Math 1172

Midterm 2

Autumn 2014

Form A

Name: ______________________

OSU username (name.nn): ______________________

Lecturer: ______________________

Recitation Instructor: ______________________

Recitation Time: ______________________

Instructions

•) SHOW ALL WORK!!! Incorrect answers with work shown may receive partial credit, but unsubstantiated answers may receive NO credit.

•) Give EXACT answers unless asked to do otherwise.

•) You do not need to simplify numerical answers such as $\frac{5}{\sqrt{8}} - \frac{5}{\sqrt{12}}$.

•) **NO CALCULATORS. NO CELL PHONES. NO ELECTRONIC DEVICES.**

•) The exam duration is 55 minutes.

•) The exam consists of 5 problems starting on page 2 and ending on page 6. Make sure your exam is not missing any pages before you start. Page 7 contains formulas. Pages 7 and 8 may be used for extra work space.

Problem Number	Maximum Point Value	Score
1	21	
2	19	
3	20	
4	18	
5	22	
Total	100	

Problem 1

[21 pts] True or False. Give a brief explanation or example to justify your answer.

True.

a) [3 pts] Suppose $f(x) = 2x^3 - 3x$. Then the 3^{rd}-order Taylor polynomial for f centered at $a = 1$ is $p_3(x) = -1 + 3(x-1) + 6(x-1)^2 + 2(x-1)^3$.

$f(1) = 2 - 3 = -1$ $\quad f'''(x) = 12 \Rightarrow f'''(1) = 12$.

$f'(x) = 6x^2 - 3 \Rightarrow f'(1) = 3$

$f''(x) = 12x \Rightarrow f''(1) = 12$ $\quad$ So $P_3(x) = -1 + \frac{3}{1!}(x-1) + \frac{12}{2!}(x-1)^2 + \frac{12}{3!}(x-1)^3$

$= -1 + 3(x-1) + 6(x-1)^2 + 2(x-1)^3$.

b) [3 pts] $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k^{10}}$ diverges.

False, because $\displaystyle\lim_{k\to\infty} \frac{2^k}{k^{10}} = +\infty$, the divergence test $\Rightarrow$ divergence.

True.

c) [3 pts] $\displaystyle\int \tan^5\theta \sec^6\theta\, d\theta = \int (u^2-1)^2 u^5\, du$, where $u = \sec\theta$. , $du = \tan\theta\sec\theta\, d\theta$.

$= \int \tan^4\theta \sec^5\theta \tan\theta\sec\theta\, d\theta$

$= \int (\sec^2\theta - 1)^2 \sec^5\theta \tan\theta\sec\theta\, d\theta = \int (u^2-1)^2 u^5\, du$.

d) [3 pts] If $\displaystyle\lim_{k\to\infty} a_k = 0$ then the series $\displaystyle\sum_{k=0}^{\infty} a_k$ converges.

False. If $\sum_{k=0}^{\infty} a_k$ converges, then $\displaystyle\lim_{k\to\infty} a_k = 0$. But the converse is not true.

Counter example: $\displaystyle\lim_{k\to\infty} \frac{1}{k} = 0$ but $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

e) [3 pts] $\displaystyle\sum_{k=2}^{\infty} \frac{6}{(-3)^k} = \frac{1}{2}$.

True.

$$\sum_{k=2}^{\infty} \frac{6}{(-3)^k} = \sum_{k=2}^{\infty} 6\left(-\frac{1}{3}\right)^k = \frac{6\cdot\left(-\frac{1}{3}\right)^2}{1-\left(-\frac{1}{3}\right)} = \frac{6\cdot\frac{1}{9}}{\frac{4}{3}} = \frac{1}{2}.$$

f) [3 pts] The sequence $a_n = \dfrac{(-1)^n n}{3n^2+1}$ converges as $n \to \infty$.

