Math 1172

Midterm 1

Autumn 2014

Form A

Name: ______

OSU username (name.nn): ______

Lecturer: ______

Recitation Instructor: ______

Recitation Time: ______

Instructions

•) SHOW ALL WORK!!! Incorrect answers with work shown may receive partial credit, but unsubstantiated answers may receive NO credit.

•) Give EXACT answers unless asked to do otherwise.

•) You do not need to simplify numerical answers such as $\frac{5}{\sqrt{8}} - \frac{5}{\sqrt{12}}$.

•) NO CALCULATORS. NO CELL PHONES. NO ELECTRONIC DEVICES.

•) The exam duration is 55 minutes.

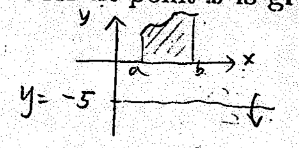
•) The exam consists of 5 problems starting on page 2 and ending on page 6. Make sure your exam is not missing any pages before you start. Page 7 contains formulas. Pages 7 and 8 may be used for extra work space.

Problem Number	Maximum Point Value	Score
1	21	
2	23	
3	22	
4	20	
5	14	
Total	100	

Problem 1

[21 pts] True or False. Give a brief explanation or example to justify your answer.

F a) [3 pts] Consider the region R bounded by the graph of $y = f(x) \geq 0$ and the x-axis on the interval $[a, b]$. If the region R is revolved about the line $y = -5$ to form a solid of revolution and you look at the cross-sections that are washers, then the outer radius of the washer at point x is given by the expression $f(x) - 5$.



outer radius $= f(x) - (-5) = f(x) + 5$

T b) [3 pts] Using integration by parts with $u = x^3$ and $dv = \sin(2x)dx$ we get $\int x^3 \sin(2x)\, dx = -\frac{1}{2}\cos(2x)x^3 + \int \frac{3}{2}\cos(2x)x^2\, dx$.

$du = 3x^2\,dx$, $v = \int \sin(2x)dx = -\frac{1}{2}\cos(2x)$

$= uv - \int v\,du = -\frac{1}{2}\cos(2x)x^3 - \int -\frac{1}{2}\cos(2x) \cdot 3x^2\, dx$ ✓

F c) [3 pts] Using the substitution $u = x+3$, $\int \frac{x+1}{x+3}\, dx = \int \frac{x+1}{u}\, du = \int (\frac{x}{u} + \frac{1}{u})\, du = \int \frac{x}{u}\, du + \int \frac{1}{u}\, du = x \int \frac{1}{u}\, du + \int \frac{1}{u}\, du = x\ln(u) + \ln(u) + C = x\ln(x+3) + \ln(x+3) + C$.

x is not a constant, can't pull it out.

F d) [3 pts] $\int \frac{1}{e^x}\, dx = \ln|e^x| + C$

$\int \frac{1}{e^x}\,dx = \int e^{-x}\,dx = -e^{-x} + C$, but $\ln|e^x| = \ln e^x = x\ln e = x$.

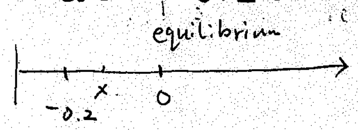
T e) [3 pts] A thin wire over $0 \leq x \leq 5$ (in *meters*) with the density function (in $\frac{kg}{m}$) $\rho(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 3 \\ 3, & \text{if } 3 < x \leq 5 \end{cases}$, has a mass of 9 *kg*.

mass $= \int_0^5 \rho(x)\,dx = \int_0^3 1\,dx + \int_3^5 3\,dx = 3 + 6 = 9$

T f) [3 pts] Assuming Hooke's law is obeyed, a spring that requires 5 J of work to be compressed 20 cm from its equilibrium position would have a spring constant of $k = 250\ N/m$.

20 cm = 0.2 cm

force $f(x) = kx = 250x$



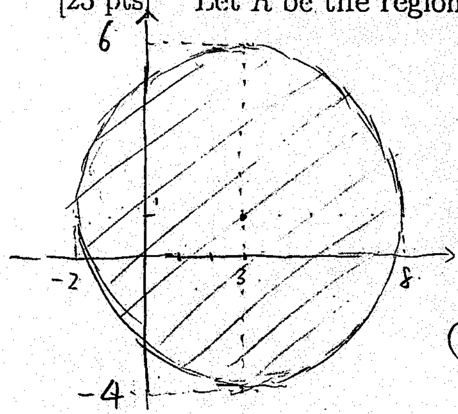
work $= \int_0^{-0.2} f(x)\,dx = \int_0^{-0.2} 250x\,dx = 125x^2 \Big|_0^{-0.2} = 5\ J$

F g) [3 pts] If it requires 30 J of work to stretch a spring, with spring constant k, from equilibrium to 10 cm, then it will take 120 J of work to stretch a spring, with spring constant $2k$, from equilibrium to 10 cm.

