Math 1172

Midterm 2

Autumn 2013

Form B

Name: ____________________

OSU username (name.nn): ____________________

Lecturer: ____________________

Recitation Instructor: Yanli Wang

Recitation Time: ____________________

Instructions

•) SHOW ALL WORK!!! Incorrect answers with work shown may receive partial credit, but unsubstantiated answers may receive NO credit.

•) Give EXACT answers unless asked to do otherwise.

•) You do not need to simplify numerical answers such as $\frac{5}{\sqrt{8}} - \frac{5}{\sqrt{12}}$.

•) Calculators are permitted EXCEPT those calculators that have computer algebra systems (CAS) or ability to communicate with others. Furthermore, all memory must be cleared and all apps must be removed. PDA's, laptops, and cell phones are prohibited. Do not have these devices out!

•) The exam duration is 55 minutes.

•) The exam consists of 5 problems starting on page 2 and ending on page 6. Make sure your exam is not missing any pages before you start. Page 7 contains formulas. Page 8 may be used for extra work space.

Problem Number	Maximum Point Value	Score
1	18	
2	22	
3	18	
4	20	
5	22	
Total	100	

Problem 1

[18 pts] True or False. Give a brief explanation or example to justify your answer.

False

a) [3 pts] The parametric equations $x = -3 + \sin(2t)$, $y = 1 + \cos(2t)$, for $0 \le t < \pi$, generate a circle centered at $(3, 1)$ and trace the circle exactly once.

General parametric equations for circle $\boxed{\text{Text book P731}}$ Detail

$x = a + r\sin(ct)$

$y = b + r\cos(ct)$

center (a, b), radius r.

C is called angular frequency. So when t changes from 0 to π, (x, y) makes a complete circle

True

b) [3 pts] $\sum_{k=1}^{\infty} \frac{2^{k+1}}{5^k} = \frac{4}{3}$

Formula: $\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}$

$\sum_{k=1}^{\infty} \frac{2^k \cdot 2}{5^k} = \sum_{k=1}^{\infty} 2 \cdot \left(\frac{2}{5}\right)^k = \frac{2 \cdot \frac{2}{5}}{1 - \frac{2}{5}} = \frac{4}{3}$

True.

c) [3 pts] $\int \frac{x^4}{(x^2+3)^{5/2}}\, dx = \int \frac{\tan^4\theta}{\sec^3\theta}\, d\theta$ where $x = \sqrt{3}\tan\theta$.

$x = \sqrt{3}\tan\theta$

$dx = \sqrt{3}\sec^2\theta\, d\theta$

$\int \frac{x^4}{(x^2+3)^{5/2}}\, dx = \int \frac{(\sqrt{3}\tan\theta)^4}{((\sqrt{3}\tan\theta)^2+3)^{5/2}} \sqrt{3}\cdot\sec^2\theta\, d\theta = \int \frac{9\tan^4\theta}{[3(\tan^2\theta+1)]^{5/2}} \sqrt{3}\sec^2\theta\, d\theta$

$= \int \frac{9\tan^4\theta}{3^{5/2}(\sec^2\theta)^{5/2}} \sqrt{3}\sec^2\theta\, d\theta = \int \frac{3^2 \tan^4\theta}{3^{5/2}\sec^5\theta} 3^{\frac{1}{2}}\sec^2\theta\, d\theta$

$= \int \frac{\tan^4\theta}{\sec^3\theta}\, d\theta$

False.

d) [3 pts] If $\sum_{k=0}^{\infty} a_k$ converges then $\lim_{k\to\infty} a_k = 1$.

Thm 9.8. (Divergence Test)

If $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$; Equivalently, if $\lim_{k\to\infty} a_k \ne 0$, then the series diverenges.

False.

e) [3 pts] $\sum_{k=2}^{\infty} \frac{4}{k^2-1} = 4$

"Telescoping"

$= \sum_{k=2}^{\infty} \frac{4}{k^2-1} = \sum_{k=2}^{\infty} \frac{4}{(k-1)(k+1)} = \sum_{k=2}^{\infty} 2\cdot\left(\frac{1}{k-1} - \frac{1}{k+1}\right) = 2\sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) =$

$= 2\cdot\Big(\underbrace{1-\cancel{\tfrac{1}{3}}}_{k=2} + \underbrace{\tfrac{1}{2}-\cancel{\tfrac{1}{4}}}_{k=3} + \underbrace{\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{5}}}_{k=4} + \underbrace{\cancel{\tfrac{1}{4}}-\cancel{\tfrac{1}{6}}}_{k=5} + \cdots\Big) = 2\cdot\left(1+\frac{1}{2}\right) = 2\cdot\frac{3}{2} = 3$

False

f) [3 pts] Given $a_1 = 0$ and $a_{n+1} = \frac{6}{3-a_n}$ for $n \ge 1$, the first four terms of the sequence $\{a_n\}_{n=1}^{\infty}$ are nonnegative

