Math 1172

Midterm 2

Spring 2014

Form A

Name: ______________________

OSU username (name.nn): ______________________

Lecturer: ______________________

Recitation Instructor: ______________________

Recitation Time: ______________________

Instructions

•) SHOW ALL WORK!!! Incorrect answers with work shown may receive partial credit, but unsubstantiated answers may receive NO credit.

•) Give EXACT answers unless asked to do otherwise.

•) You do not need to simplify numerical answers such as $\frac{5}{\sqrt{8}} - \frac{5}{\sqrt{12}}$.

•) Calculators are permitted EXCEPT those calculators that have computer algebra systems (CAS) or ability to communicate with others. Furthermore, all memory must be cleared and all apps must be removed. PDA's, laptops, and cell phones are prohibited. Do not have these devices out!

•) The exam duration is 55 minutes.

•) The exam consists of 5 problems starting on page 2 and ending on page 6. Make sure your exam is not missing any pages before you start. Page 7 contains formulas. Page 8 may be used for extra work space.

Problem Number	Maximum Point Value	Score
1	18	
2	21	
3	21	
4	18	
5	22	
Total	100	

Problem 1

[18 pts] True or False. Give a brief explanation or example to justify your answer.

False . a) [3 pts] If $\lim_{k\to\infty} a_k = 0$ then the series $\sum_{k=1}^{\infty} a_k$ converges.

conter example: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

False b) [3 pts] $e = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$e^1 = \underline{1+1}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}$

↑ There are two "1"

False c) [3 pts] If the power series $\sum_{k=0}^{\infty} c_k(x+2)^k$ converges for $x = -4$, then it will converge for $x = 1$.

Conter example: $\sum_{k=0}^{\infty} \left(\frac{2}{5}\right)^k (x+2)^k$

$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{\left(\frac{2}{5}\right)^{k+1}(x+2)^{k+1}}{\left(\frac{2}{5}\right)^k (x+2)^k}\right| = \lim_{k\to\infty} \left|\frac{2}{5}\cdot(x+2)\right|$

$= \frac{2}{5}|x+2| \lim_{k\to\infty} 1$

$= \frac{2}{5}|x+2| < 1$

$|x+2| < \frac{5}{2} = 2.5$

Interval of convengence is

$-2.5 < x+2 < 2.5$

$-4.5 < x < 0.5$

$(-4.5, 0.5) = I$

We can see that 4 is in I but 1 is not in I

So the series converges $x=-4$ but diverges at $x=1$.

True . d) [3 pts] If $a_n = n^2 + 2$ for $n \geq 0$ then $a_{n+1} = a_n + 2n + 1$ for $n \geq 0$.

$a_n = n^2+2$

$a_{n+1} = (n+1)^2+2 = n^2+2n+1+2 = \underline{n^2+2}+2n+1 = a_n+2n+1$

True e) [3 pts] The improper integral $\int_0^1 \frac{1}{x^2}\,dx$ diverges.

$\int_0^1 \frac{1}{x^2}dx = \lim_{a\to 0} \int_a^1 \frac{1}{x^2}dx = \lim_{a\to 0} \int_a^1 x^{-2}dx = \lim_{a\to 0} \left(-x^{-1}\Big|_a^1\right)$

$= \lim_{a\to 0} \left(-1+\frac{1}{a}\right) = \infty$

True f) [3 pts] Suppose $f(x) = 2x^3 - 3x$. Then the 3^{rd}-order Taylor polynomial for f centered at $a = 1$ is $p_3(x) = -1 + 3(x-1) + 6(x-1)^2 + 2(x-1)^3$.

$f(a) = f(1) = 2-3 = -1$

$f'(x) = 6x^2-3 \quad f'(1) = 6-3 = 3$

$f''(x) = 12x \quad f''(1) = 12$

$f'''(x) = 12 \quad f'''(1) = 12$

$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$

$= -1 + 3(x-1) + 6(x-1)^2 + 2(x-1)^3$

Problem 2

[21 pts] Series and Power Series.

a) [7 pts] Determine if the following series converges or diverges. If the series converges, find its value.

$$\sum_{k=1}^{\infty} \frac{3^{k+2}}{7^k} = \frac{\text{1st term}}{1-r} = \frac{\frac{3^3}{7^1}}{1-\frac{3}{7}} = \frac{\frac{27}{7}}{\frac{4}{7}} = \frac{27}{4}$$

$$= \sum_{k=1}^{\infty} \frac{3^2 \cdot 3^k}{7^k}$$

$$= \sum_{k=1}^{\infty} 9 \cdot \left(\frac{3}{7}\right)^k$$

$-1 < r = \frac{3}{7} < 1$

This geometric series converges

b) [7 pts] Determine if the following series converges or diverges. If the series converges, find its value.