$30 = \int_0^{0.1} kx\,dx$; new work $= \int_0^{0.1} 2kx\,dx = 2\int_0^{0.1} kx\,dx = 2 \cdot 30 = 60\ J$

Problem 2

[23 pts] Let R be the region bounded by the circle $(x-3)^2+(y-1)^2=25$



center $(3, 1)$, radius 5

$$(x-3)^2 = 25-(y-1)^2$$

$$x-3 = \pm\sqrt{25-(y-1)^2}$$

$$x = 3 \pm \sqrt{25-(y-1)^2} \quad \begin{cases} + \text{ for the right curve} \\ - \text{ for the left curve} \end{cases}$$

$$(y-1)^2 = 25-(x-3)^2$$

$$y-1 = \pm\sqrt{25-(x-3)^2}$$

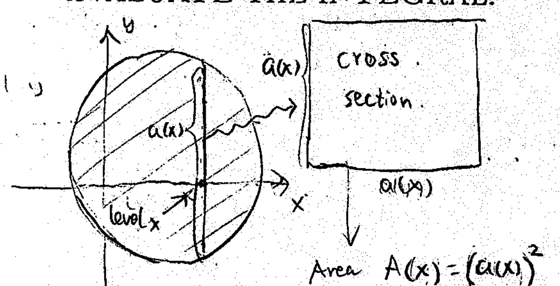
$$y = 1 \pm \sqrt{25-(x-3)^2} \quad \begin{cases} + \text{ for the upper curve} \\ - \text{ for the lower curve} \end{cases}$$

a) [7 pts] Set up an integral with respect to y that represents the area of the region R. DO NOT EVALUATE THE INTEGRAL.

$$\text{area}(R) = \int_{-4}^{6} \text{right} - \text{left}\, dy$$

$$= \int_{-4}^{6} \left(3+\sqrt{25-(y-1)^2}\right) - \left(3-\sqrt{25-(y-1)^2}\right) dy$$

b) [8 pts] Set up an integral that represents the volume of the solid whose base is the region R and whose cross sections perpendicular to the x-axis are squares. DO NOT EVALUATE THE INTEGRAL.



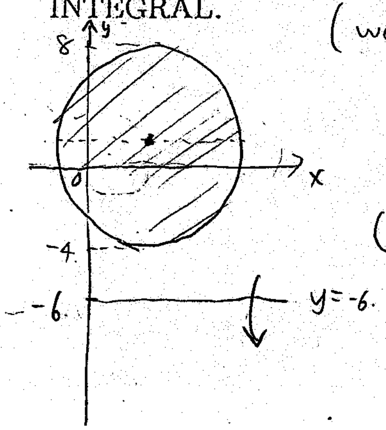
$$a(x) = \text{upper} - \text{lower} = 1+\sqrt{25-(x-3)^2} - \left(1-\sqrt{25-(x-3)^2}\right)$$

Hence,

$$\text{vol} = \int_{-2}^{8} A(x)\, dx$$

$$= \int_{-2}^{8} \left[\left(1+\sqrt{25-(x-3)^2}\right) - \left(1-\sqrt{25-(x-3)^2}\right)\right]^2 dx$$

c) [8 pts] Set up an integral that represents the volume of the solid of revolution that is formed by revolving the region R about the line $y=-6$. DO NOT EVALUATE THE INTEGRAL.



(washer method)

$$\text{vol} = \int_{-2}^{8} \pi\left(\text{outer radius}^2 - \text{inner radius}^2\right) dx$$

$$= \int_{-2}^{8} \pi\left[\left(1+\sqrt{25-(x-3)^2}-(-6)\right)^2 - \left(1-\sqrt{25-(x-3)^2}-(-6)\right)^2\right] dx$$

(shell method)

$$\text{vol} = \int_{-4}^{8} 2\pi\, \text{radius} \cdot \text{height}\, dy$$

$$= \int_{-4}^{8} 2\pi\,(y-(-6))\left[3+\sqrt{25-(y-1)^2} - \left(3-\sqrt{25-(y-1)^2}\right)\right] dy$$

Problem 3

[22 pts] Applications. **Show your work.**

a) [12 pts] *Find the length of the curve* $y = \frac{x^7}{7} + \frac{x^{-5}}{20}$ *on the interval* $[1, 2]$. *You do not need to simplify your final answer.*

$$y' = \frac{7x^6}{7} + \frac{-5x^{-6}}{20} = x^6 - \frac{1}{4}x^{-6}$$

$$(y')^2 = (x^6 - \tfrac{1}{4}x^{-6})^2 = (x^6)^2 - 2\cdot x^6 \cdot \tfrac{1}{4}x^{-6} + (\tfrac{1}{4}x^{-6})^2$$

$$= (x^6)^2 - \tfrac{1}{2} + (\tfrac{1}{4}x^{-6})^2$$

$$1 + (y')^2 = (x^6)^2 + \tfrac{1}{2} + (\tfrac{1}{4}x^{-6})^2 = (x^6 + \tfrac{1}{4}x^{-6})^2$$