$a_1 = 0$

$a_2 = \frac{6}{3-a_1} = \frac{6}{3-0} = 2$

$a_3 = \frac{6}{3-a_2} = \frac{6}{3-2} = \frac{6}{1} = 6$

$a_4 = \frac{6}{3-a_3} = \frac{6}{3-6} = \frac{6}{-3} = -2$

Problem 2

[22 pts] Determine whether the following improper integral converges or diverges. If it converges, find its value.

$$\int_4^{\infty} \frac{2x+6}{x(x^2-2x-3)}\,dx.$$

$$= \lim_{b\to\infty} \int_4^b \frac{2x+6}{x(x^2-2x-3)}\,dx$$ partial fraction.

$$\frac{2x+6}{x(x^2-2x-3)} = \frac{2x+6}{x(x-3)(x+1)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+1}$$

$$2x+6 = A(x-3)\cdot(x+1) + Bx(x+1) + C\,x\,(x-3)$$

Plug in $x=0$: $6 = -3A \Rightarrow A = -2$

Plug in $x=3$: $12 = 12B \Rightarrow B = 1$

Plug in $x=-1$: $4 = 4C \Rightarrow C = 1$

$$= \lim_{b\to\infty} \int_4^b \frac{-2}{x} + \frac{1}{x-3} + \frac{1}{x+1}\,dx$$

$$= \lim_{b\to\infty} \left(-2\ln|x| + \ln|x-3| + \ln|x+1|\right)\Big|_4^b$$

$$= \lim_{b\to\infty} \underline{-2\ln b + \ln(b-3) + \ln(b+1)} + 2\ln 4 - \ln 1 + \ln 5$$

$$= \lim_{b\to\infty} \ln\left(\frac{(b-3)(b+1)}{b^2}\right) + 2\ln 4 - \ln 1 + \ln 5$$

$$= \ln 1 + 2\ln 4 - \ln 1 + \ln 5$$

$$= 2\ln 4 + \ln 5$$

Remark: $\lim_{b\to\infty} \frac{(b-3)(b+1)}{b^2} = 1$.

Since denominator and numerator have the same degree, the limit = the ratio of coefficient of the leading terms

$= \frac{1}{1} = 1$.

Or use L'Hospital's Rule or multiplying by 1

Problem 3

[18 pts] Power Series.

a) [12 pts] Find the radius and interval of convergence for the power series.

$$\sum_{k=0}^{\infty} \frac{k^2(x+3)^k}{5^{2k}}$$

Ratio Test !

$$r = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \left|\frac{(k+1)^2(x+3)^{k+1}}{5^{2k+2}}\right| \cdot \left|\frac{5^{2k}}{k^2(x+3)^k}\right|$$

↑ Be careful !

Pull out

$$= \lim_{k\to\infty} \left|\left(\frac{k+1}{k}\right)^2 \cdot (x+3) \cdot \frac{1}{5^2}\right|$$

$$= |x+3| \lim_{k\to\infty} \left|\frac{1}{25} \left(\frac{k+1}{k}\right)^2\right|$$

$$= |x+3| \frac{1}{25}$$

By ratio test, if $r<1$, then it converges.

So $|x+3|\frac{1}{25} < 1$ i.e. $|x+3|<25$ radius of convergence 25

$-25 < x+3 < 25 \Rightarrow -28 < x < 22$ interval of convergence $(-28, 22)$

↑ open.

b) [6 pts] **Multiple Choice.** Choose the function that is represented by the series. [Note: For this problem you do not need to show your work. There is no partial credit for this problem.]

$$\sum_{k=0}^{\infty} \frac{3^k x^{k+1}}{k+1}.$$

$$\sum_{k=0}^{\infty} \frac{3^k x^{k+1}}{k+1} = \frac{1}{3}\sum_{k=0}^{\infty} \frac{(3x)^{k+1}}{(k+1)} = \frac{1}{3}\sum_{k=1}^{\infty} \frac{(3x)^k}{k} = -\frac{1}{3}\ln|1-3x|$$

By Table 10.5

(i) $-\ln|1-3x|$ (ii) $\dfrac{x}{1-3x}$ (iii) $-3\ln|1-3x|$ (iv) $\dfrac{3}{1-3x}$

(v) $\dfrac{-1}{3}\ln|1-3x|$ ✓ (vi) $\dfrac{1}{3-x}$ (vii) $\dfrac{x}{1+3x}$ (viii) xe^{3x}

Problem 4

[20 pts]. Taylor Series. Find the first four nonzero $\boxed{\text{terms}}$ of the Taylor series for the given function centered at a.

