Spherical varieties and tropical geometry (continued)

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The plan for the lectures

1. Advertisement
2. Toric varieties
3. Flag varieties
4. Spherical varieties
5. Jason Miller's research program
6. Tropical geometry
7. Tropical geometry in the spherical world

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- This is intended as a prospectus of research, and I'm looking for students who want to work on this project.
- Lecture slides are available at *u.osu.edu/kennedy.28* .

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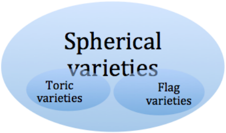
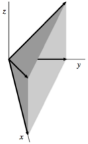
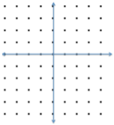
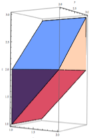
- The theme: to study aspects of algebraic geometry through combinatorial and piecewise linear methods.

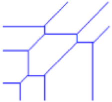




Cones


Fans


Convex bodies


Tropical varieties

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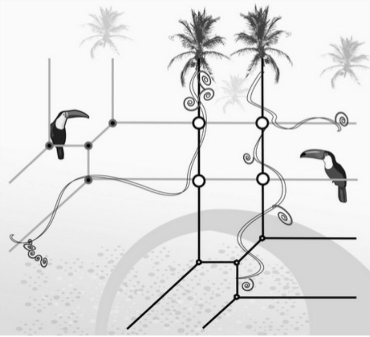
- The goal: To find notions akin to those of *tropical geometry* in the wider world of *spherical varieties* .



(image by Cowdery and Challas, featured in June 2009 Mathematics Magazine)

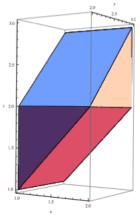
- A *toric variety* X is an irreducible algebraic variety containing an open subset which is an *algebraic torus* $T = (\mathbf{C}^*)^n$, such that the action of $(\mathbf{C}^*)^n$ on itself extends to an action on X.
- The study of toric varieties leads one to *cones* , *fans* , and *polytopes* .

Flag varieties

- Let $G = SL_n$ and $B = \{\text{upper triangular matrices}\}$. Then G/B parametrizes *complete flags* in $\mathbf{C}^n$:

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1}.$$

- In G/B there are interesting *Schubert varieties* , studied via the classical *Schubert calculus* .
- Recently Kiritchenko et al have found a way to understand the Schubert calculus via an associated polytope.



(Figure from Kiritchenko, Smirnov, Timorin)

Spherical varieties

- A complex algebraic variety is a *spherical variety* if it's acted upon by a reductive group G and there is a dense orbit under the action of a Borel subgroup B.
- From a certain viewpoint, toric varieties and flag varieties are two opposite extreme cases.
- There's an active area of research devoted to extending combinatorial methods to the wider world of spherical varieties.

Jason Miller's research program

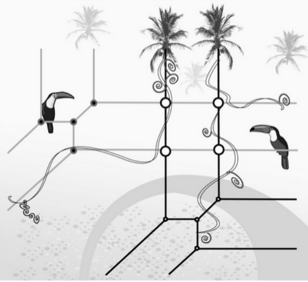
- Jason Miller (in his 2014 Ohio State dissertation) has helped to extend the theory of associated polytopes to this wider context.

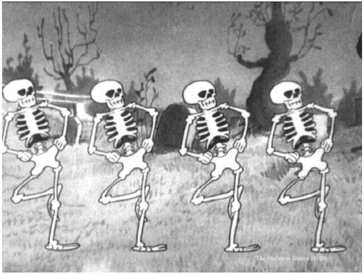
Tropical geometry

- A new subject *tropical geometry* has emerged out of discrete math, optimization, and computer science.
- It's named in honor of Brazilian mathematician Imre Simon (1943–2009).

www.ime.usp.br

- It's a piecewise linear or skeletonized version of algebraic geometry.





- Soon-to-be-standard textbook: *Introduction to Tropical Geometry* by Maclagan & Sturmfels, Graduate Studies in Math, AMS, to appear in Jan. 2015.

Mathematics Magazine, June 2009

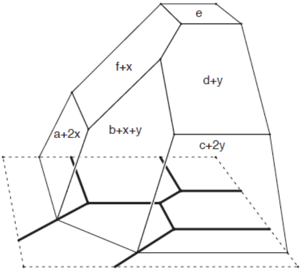
from Disney's 1929 "Skeleton Dance,"

A tropical plane curve

- Start with a polynomial $p(x, y) = ax^2 + bxy + cy^2 + dy + e + fx$.
- *Tropicalize* it by replacing all multiplications by $\otimes$ and additions by $\oplus$, where
  - $\otimes$ means $+$
  - $\oplus$ means "take the minimum."
- $\text{trop}(p) = \min\{a + 2x, b + x + y, c + 2y, d + y, e, f + x\}$
- What does the graph look like?