So

$$\text{length} = \int_1^2 \sqrt{1+(y')^2}\, dx = \int_1^2 x^6 + \tfrac{1}{4}x^{-6}\, dx$$

$$= \frac{x^7}{7} + \frac{1}{20}x^{-5} \Big|_1^2$$

$$= \frac{2^7}{7} - \frac{1}{20}\cdot 2^{-5} - \left(\frac{1}{7} - \frac{1}{20}\right).$$

b) [10 pts] Find the mass of the thin wire over the interval $[e, e^2]$ (in *meters*) that has the density function (in $\frac{kg}{m}$) $\rho(x) = x^5 \ln x$. You do not need to simplify your answer.

$$\text{mass} = \int_a^b \underset{\text{dencity}}{\rho(x)}\, dx = \int_e^{e^2} x^5 \ln x\, dx \quad (\text{need integration by parts})$$

$$\begin{cases} u = \ln x & du = \frac{1}{x}dx \\ dv = x^5 dx & v = \int x^5 dx = \frac{x^6}{6} \end{cases}$$

$$\int x^5 \ln x\, dx = \int u\,dv = uv - \int v\,du = \tfrac{1}{6}x^6 \ln x - \int \frac{x^6}{6}\cdot\frac{1}{x}\, dx$$

$$= \tfrac{1}{6}x^6 \ln x - \tfrac{1}{6}\int x^5\, dx = \tfrac{1}{6}x^6 \ln x - \tfrac{1}{36}x^6 + C.$$

So

$$\text{mass} = \tfrac{1}{6}x^6 \ln x - \tfrac{1}{36}x^6 \Big|_e^{e^2} = \tfrac{1}{6}e^{12}\ln(e^2) - \tfrac{1}{36}e^{12} - \left(\tfrac{1}{6}e^6 \ln e - \tfrac{1}{36}e^6\right)$$

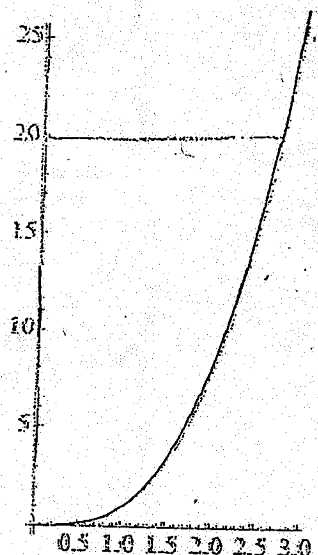
$$= \left(\tfrac{1}{3} - \tfrac{1}{36}\right)e^{12} - \left(\tfrac{1}{6} - \tfrac{1}{36}\right)e^6$$

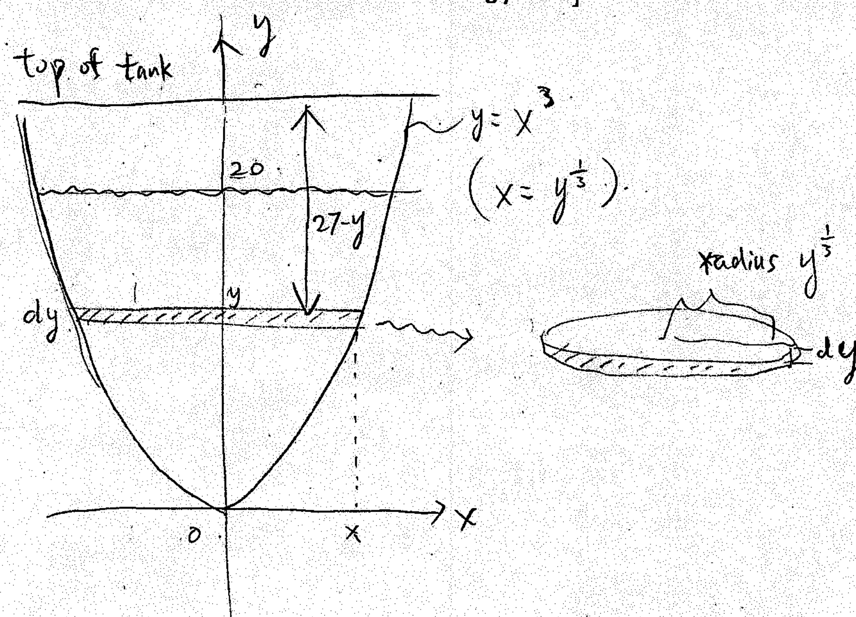
$\ln(a^b) = b\ln a$

$\ln e = 1$

Problem 4

[20 pts] A tank is formed by revolving the graph of $y = x^3$ for $0 \le x \le 3$ (in *meters*) about the y-axis. The tank is filled with water to a height of 20 *meters*. How much work is required to pump all of the water to an exit pipe at the top of the tank? You do not need to simplify your answer. [Note: The density of water is 1000 kg/m^3.]





cut a slice horizontally at level y with width dy.

