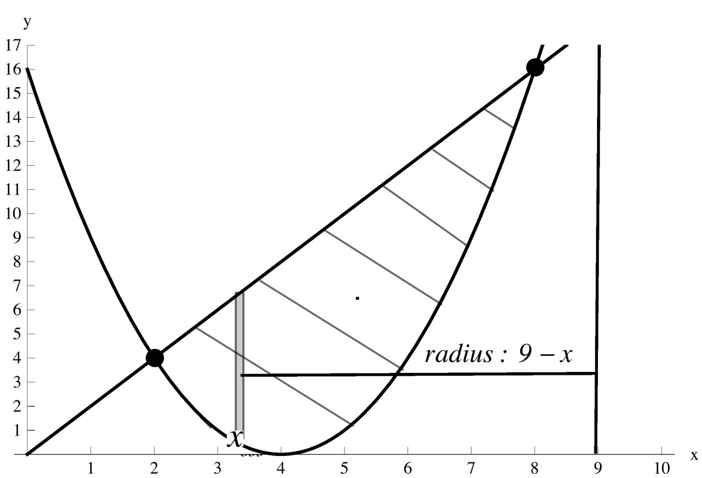
$$= \left[x^2 - \frac{(x-4)^3}{3}\right]_2^8 =$$

$$= \frac{-64}{3} + 64 - \frac{8}{3} - 4 = \text{(full credit !)}$$

$$= 60 - 24 = 36$$

(d) (6 pts) Set up an integral that represents the volume of the solid of revolution formed by revolving the region R about the line x = 9. DO NOT EVALUATE THE INTEGRAL ! ! !

SOLUTION :



$$V = 2\pi \int_2^8 \text{radius} * \text{height} * \text{thickness} = 2\pi \int_2^8 (9-x) * \left[2x - (x-4)^2\right] * dx$$

2. (18 pts) Consider the curve $y = \frac{1}{2} x^3 + \frac{1}{6x}$ on $1 \le x \le 3$.

Find the length of the given curve.

SOLUTIONS :

$$[y']^2 = \left[\frac{3}{2} x^2 - \frac{1}{6x^2}\right]^2 = \frac{9}{4} x^4 - \frac{1}{2} + \frac{1}{36x^4}$$

$$L = \int_1^3 \sqrt{1 + [y']^2}\, dx =$$

$$= \int_1^3 \sqrt{1 + \frac{9}{4} x^4 - \frac{1}{2} + \frac{1}{36x^4}}\, dx =$$

$$= \int_1^3 \sqrt{\frac{9}{4} x^4 + \frac{1}{2} + \frac{1}{36x^4}}\, dx =$$

$$= \int_1^3 \sqrt{\left[\frac{3}{2} x^2 + \frac{1}{6x^2}\right]^2}\, dx =$$

$$= \int_1^3 \left[\frac{3}{2} x^2 + \frac{1}{6x^2}\right] dx =$$

$$= \left[\frac{1}{2} x^3 - \frac{1}{6x}\right]_1^3 =$$

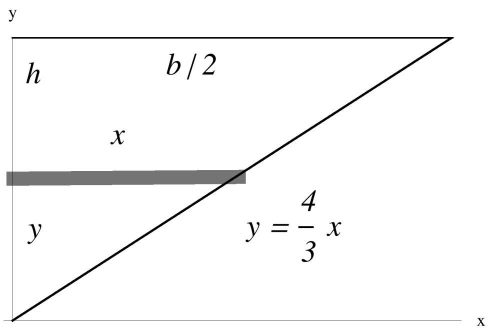
$$= \left[\frac{27}{2} - \frac{1}{18} - \frac{1}{2} + \frac{1}{6}\right] = \text{(FULL CREDIT !)}$$

$$= \frac{118}{9}$$

3. (18 pts) A water trough is full of water.
The length, l, of the trough is 4 m (see figure).
The cross section of the trough is an isosceles triangle
with a base b, where b = 1.2 m and a height h, where h = 0.8 m.
How much work is required to pump all of the water out of the trough
(to the level of the top of the trough)?
(Note : The density of water is 1000 kg / m^3.)

SOLUTIONS :

V (y) = 2 * x * l * dy, where



$$F(y) = 1000 * g * V(y) = 1000 * g * 2 * x * l * dy = 2000\, g * \frac{3\,y}{4} * 4 * dy$$

$$W(y) = F(y) * (h - y) = F(y) * (0.8 - y) = 2000\, g * \frac{3\,y}{4} * 4 * dy * (0.8 - y)$$