A tropical plane curve

- Assuming $2b < a + c, 2d < e + c, 2f < a + e$, here's the graph.

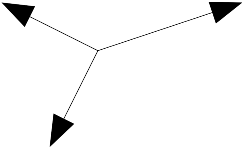


- It's linear on large regions, away from the locus where two (or more) of the six functions tie for achieving the minimum. This locus is called the *tropical curve* defined by trop(p).

Maclagan & Sturmfels

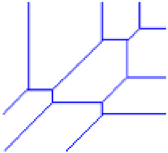
Tropical plane curves

- At each node, a *balancing condition* is satisfied: taking on each outward ray the smallest integer vector, these vectors sum to the zero vector.



Tropical plane curves

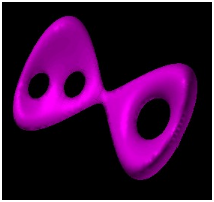
- The tropical curve consists of line segments and rays.
- The rays point north, east, and southwest, and there are the same number in each direction.
- This number is called the *degree* of the tropical plane curve.
- Here is a tropical cubic curve.



Sottile, Tropical interpolation

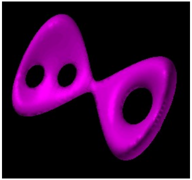
Classical facts about plane curves

- If you look at all the complex-valued points of a plane curve it's really a surface. (Basic idea: one complex dimension is the same as two real dimensions.)
- The surface will have a number of holes. This is called its *genus* .



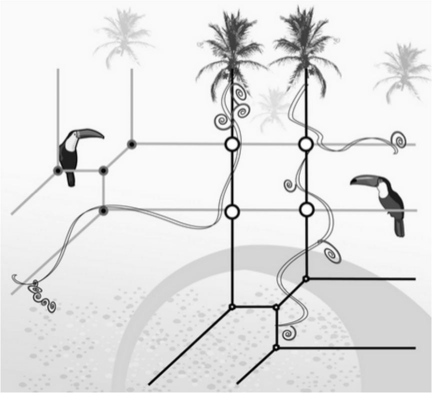
Classical facts about plane curves

- The genus of a curve of degree d is $\frac{(d-1)(d-2)}{2}$.



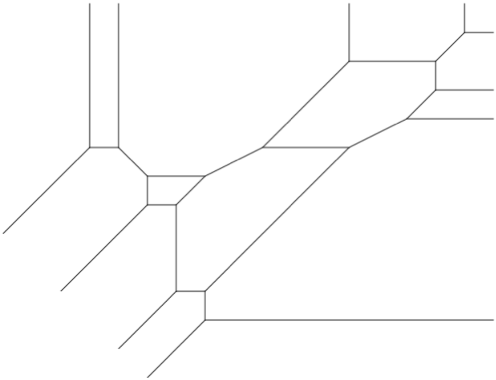
- Bézout's Theorem: Curves of degrees d and e meet in de points.
- In particular a curve of degree d has *self-intersection* d^2.
  - This is true, if you can properly interpret the notion: for example, as the degree of the normal bundle.

- Tropical Bézout

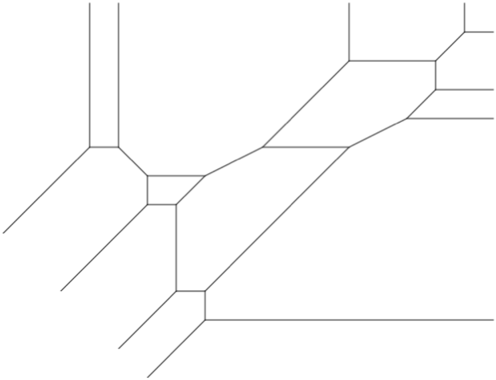


Tropical versions

- Tropical genus formula: by graph theory or Euler characteristic, the number of bounded polygons is $\frac{(d-1)(d-2)}{2}$.