It is approximately a disk of radius $y^{\frac{1}{3}}$, and width dy.

So its mass is

$$\underbrace{1000}_{\text{density}} \cdot \underbrace{\pi \cdot (y^{\frac{1}{3}})^2 \, dy}_{\text{volume}}$$

It is lifted by a distance of $27 - y$. So the total work is

$$\int_0^{20} 9.8 \times 1000 \times \pi \times (y^{\frac{1}{3}})^2 \times (27 - y) \, dy.$$

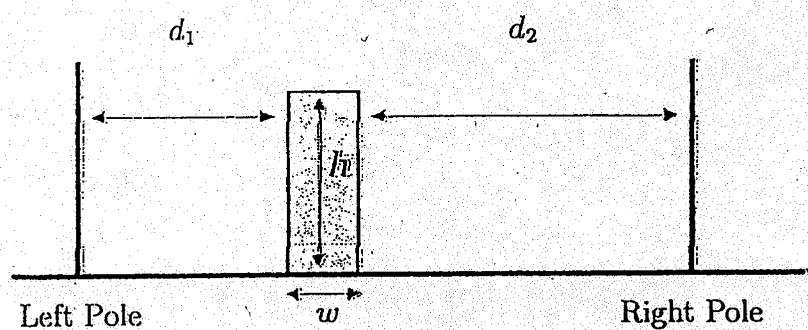
$$= 9800\pi \int_0^{20} 27 y^{\frac{2}{3}} - y^{\frac{5}{3}} \, dy = 9800\pi \left(27 \cdot \frac{3}{5} y^{\frac{5}{3}} - \frac{3}{8} y^{\frac{8}{3}} \right) \Big|_0^{20}$$

$$= 9800\pi \left(27 \cdot \frac{3}{5} \cdot 20^{\frac{5}{3}} - \frac{3}{8} 20^{\frac{8}{3}} \right) \text{ J}$$

Problem 5

[14 pts] Slice and Sum Method. You have decided to make up your own measurment system and call the units Goonies. The measurement, you have decided, relies on the placement of two fixed poles: a left pole and a right pole. Then, to compute the number of Goonies, G, in a rectangle you use the following formula based on the diagram below.

$G = \frac{d_1}{d_2} hw$, where h is the hieght of the rectangle, w is the width, d_1 is the distance from the rectangle to the left pole, and d_2 is the distance from the rectangle to the right pole.



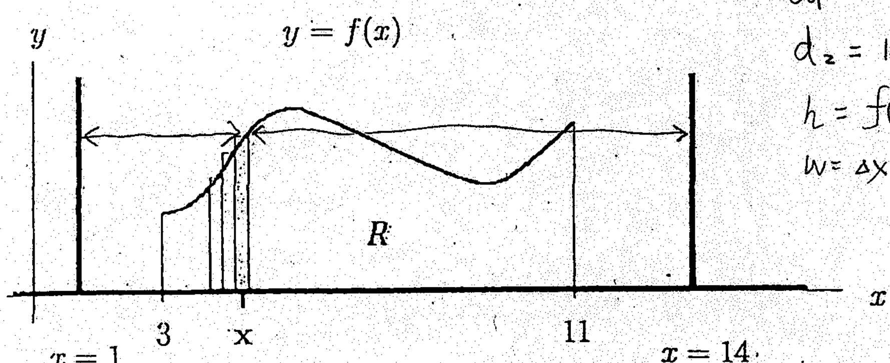
a) [8 pts] Consider the region R that lies under the graph of $y = f(x) > 0$ over the interval $[3, 11]$. The region R is between two fixed poles. The left pole is at $x = 1$ and the right pole is at $x = 14$. In the graph below, approximate the number of Goonies in the single grey rectangle positioned at x and express it in terms of x and Δx. Note: Δx is the width of the rectangle. Moreover, since Δx is small, you may simply use the position x as an approximation to the location of the sides of the rectangle.

$d_1 = x - 1$

$d_2 = 14 - x$

$h = f(x)$

$w = \Delta x$



So $G = \frac{d_1}{d_2} hw$

$= \frac{x-1}{14-x} f(x) \Delta x$

b) [6 pts] Use your result from part a to develop an integral that represents the exact number of Goonies for the region R. You do not need to evaluate the integral.

$$\text{Goonies}(R) = \int_3^{11} \frac{x-1}{14-x} f(x)\, dx$$