$$\sum_{k=1}^{\infty} \cos\left(\frac{k+3}{k^2+4}\right)$$

By divergence test, it diverges

$$\lim_{k\to\infty} \cos\left(\frac{k+3}{k^2+4}\right)$$

$= \cos(0)$

$= 1 \neq 0$

c) [7 pts] Find the radius of convergence for the power series.

$$\sum_{k=0}^{\infty} \frac{k^2}{7^k}(x-3)^k$$

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)^2(x-3)^{k+1}}{7^{k+1}} \cdot \frac{7^k}{k^2(x-3)^k}\right|$$

$$= \lim_{k\to\infty} \left|\frac{(x-3)(k+1)^2}{7 \cdot k^2}\right|$$

$$= \frac{|k-3|}{7} \lim_{k\to\infty} \left|\frac{(k+1)^2}{k^2}\right|$$

$= \frac{|x-3|}{7} \cdot 1 < 1$

$|x-3| < 7$

So the radius of convergence is 7.

Problem 3

[21 pts] Consider the sequence $b_n = 5$ for $n \geq 0$.

Also, consider the convergent series $\sum_{k=0}^{\infty} a_k = L$.

Evaluate the following. [There is no partial credit for problems on this page.]

a) [3 pts] $\lim_{n\to\infty} \frac{n^3 + b_n}{n!}$ $= \lim_{n\to\infty} \frac{n^3+5}{n!} = 0$, since $\{n^3+5\} << \{n!\}$ "Growth order"

b) [3 pts] $\lim_{n\to\infty} (a_n + b_n)$ $= \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = 0+5 = 5$

$\lim_{n\to\infty} a_n = 0$, since $\sum_{n=0}^{\infty} a_n$ converges

c) [3 pts] $\lim_{n\to\infty} \sum_{k=0}^{n} a_k$ $= \lim_{n\to\infty} S_n = L$, since $\sum_{n=0}^{\infty} a_n$ converges

d) [3 pts] $\sum_{k=0}^{\infty} b_k$ $= \sum_{k=0}^{\infty} 5 = \infty$

e) [3 pts] Expand and simplify the sum $\sum_{k=0}^{n} (a_{k+1} - a_k)$.

$\sum_{k=0}^{n} (a_{k+1} - a_k) = \cancel{a_1} - a_0 + \cancel{a_2} - \cancel{a_1} + \cancel{a_3} - \cancel{a_2} + \cdots \quad \cancel{a_n} - a_{n-1} + a_{n+1} - \cancel{a_n}$

$= a_{n+1} - a_0$

f) [3 pts] $\sum_{k=0}^{\infty} (a_{k+1} - a_k)$ $= \lim_{n\to\infty} \sum_{k=0}^{n} (a_{k+1} - a_k) = \lim_{n\to\infty} (a_{n+1} - a_0) = -a_0$

Since $\lim_{n\to\infty} a_{n+1} = 0$

g) [3 pts] $\sum_{k=0}^{\infty} (b_{k+1} - b_k)$ $= \sum_{k=0}^{\infty} (5-5) = \sum_{k=0}^{\infty} 0 = 0$

Problem 4

[18 pts] Taylor Series.

a) [8 pts] Find the first four nonzero terms of the Taylor series for $f(x)$ centered at a.

$f(x) = \sin x, \quad a = \pi$

$f(a) = f(\pi) = \sin(\pi) = 0$

$f'(x) = \cos x \quad f'(a) = f'(\pi) = \cos(\pi) = -1$

① The first nonzero term is $f'(a)(x-a) = -(x-\pi)$

$f''(x) = -\sin x \quad f''(a) = f''(\pi) = -\sin(\pi) = 0$

$f'''(x) = -\cos x \quad f'''(a) = f'''(\pi) = -\cos(\pi) = 1$

② The second nonzero term is $\frac{f'''(a)(x-a)^3}{3!} = \frac{1}{6}(x-\pi)^3$

$f^{(4)}(x) = \sin x \quad f^{(4)}(a) = \sin(\pi) = 0$

$f^{(5)}(x) = \cos x \quad f^{(5)}(a) = \cos(\pi) = -1$

③ The third nonzero term is $\frac{f^{(5)}(x)(x-a)^5}{5!} = -\frac{1}{120}(x-\pi)^5$