True. $\displaystyle\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{3n^2+1} = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

True

g) [3 pts] $x = 2 - 4t$, $y = 3 + 8t$, for $-\infty < t < \infty$ are parametric equations for a line with $\dfrac{dy}{dx} = -2$.

$\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{8}{-4} = -2$ for all t. So we have a line w/ slope -2.

Or can see it this way: $x = 2 - 4t \Rightarrow 4t = 2 - x$, $\Rightarrow y = 3 + 2\cdot 4t = 3 + 2\cdot(2-x) = -2x + 7$

Problem 2

[19 pts] Integrals and Series.

a) [10 pts] Evaluate the indefinite integral.

$$\int \frac{1}{x^2+6x+8}\,dx$$

$$= \int \frac{1}{x^2+2\cdot 3x+3^2-3^2+8}\,dx$$

$$= \int \frac{1}{(x+3)^2-1}\,dx$$

$(u=x+3)$

$$= \int \frac{du}{u^2-1}$$

$$= \int \frac{du}{(u-1)(u+1)}$$

$$\left[\ \frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1} = \frac{A(u+1)+B(u-1)}{(u-1)(u+1)} = \frac{(A+B)u+(A-B)}{(u-1)(u+1)} \right.$$

$$\left. \text{so } \begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow A=\frac{1}{2},\ B=-\frac{1}{2}. \ \right]$$

$$= \int \frac{1}{2}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1}\,du$$

$$= \frac{1}{2}\int \frac{1}{u-1} - \frac{1}{u+1}\,du$$

$$= \frac{1}{2}\left(\ln|u-1| - \ln|u+1|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{x+2}{x+4}\right| + C.$$

b) [9 pts] Determine whether the series converges or diverges. If it converges, find the value.

$$\sum_{k=1}^{\infty}\frac{1}{k^2+6k+8} = \sum_{k=1}^{\infty}\left(\frac{1}{2}\frac{1}{(k+3-1)} - \frac{1}{2(k+3+1)}\right)$$

$$= \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{k+2} - \frac{1}{k+4}\right).$$

$$S_n = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k+2} - \frac{1}{k+4}\right) = \frac{1}{2}\left(\frac{1}{3} - \cancel{\frac{1}{5}} + \frac{1}{4} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{6}} - \cancel{\frac{1}{8}} + \cdots\right.$$

$$\left. + \cdots + \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+2}} + \cancel{\frac{1}{n+1}} - \frac{1}{n+3} + \cancel{\frac{1}{n+2}} - \frac{1}{n+4}\right)$$

$$= \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4} - \frac{1}{n+3} - \frac{1}{n+4}\right).$$

$$\lim_{n\to\infty} S_n = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}-0-0\right) = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}\right) = \frac{7}{24}$$

So the series converges to

Problem 3

[20 pts] Consider the function f and its power series $f(x) = \sum_{k=0}^{\infty} \frac{5}{3^k}(x+1)^k$.

a) [2 pts] Compute $f(-1)$.

$f(x) = 5 + \frac{5}{3}(x+1) + \frac{5}{9}(x+1)^2 + \cdots$; so $f(-1) = 5$.

b) [8 pts] Find the radius of convergence for the power series.

$$r = \lim_{k\to\infty} \left| \frac{\frac{5}{3^{k+1}}(x+1)^{k+1}}{\frac{5}{3^k}(x+1)^k} \right| = \lim_{k\to\infty} \left| \frac{5(x+1)^k}{3^{k+1}} \cdot \frac{3^k}{5(x+1)^k} \right| = \lim_{k\to\infty} \left| \frac{x+1}{3} \right| = \frac{|x+1|}{3}.$$

By the ratio test, $\sum_{k=0}^{\infty} \frac{5}{3^k}(x+1)^k$ converges absolutely if $\frac{|x+1|}{3} < 1$, which is equivalent to $|x+1| < 3$. So the radiu of convergence is 3.

c) [4 pts] Find the interval of convergence for the power series.