Therefore,

$$W = \int_0^{0.8} 2000\, g * \frac{3\,y}{4} * 4 * (0.8 - y)\, dy == 2000\, g \int_0^{0.8} 3\,y * (0.8 - y)\, dy =$$

$$= 6000\, g \int_0^{0.8} \left(0.8\,y - y^2\right) dy = 6000\, g \left(0.8\,\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^{0.8} =$$

$$= 6000\, g \left(\frac{0.8^3}{2} - \frac{0.8^3}{3}\right) = 6000\, g\, \frac{0.8^3}{6}\ \text{Nm} = \text{(FULL CREDIT!)}$$

$$= \left(1000\, g\ 0.8^3\right) \text{Nm} = (512\, g)\ \text{Nm}$$

4. (24 pts) **Evaluate the following integrals. Show your work!**

(a) (12 pts) $$\int_0^{\frac{\pi}{4}} \frac{1}{1 - \sin\theta}\, d\theta$$

SOLUTIONS :

$$\int_0^{\frac{\pi}{4}} \frac{1}{1 - \sin\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \frac{1 + \sin\theta}{(1 - \sin\theta)\,(1 + \sin\theta)}\, d\theta =$$

$$= \int_0^{\frac{\pi}{4}} \frac{1 + \sin\theta}{1 - \sin^2\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \frac{1 + \sin\theta}{\cos^2\theta}\, d\theta =$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2\theta} + \frac{\sin\theta}{\cos^2\theta}\right) d\theta = \int_0^{\frac{\pi}{4}} \left(\sec^2\theta + \tan\theta\sec\theta\right) d\theta =$$

$$= [\tan\theta + \sec\theta]_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} + \sec\frac{\pi}{4} - \tan 0 - \sec 0 =$$

$$= 1 + \sqrt{2} - 0 - 1 = \sqrt{2}$$

(b) (12 pts) $$\int_0^1 x^2 e^{3x}\, dx$$

SOLUTIONS :

$$\int_0^1 x^2 e^{3x}\, dx = \begin{cases} u = x^2 & dv = e^{3x}\, dx \\ du = 2x\, dx & v = \dfrac{e^{3x}}{3} \end{cases}$$

$$= \left(x^2 \frac{e^{3x}}{3}\right)_0^1 - \int_0^1 2x \frac{e^{3x}}{3}\, dx =$$

$$\frac{e^3}{3} - \frac{2}{3}\int_0^1 x\ e^{3x}\, dx = \begin{cases} u = x & dv = e^{3x}\, dx \\ du = dx & v = \frac{e^{3x}}{3} \end{cases}$$

$$= \frac{e^3}{3} - \frac{2}{3}\left[\left(x\frac{e^{3x}}{3}\right)_0^1 - \int_0^1 \frac{e^{3x}}{3}\, dx\right] =$$

$$= \frac{e^3}{3} - \frac{2}{3}\left[\frac{e^3}{3} - \frac{1}{3}\int_0^1 e^{3x}\, dx\right] =$$

$$= \frac{e^3}{3} - \frac{2}{9}e^3 + \frac{2}{9}\int_0^1 e^{3x}\, dx =$$

$$= \frac{e^3}{3} - \frac{2}{9}e^3 + \frac{2}{9}\left(\frac{e^{3x}}{3}\right)_0^1 =$$

$$\frac{e^3}{3} - \frac{2}{9}e^3 + \frac{2}{9}\left(\frac{e^3}{3} - \frac{1}{3}\right) = \text{(FULL CREDIT)}$$

$$= \frac{e^3}{3} - \frac{2}{9}e^3 + \frac{2}{27}e^3 - \frac{2}{27} = \frac{9-6+2}{27}e^3 - \frac{2}{27} =$$

$$= \frac{5e^3 - 2}{27}$$

5. (20 pts) **Evaluate the integral. Show your work!**

$$\int \frac{x^3}{\sqrt{25 - x^2}}\, dx$$

SOLUTIONS :

We use the following trig substitution

$$x = 5 \sin\theta$$

$$dx = 5 \cos\theta \, d\theta$$

$$\sqrt{25 - x^2} = \sqrt{25 - 25 \sin^2\theta} = \sqrt{25\left(1 - \sin^2\theta\right)} =$$

$$= \sqrt{25 \cos^2\theta} = 5 \cos\theta$$

Therefore,

$$\int \frac{x^3}{\sqrt{25 - x^2}}\, dx = \int \frac{5^3 \sin^3\theta}{5 \cos\theta}\, 5 \cos\theta \, d\theta =$$

$$= \int \frac{5^3 \sin^3\theta}{5 \cos\theta}\, 5 \cos\theta \, d\theta = \int 5^3 \sin^3\theta \, d\theta =$$

$$= \int 5^3 \sin\theta \sin^2\theta \, d\theta = \int 5^3 \sin\theta \left(1 - \cos^2\theta\right) d\theta =$$

$$= 5^3 \int \left(\sin\theta - \sin\theta \cos^2\theta\right) d\theta =$$

$$= 5^3 \left[-\cos\theta + \frac{\cos^3\theta}{3}\right] + C =$$

$$= 5^3\left[\frac{-\sqrt{25 - x^2}}{5} + \frac{\left(\sqrt{25 - x^2}\right)^3}{3 * 5^3}\right] + C =$$

$$= -25\sqrt{25 - x^2} + \frac{\left(\sqrt{25 - x^2}\right)^3}{3} + C =$$