Math 1172 — Name: ____________________

Midterm 2 — OSU username (name.nn): ____________________

Spring 2015 — Lecturer: ____________________

Recitation Instructor: ____________________

Form A — Recitation Time: ____________________

Instructions

•) SHOW ALL WORK!!! Incorrect answers with work shown may receive partial credit, but unsubstantiated answers may receive NO credit.

•) Give EXACT answers unless asked to do otherwise.

•) You do not need to simplify numerical answers such as $\frac{5}{\sqrt{8}} - \frac{5}{\sqrt{12}}$ unless asked to do otherwise.

•) NO CALCULATORS. NO CELL PHONES. NO ELECTRONIC DEVICES.

•) The exam duration is 55 minutes.

•) The exam consists of 5 problems starting on page 2 and ending on page 6. Make sure your exam is not missing any pages before you start. Page 7 contains formulas. Pages 7 and 8 may be used for extra work space.

Problem Number	Maximum Point Value	Score
1	18	
2	18	
3	22	
4	22	
5	20	
Total	100	

Problem 1

[18 pts] True or False. You do not need to show work for problems on this page.

F a) [2 pts] $\frac{50}{3} = 10 - 4 + \frac{8}{5} - \frac{16}{25} + \cdots$

False, because we don't know what is "..."

T b) [2 pts] The second order Taylor polynomial for $f(x) = \tan^{-1} x$ centered at 0 is $p_2(x) = x$.

$P_2(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 = 0 + x + 0 \cdot x^2 = x$.

$\left(f(0) = \tan^{-1}(0) = 0, \quad f'(0) = \frac{1}{1+x^2}\Big|_{x=0} = 1, \quad f''(0) = -\frac{2x}{(1+x^2)}\Big|_{x=0} = 0. \right)$

T c) [2 pts] The sequence $a_n = \frac{(-1)^n + n}{2n-1}$ converges and $\lim_{n\to\infty} a_n = \frac{1}{2}$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n + n}{2n-1} = \lim_{n\to\infty}\left(\frac{(-1)^n}{2n-1} + \frac{n}{2n-1}\right) = 0 + \frac{1}{2} = \frac{1}{2}$.

T d) [2 pts] If the series $\sum_{k=10}^{\infty} a_k$ converges then the series $\sum_{k=100}^{\infty} a_k$ converges.

True, because the convergence of a series does not depend on the first few (finitely many) terms.

T e) [2 pts] Given that $f(x) = \frac{x}{5x+1}$, then $\int_1^{\infty} f'(x)\,dx = \frac{1}{5} - \frac{1}{6}$.

$\int_1^{\infty} f'(x)\,dx = \lim_{b\to\infty} \int_1^b f'(x)\,dx = \lim_{b\to\infty}\left(f(x)\Big|_1^b\right) = \lim_{b\to\infty}\left(\frac{x}{5x+1}\Big|_1^b\right) = \lim_{b\to\infty}\left(\frac{b}{5b+1} - \frac{1}{6}\right)$

$= \frac{1}{5} - \frac{1}{6}$.

F f) [2 pts] $\lim_{n\to\infty}\left(1 + \frac{2}{n}\right)^n = 2e$

$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty}\left(\left(1 + \frac{1}{\left(\frac{n}{2}\right)}\right)^{\frac{n}{2}}\right)^2 = e^2 \neq 2e$

F g) [2 pts] $0 = 1 - 1 + 1 - 1 + \cdots$

First, we don't know "..."; Second, even if "..." repeats the pattern, we get the right side $= \sum_{n=0}^{\infty}(-1)^n$ which does not converge, since $\lim_{n\to\infty}(-1)^n$ DNE.

T h) [2 pts] Given $a_1 = 4$ and $a_{n+1} = 12 - 2a_n$ for $n \geq 1$, the first four terms of the sequence $\{a_n\}_{n=1}^{\infty}$ are positive.

$a_2 = 12 - 2a_1 = 12 - 8 = 4 > 0$

$a_3 = 12 - 2a_2 = 12 - 8 = 4 > 0$

$a_4 = 12 - 2a_3 = 12 - 8 = 4 > 0$.

F i) [2 pts] $\int \frac{1}{x\sqrt{4-x^2}}\,dx = \int \csc\theta\,d\theta$, where $x = 2\sin\theta$.