1° $\frac{|x+1|}{3} < 1 \iff |x+1| < 3 \iff -3 < x+1 < 3 \iff -4 < x < 2$

2° $x=2 \Rightarrow \sum_{k=0}^{\infty} \frac{5 \cdot 3^k}{3^k} = \sum_{k=0}^{\infty} 5$ diverges ; $x=4 \Rightarrow \sum_{k=0}^{\infty} \frac{5(-3)^k}{3^k} = 5\sum_{k=0}^{\infty}(-1)^k$ diverges

$\Rightarrow$ the interval of convergence is $(-4, 2)$

d) [4 pts] Find a power series representation for $f'(x)$.

$$f'(x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{5(x+1)^k}{3^k}\right) = \frac{d}{dx}\left(5 + \sum_{k=1}^{\infty} \frac{5(x+1)^k}{3^k}\right)$$

$$= \sum_{k=1}^{\infty} \frac{d}{dx}\left(\frac{5(x+1)^k}{3^k}\right) = \sum_{k=1}^{\infty} \frac{5k(x+1)^{k-1}}{3^k} = \sum_{k=0}^{\infty} \frac{5(k+1)(x+1)^k}{3^{k+1}}$$

e) [2 pts] Compute $f'(-1)$.

By (d), $f'(x) = \frac{5}{3} + \sum_{k=1}^{\infty} \frac{5(k+1)(x+1)^k}{3^{k+1}}$; $f'(0) = \frac{5}{3}$.

Problem 4

[18 pts] Taylor Series.

a) [10 pts] Find the first four nonzero terms of the Taylor series for $f(x) = \frac{1}{x^2}$ centered at $a = 1$.

$f^{(0)}(x) = f(x) = \frac{1}{x^2} = x^{-2}$, $\quad f^{(0)}(1) = 1$

$f^{(1)}(x) = f'(x) = -2x^{-3}$, $\quad f^{(1)}(1) = -2$

$f^{(2)}(x) = f''(x) = (-2)(-3)x^{-4} = 6x^{-4}$, $\quad f^{(2)}(1) = 6$

$f^{(3)}(x) = 6 \cdot (-4)x^{-5} = -24x^{-5}$, $\quad f^{(3)}(1) = -24.$

So the first four nonzero terms are

$f^{(0)}(1) = 1$, $\quad \frac{f^{(1)}(1)}{1!}(x-1) = -2(x-1)$,

$\frac{f^{(2)}(1)}{2!}(x-1)^2 = \frac{6}{2}(x-1)^2 = 3(x-1)^2$,

and $\frac{f^{(3)}(1)}{3!}(x-1)^3 = \frac{-24}{3!}(x-1)^3 = -4(x-1)^3$.

b) [8 pts] Use the Taylor series $\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k$ to find the first four nonzero terms of the Taylor series centered at 0 for the function $g(x) = \frac{x^2}{1+5x^3}$.

$$g(x) = x^2 \cdot \frac{1}{1+(5x^3)} = x^2 \sum_{k=0}^{\infty} (-1)^k (5x^3)^k$$

$$= x^2 \sum_{k=0}^{\infty} (-1)^k 5^k x^{3k}$$

$$= \sum_{k=0}^{\infty} (-1)^k 5^k x^{3k+2}$$

Problem 5

[22 pts] Integration.

a) [14 pts] Evaluate the integral. Write your answer in terms of w and **simplify**.

$$\int \frac{1}{w(a^2+w^2)}\,dw, \quad \text{where } a > 0 \text{ is a constant.}$$