a) [10 pts] $f(x) = \dfrac{1}{(1+x)^3}$, $a = 1$

$f(x) = (1+x)^{-3}$ $\quad f(1) = 2^{-3} = \frac{1}{8}$

$f'(x) = -3(1+x)^{-4}$ $\quad f'(1) = \frac{-3}{2^4} = -\frac{3}{16}$

$f''(x) = 12(1+x)^{-5}$ $\quad f''(1) = \frac{12}{2^5} = \frac{3}{8}$

$f'''(x) = -60(\cancel{x}+x)^{-6}$ $\quad f'''(1) = \frac{-60}{2^6} = -\frac{15}{16}$

Then. $P_3(x) = \underbrace{\frac{1}{8}}_{\text{1st}} - \underbrace{\frac{3}{16}(x-1)}_{\text{2nd}} + \underbrace{\frac{3}{8}\cdot\frac{1}{2!}(x-1)^2}_{\text{3rd}} - \underbrace{\frac{15}{16}\cdot\frac{1}{3!}(x-1)^3}_{\text{4th}}$

b) [10 pts] $g(x) = x^3 - 5x^2 + 2$, $a = -1$

$g(x) = x^3 - 5x^2 + 2$ $\quad g(-1) = (-1)^3 - 5(-1)^2 + 2 = -1 - 5 + 2 = -4$

$g'(x) = 3x^2 - 10x$ $\quad g'(-1) = 3\cdot(-1)^2 - 10\cdot(-1) = 3 + 10 = 13$

$g''(x) = 6x - 10$ $\quad g''(-1) = -6 - 10 = -16$

$g'''(x) = 6$ $\quad g'''(-1) = 6$

$P_3(x) = \underbrace{-4}_{\text{1st}} + \underbrace{13(x+1)}_{\text{2nd}} - \underbrace{16\frac{1}{2!}(x+1)^2}_{\text{3rd}} + \underbrace{6\cdot\frac{1}{3!}(x+1)^3}_{\text{4th}}$

Problem 5

[22 pts] Parametric and Polar equations.

a) [10 pts] Find an equation of the line tangent to the curve at the point corresponding to the given value of t.

$x = t^3 + t, \quad y = t^4 - t; \quad t = 1$

Slope $= \left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1} = \left.\frac{(t^4-t)'}{(t^3+t)'}\right|_{t=1} = \left.\frac{4t^3-1}{3t^2+1}\right|_{t=1} = \frac{4-1}{3+1} = \frac{3}{4}$

Point. $x_0 = t^3+t|_{t=1} = 1+1 = 2$

$y_0 = t^4 - t|_{t=1} = 1-1=0.$

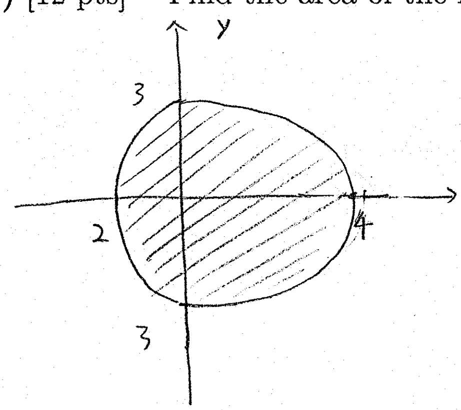
Tangent line:

$y - y_0 = k\,(x - x_0)$

So $y - 0 = \frac{3}{4}(x-2).$

$\boxed{y = \frac{3}{4}x - \frac{3}{2}}.$

b) [12 pts] Find the area of the region bounded by the polar curve $r = 3 + \cos\theta$.



$A = \int_0^{2\pi} \frac{1}{2}(3+\cos\theta)^2\, d\theta$

$= \frac{1}{2}\int_0^{2\pi} (9 + \cos^2\theta + 18\cos\theta)\, d\theta$

double angle

$= \frac{1}{2}\int_0^{2\pi} 9 + \frac{1+\cos 2\theta}{2} + 18\cos\theta \, d\theta$

$= \frac{1}{2}\int_0^{2\pi} 9\frac{1}{2} + \frac{1}{2}\cos 2\theta + 18\cos\theta \, d\theta$

$= \frac{1}{2}\left(9\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + 18\sin\theta\right)\Big|_0^{2\pi}$

$= 9\frac{1}{2}\pi + \frac{1}{8}\cancel{\sin 4\pi} + 9\cancel{\sin 2\pi} - \cancel{\frac{1}{8}\sin 0} - 9\cancel{\sin 0}$

$= 9\frac{1}{2}\pi$

A Few Trigonometric Identities

1) $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$

2) $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$

3) $\cos^2\theta + \sin^2\theta = 1$

4) $\sec^2\theta - \tan^2\theta = 1$

5) $\csc^2\theta - \cot^2\theta = 1$

A Few Reduction Formulas

Assume n is a positive integer.

1) $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$

2) $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

3) $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx, \quad n \neq 1$

4) $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx, \quad n \neq 1$

EXTRA WORKSPACE
Do not remove this page.