$\int \frac{1}{x\sqrt{4-x^2}}\,dx = \int \frac{2\cos\theta\,d\theta}{2\sin\theta \cdot 2\cos\theta}$
$= \frac{1}{2}\int \frac{d\theta}{\sin\theta} = \frac{1}{2}\int \csc\theta\,d\theta$

$\Leftarrow \begin{cases} dx = 2\cos\theta\,d\theta \\ 4 - x^2 = 4 - 4\sin^2\theta = 4(1-\sin^2\theta) = 4\cos^2\theta \\ \sqrt{4-x^2} = 2\cos\theta. \end{cases}$

Problem 2

[18 pts] Determine whether the series converges or diverges. If it converges, find its value. State any tests that you use.

a) [6 pts] $\sum_{k=0}^{\infty} \frac{8^k}{3^{2k}} = \sum_{k=0}^{\infty} \left(\frac{8}{9}\right)^k$ is a geometric series with ratio $\frac{8}{9}$.

it converges because $\left|\frac{8}{9}\right| < 1$.

So $\sum_{k=0}^{\infty} \frac{8^k}{3^{2k}} = \frac{1}{1-\frac{8}{9}} = \frac{1}{\frac{1}{9}} = 9$.

b) [6 pts] $\sum_{k=1}^{\infty} \left(\cos\left(\frac{\pi}{k}\right) - \cos\left(\frac{\pi}{k+1}\right)\right)$ is a telescoping series.

The sum of first n terms is

$$S_n = \sum_{k=1}^{n} \left(\cos\left(\frac{\pi}{k}\right) - \cos\left(\frac{\pi}{k+1}\right)\right) = \cancel{\text{co}}$$

$$= \left(\cos\frac{\pi}{1} - \cancel{\cos\frac{\pi}{2}}\right) + \left(\cancel{\cos\frac{\pi}{2}} - \cos\frac{\pi}{3}\right) + \cdots + \left(\cos\frac{\pi}{n-1} - \cancel{\cos\frac{\pi}{n}}\right) + \left(\cancel{\cos\frac{\pi}{n}} - \cos\frac{\pi}{n+1}\right)$$

$$= \cos\pi - \cos\frac{\pi}{n+1} = -1 - \cos\frac{\pi}{n+1}$$

$\lim_{n\to\infty} S_n = -1 - \cos 0 = -2$, So the series converges to -2.

c) [6 pts] $\sum_{k=1}^{\infty} \left(1 + \frac{3}{k}\right)^k$

Let $a_k = \left(1+\frac{3}{k}\right)^k$

Then $\lim_{k\to\infty} a_k = \lim_{k\to\infty}\left(1+\frac{3}{k}\right)^k = \lim_{k\to\infty}\left(1+\frac{1}{\left(\frac{k}{3}\right)}\right)^{\frac{k}{3}\cdot 3} = e^3 \neq 0$

So the divergence test implies that the series $\sum_{k=1}^{\infty}\left(1+\frac{3}{k}\right)^k$ diverges.

Problem 3

[22 pts] Consider the function f and its Taylor series $f(x) = \sum_{k=0}^{\infty} \frac{3^k x^k}{(k+1)!}$.

a) [8 pts] Find the radius of convergence for the power series.

$$r = \lim_{k\to\infty} \left| \frac{\frac{3^{k+1}x^{k+1}}{(k+1)!}}{\frac{3^k \cdot x^k}{k!}} \right| = \lim_{k\to\infty} \left| \frac{3^{k+1} x^{k+1}}{(k+1)!} \cdot \frac{k!}{3^k x^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{3x}{k+1} \right| = \lim_{k\to\infty} \frac{3}{k+1} \cdot |x| = 0 < 1 \quad \text{for all } x.$$

So the ratio test implies that $\sum_{k=0}^{\infty} \frac{3^k x^k}{(k+1)!}$ converges absolutely for all x. Thus, the radius of convergence is ∞.

b) [6 pts] By inspecting the coefficients of the Taylor series for f centered at 0 (given above), what are the values of $f(0)$, $f'(0)$, and $f''(0)$?

$$f(x) = \sum_{k=0}^{\infty} \frac{3^k x^k}{(k+1)!} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \Rightarrow f^{(k)}(0) = \frac{k!\,3^k}{(k+1)!} = \frac{3^k}{k+1}$$

$$\Rightarrow f(0) = f^{(0)}(0) = \frac{3^0}{0+1} = 1,\quad f'(0) = \frac{3^1}{1+1} = \frac{3}{2},\quad f''(0) = \frac{3^2}{2+1} = 3$$

c) [4 pts] Find a power series representation for $f'(x)$.

$$f(x) = \underset{k=0}{\underset{\uparrow}{1}} + \sum_{k=1}^{\infty} \frac{3^k x^k}{(k+1)!} \Rightarrow f'(x) = (1)' + \sum_{k=1}^{\infty} \left(\frac{3^k x^k}{(k+1)!} \right)'$$

$$= \sum_{k=1}^{\infty} \frac{3^k k x^{k-1}}{(k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{3^{k+1}(k+1) x^k}{(k+2)!}$$

d) [4 pts] Find a power series representation for $g(x) = x^6 f(3x)$.