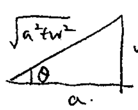
$$\left(\begin{array}{l} w = a\tan\theta, \quad dw = a\sec^2\theta\, d\theta \\ a^2+w^2 = a^2\sec^2\theta, \end{array} \right)$$

$$= \int \frac{a\sec^2\theta\, d\theta}{a\tan\theta\,(a^2\sec^2\theta)} = \frac{1}{a^2}\int \frac{d\theta}{\tan\theta}$$

$$= \frac{1}{a^2}\int \frac{\cos\theta}{\sin\theta}\,d\theta \qquad \left(\begin{array}{l} u = \sin\theta \\ du = \cos\theta\, d\theta \end{array} \right)$$

$$= \frac{1}{a^2}\int \frac{du}{u} = \frac{1}{a^2}\ln|u| + C$$

$$= \frac{1}{a^2}\ln|\sin\theta| + C$$



$$\tan\theta = \frac{w}{a} \Rightarrow \sin\theta = \frac{w}{\sqrt{a^2+w^2}}$$

$$= \frac{1}{a^2}\ln\left|\frac{w}{\sqrt{a^2+w^2}}\right| + C.$$

Method 2.

$$\frac{1}{w(a^2+w^2)} = \frac{A}{w} + \frac{Bw+C}{a^2+w^2}$$

$$= \frac{A(a^2+w^2)+w(Bw+C)}{w(a^2+w^2)} = \frac{(A+B)w^2+Cw+Aa^2}{w(a^2+w^2)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ C=0 \\ Aa^2=1 \end{cases} \Rightarrow A = \frac{1}{a^2},\ B = -\frac{1}{a^2},\ C=0$$

$$\Rightarrow \int \frac{1}{w(a^2+w^2)}\,dw = \frac{1}{a^2}\int\frac{dw}{w} - \frac{1}{a^2}\int\frac{w\,dw}{a^2+w^2}$$

$$= \frac{1}{a^2}\ln|w| - \underbrace{\frac{1}{a^2}\int\frac{w\,dw}{a^2+w^2}}_{\text{use } u=a^2+w^2,\ du=2w\,dw}$$

$$= \frac{1}{a^2}\ln|w| - \frac{1}{2a^2}\int\frac{du}{u}$$

$$= \frac{1}{a^2}\ln|w| - \frac{1}{2a^2}\ln|u| + C$$

$$= \frac{1}{a^2}\ln|w| - \frac{1}{2a^2}\ln|a^2+w^2| + C.$$

same answer ☺

b) [8 pts] Determine whether the improper integral converges or diverges.
If it converges, find its limit (which may involve the constant a.)

$$\int_a^\infty \frac{1}{w(a^2+w^2)}\,dw, \quad \text{where } a > 0 \text{ is a constant.}$$

$$= \lim_{b\to\infty}\int_a^b \frac{1}{w(a^2+w^2)}\,dw = \lim_{b\to\infty}\left(\frac{1}{a^2}\ln\left|\frac{w}{\sqrt{a^2+w^2}}\right|\Bigg|_a^b\right) = \lim_{b\to\infty}\frac{1}{a^2}\left(\ln\frac{b}{\sqrt{a^2+b^2}} - \ln\frac{a}{\sqrt{a^2+a^2}}\right)$$

$$= \frac{1}{a^2}\lim_{b\to\infty}\ln\left(\frac{b}{\sqrt{a^2+b^2}}\right) - \frac{1}{a^2}\ln\frac{a}{\sqrt{2}\,a} = \frac{1}{a^2}\ln 1 - \frac{1}{a^2}\ln\frac{1}{\sqrt{2}}$$

$$= \frac{\ln\sqrt{2}}{a^2} = \frac{\ln 2}{2a^2}.$$

So it converges to $\frac{\ln 2}{2a^2}$.

A Few Trigonometric Identities

1) $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$

2) $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$

3) $\cos^2\theta + \sin^2\theta = 1$

4) $\sec^2\theta - \tan^2\theta = 1$

5) $\csc^2\theta - \cot^2\theta = 1$

A Few Reduction Formulas

Assume n is a positive integer.

1) $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$

2) $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

3) $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx, \quad n \neq 1$

4) $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx, \quad n \neq 1$