$f^{(6)}(x) = -\sin x \quad f^{(6)}(a) = -\sin\pi = 0$

$f^{(7)}(x) = -\cos x$

$f^{(7)}(a) = -\cos(\pi) = 1$

④ The fourth nonzero term is

$\frac{f^{(7)}(a)(x-a)^7}{7!} = \frac{1}{7!}(x-\pi)^7$

b) [10 pts] Use the first four nonzero terms of a Taylor series centered at $a = 0$ to approximate the definite integral.

$$\int_0^{0.2} \frac{1}{1+x^3}\,dx$$

$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k$

$\frac{1}{1+x^3} = \sum_{k=0}^{\infty} (-x^3)^k$

The first 4 nonzero terms are

$1 - x^3 + x^6 - x^9$

Then $\int_0^{0.2} \frac{1}{1+x^3}\,dx \approx \int_0^{0.2} (1 - x^3 + x^6 - x^9)\,dx$

$= x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{x^{10}}{10} \Big|_0^{0.2}$

$= 0.2 - \frac{1}{4}\cdot(0.2)^4 + \frac{1}{7}(0.2)^7 - \frac{1}{10}(0.2)^{10}$

Problem 5

[22 pts] Integration.

a) [14 pts] Evaluate the integral.

$$\int \frac{-16x}{(x-1)(x-3)^2}\,dx$$

$$\frac{-16x}{(x-1)(x-3)^2} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

$$-16x = A(x-3)^2 + B(x-1)(x-3) + C(x-1)$$

$$-16x = A(x^2-6x+9) + B(x^2-4x+3) + C(x-1)$$

So $\begin{cases} A+B=0 \\ -6A-4B+C=-16 \\ 9A+3B-C=0 \end{cases} \Rightarrow \begin{cases} A=-4 \\ B=4 \\ C=-24 \end{cases}$

Then $\dfrac{-16x}{(x-1)(x-3)^2} = \dfrac{-4}{x-1} + \dfrac{4}{x-3} - \dfrac{24}{(x-3)^2}$

$$\int \frac{-16x}{(x-1)(x-3)^2}\,dx = \int \frac{-4}{x-1} + \frac{4}{x-3} - \frac{24}{(x-3)^2}\,dx$$

$$= -4\ln|x-1| + 4\ln|x-3| - 24\int (x-3)^{-2}\,dx$$

$$= -4\ln|x-1| + 4\ln|x-3| - 24\,\frac{(x-3)^{-1}}{-2+1}$$

$$= -4\ln|x-1| + 4\ln|x-3| + \frac{24}{x-3} + C$$

$$= -\ln|x-1|^4 + \ln|x-3|^4 + \frac{24}{x-3} + C$$

$$= \ln\left|\frac{x-3}{x-1}\right|^4 + \frac{24}{x-3} + C$$

b) [8 pts] Determine whether the improper integral converges or diverges. If it converges, find its limit.

$$\int_4^\infty \frac{-16x}{(x-1)(x-3)^2}\,dx$$

$$= \lim_{b\to\infty} \int_4^b \frac{-16x}{(x-1)(x-3)^2}\,dx$$

$$= \lim_{b\to\infty} \left(\ln\left|\frac{x-3}{x-1}\right|^4 + \frac{24}{x-3}\right)\Bigg|_4^b$$

$$= \lim_{b\to\infty} \ln\left(\frac{b-3}{b-1}\right)^4 + \frac{24}{b-3} - \left(\ln\left(\frac{4-3}{4-1}\right)^4 + \frac{24}{4-3}\right)$$

$$= \lim_{b\to\infty} 4\ln\left(\frac{b-3}{b-1}\right) + \lim_{b\to\infty}\frac{24}{b-3} - 4\ln\frac{1}{3} - 24$$

$$= 0+0-4\ln\frac{1}{3} - 24$$

$$= -4\ln\frac{1}{3} - 24$$

A Few Trigonometric Identities

1) $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$

2) $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$

3) $\cos^2\theta + \sin^2\theta = 1$

4) $\sec^2\theta - \tan^2\theta = 1$

5) $\csc^2\theta - \cot^2\theta = 1$

A Few Reduction Formulas

Assume n is a positive integer.

1) $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$

2) $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

3) $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx, \quad n \neq 1$

4) $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx, \quad n \neq 1$