$$g(x) = x^6 \sum_{k=0}^{\infty} \frac{3^k (3x)^k}{(k+1)!} = x^6 \sum_{k=0}^{\infty} \frac{3^k \cdot 3^k x^k}{(k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{3^{2k} x^{k+6}}{(k+1)!} = \sum_{k=6}^{\infty} \frac{3^{2(k-6)} x^k}{(k-5)!}$$

Problem 4

[22 pts] Integration.

a) [14 pts] Evaluate the integral.

$$\int \frac{5x+7}{x^2+2x-3}\,dx \qquad x^2+2x-3 = (x-1)(x+3).$$

So $\dfrac{5x+7}{x^2+2x-3} = \dfrac{A}{x-1} + \dfrac{B}{x+3} = \dfrac{A(x+3)+B(x-1)}{(x-1)(x+3)}$.

$\Rightarrow 5x+7 = A(x+3)+B(x-1) = (A+B)x + (3A-B)$

$\Rightarrow \begin{cases} A+B=5 & \cdots ① \\ 3A-B=7 & \cdots ② \end{cases}$ $\quad ①+② \Rightarrow 4A=12 \Rightarrow A=3.$

plug $A=3$ in ① $\Rightarrow B=5-A=5-3=2.$

So $\displaystyle\int \frac{5x+7}{x^2+2x-3}\,dx = \int \frac{3}{x-1}+\frac{2}{x+3}\,dx.$

$= 3\ln|x-1| + 2\ln|x+3| + C.$

b) [8 pts] Determine whether the improper integral converges or diverges. If it converges, find its limit.

$$\int_{-3}^{0} \frac{5x+7}{x^2+2x-3}\,dx = \int_{-3}^{0}\frac{5x+7}{(x-1)(x+3)}\,dx$$ is improper b/c

$x=-3$ gives zero denominator

$= \lim\limits_{a\to -3^+} \displaystyle\int_a^0 \frac{5x+7}{x^2+2x-3}\,dx$

$= \lim\limits_{a\to -3^+} \left[\left(3\ln|x-1| + 2\ln|x+3|\right)\Big|_a^0 \right]$

$= \lim\limits_{a\to -3^+} \left[(3\ln 1 + 2\ln 3) - (3\ln|a-1| + 2\ln|a+3|) \right]$

diverges b/c $\lim\limits_{a\to -3^+} \ln|a-1| = \ln 4$ but $\lim\limits_{a\to -3^+} \ln|a+3| = -\infty.$

Problem 5

[20 pts] Evaluate the indefinite integral. Write your final answer in terms of w.

[You may find this identity useful at some point: $\sin(2\theta) = 2\sin\theta\cos\theta$.]

$$\int \frac{w^2}{(a^2+w^2)^2}\,dw \quad , \text{ where } a > 0 \text{ is a constant}$$

Use trig. substitution $w = a\tan\theta$.

$dw = a\sec^2\theta\, d\theta$, $a^2+w^2 = a^2(1+\tan^2\theta) = a^2\sec^2\theta$

So $$\int \frac{w^2\,dw}{(a^2+w^2)^2} = \int \frac{a^2\tan^2\theta \cdot a\sec^2\theta\, d\theta}{(a^2\sec^2\theta)^2}$$

$$= \int \frac{a^3\tan^2\theta\cdot\sec^2\theta}{a^4\sec^4\theta}\,d\theta$$

$$= \frac{1}{a}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

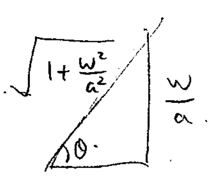
$$= \frac{1}{a}\int \sin^2\theta\, d\theta$$

$$= \frac{1}{a}\int \frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= \frac{1}{2a}\int 1-\cos 2\theta\, d\theta$$

$\theta = \tan^{-1}\left(\frac{w}{a}\right)$

$$= \frac{1}{2a}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = \frac{1}{2a}\left(\theta - \sin\theta\cos\theta\right) + C$$



$$= \frac{1}{2a}\left(\tan^{-1}\left(\frac{w}{a}\right) - \frac{w}{\sqrt{a^2+w^2}}\,\frac{a}{\sqrt{a^2+w^2}}\right) + C$$

$$\sin\theta = \frac{\frac{w}{a}}{\sqrt{1+\frac{w^2}{a^2}}} = \frac{w}{\sqrt{a^2+w^2}}$$

$$\cos\theta = \frac{1}{\sqrt{1+\frac{w^2}{a^2}}} = \frac{a}{\sqrt{a^2+w^2}}$$