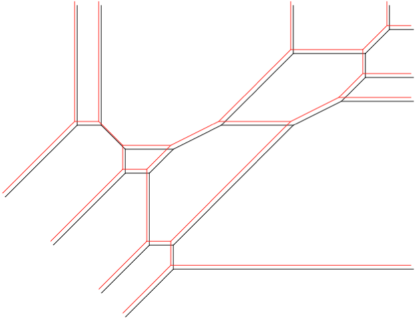


- Tropical self-intersection: in this example, there are supposed to be 16 self-intersection points. Where are they?



Tropical versions

- Idea of the calculation: jiggle the picture



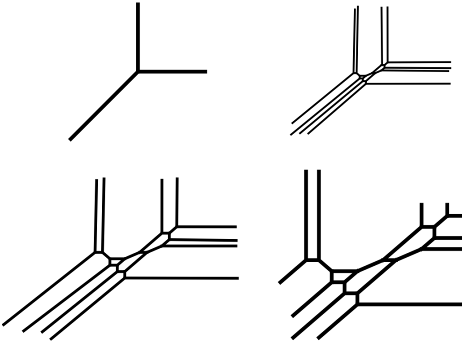
- The moral: tropical geometry may make your life easier!

Tropical geometry in the spherical world

- The rest of the talk is speculation.
- I claim: Starting with a spherical variety, one can associate something like a tropical variety, which records information about the variety together with the action.
- There is lots of circumstantial support for this claim. I will explain two pieces of evidence.

From a tropical curve to a fan

- If you look at this tropical quartic curve from far away, you just see three rays.



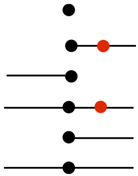
- This is the fan of the projective plane $\mathbf{P}^2$.
- This tells us that the tropical curve is a curve in $\mathbf{P}^2$.

Colored fans

- For spherical varieties we also have fans.
- The fans carry additional information, akin to information about Schubert varieties, and about how the Schubert varieties are related to the action. This additional information is called *colors* , and the fans are thus called *colored fans* .
- I'll show an example from *Introduction to spherical varieties and description of special classes of spherical varieties* by Boris Pasquier.

A colored fan

- Consider the natural action of SL_2 on $\mathbf{C}^2 \setminus \mathbf{0}$. The Borel subgroup $B = \{\text{upper triangular matrices}\}$ has a dense orbit, so this is a spherical variety. There are six possible colored fans, giving six ways of naturally adding on other orbits.



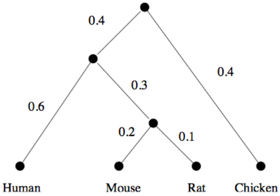
	SL_2/U-embedding X	SL_2-stable divisor(s)	closed SL_2-orbit(s)	color of X
1/	$\mathbb{C}^2$	$\varnothing$	$\{0\}$	D_α
2/	$\mathbb{P}^2 \setminus \{0\}$	D	D	$\varnothing$
3/	$\mathbb{P}^2$	D	D and $\{0\}$	D_α
4/	blow-up of 0 in $\mathbb{C}^2$	E	E	$\varnothing$
5/	blow-up of 0 in $\mathbb{P}^2$	D and E	D and E	$\varnothing$

Speculations

- There should be a theory of *colored tropical varieties* . They should somehow combine ordinary tropical varieties with colors.
- Whatever they may be, when viewed “from infinity” they ought to turn into colored fans.

You can tropicalize anything!

- The second bit of support for the desired theory is that there already is a notion of tropicalization for arbitrary varieties.
- You can tropicalize, e.g., the Grassmannian variety representing two-dimensional vector subspaces of $\mathbf{C}^4$. A point of the *tropical Grassmannian* is a metric tree with four leaves (a.k.a. a *phylogenetic tree*).



Speyer & Sturmfels, Tropical matheamatics

Huh?

- But wait — if there already is a theory, then why do I say we still need another theory?
- These general tropical varieties don't reflect anything about the group action. And thus they are way too big! In the tropical theory I'm looking for, things like flag varieties or Grassmannians should lead to single points. (We still have the colors, but the fans are gone.)
- This suggests that we might get to a useful theory by "cutting down" or "taking quotients" of the general tropical varieties.

How big should they be?

- For a toric variety, its dimension is the same as the dimension of the fan. Note in Pasquier's example above that the colored fans are one-dimensional, but the varieties are two-dimensional, and the group SL_2 is three-dimensional.
- For the sorts of objects I want, I think the expected dimension is the dimension of the maximal torus inside the group.

A reminder

- Lecture slides are available at *u.osu.edu/kennedy.28* .
- Also there, look for the “Reading list for tropical geometry and spherical varieties.”