The Gordon-Litherland pairing for knots and links in thickened surfaces

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1. **Knot Signature.**
In Cooper [1982 PhD thesis], Mandelbaum-Moishezon [1983], and Cimasoni-Turaev [2007, Osaka J Math], signatures are defined for homologically trivial knots in 3-manifolds.

In Im, Lee, Lee [2010, JKTR] and B-, Chrisman, Gaudreau [2020, Indiana Univ J Math], signature-type invariants are defined on various subcategories of virtual knots and links.

**Goal 1:** Provide more general definitions of signature invariants for knots and links in 3-manifolds, and for virtual knots and links.
In Greene [2017, Duke Math J], the GL pairing is extended to $\mathbb{Z}_2$ homology 3-spheres. He used it to give a geometric characterization of alternating links (cf. Howie [2017, Geom Topol]), and a new proof of the Tait conjectures.

Goal 2: Extend the GL pairing to more general 3-manifolds.
Use it to characterize alternating knots and links in 3-manifolds.
There are at least three ways to define the knot signature for classical knots.

1. [Trotter, Murasugi]
Let $K$ be a knot. Choose a Seifert surface $F$. The Seifert form $\Theta$ is given by $\Theta(\alpha, \beta) = \text{lk}(\alpha^-, \beta)$ for $\alpha, \beta \in H_1(F)$. Any matrix $V$ representing $\Theta$ on a basis for $H_1(F)$ is called a Seifert matrix. It is well-defined up to unimodular congruence.

The signature of $V + V^T$ is invariant under unimodular congruence and independent of choice of $F$.

**Definition**

The knot signature is given by $\sigma(K) = \text{sig}(V + V^T)$.
2. [Kauffman-Taylor] View $K \subset S^3 = \partial B^4$. Push $F$ into $D^4$, and let $M_F$ be the double cover of $D^4$ branched along $F$. Then $\partial M_F = X_2$, the double cover of $S^3$ branched along $K$. Note that $X_2$ is a $\mathbb{Z}_2$ homology 3-sphere.

The intersection form $Q: H_2(M_F) \times H_2(M_F) \to \mathbb{Z}$ is non-degenerate.

**Definition**

The knot signature is given by $\sigma(K) = \text{sig}(Q)$. 
3. [Gordon-Litherland]
Let $F$ be a spanning surface for $K$, not necessarily oriented.

*Gordon and Litherland* define a symmetric, bilinear pairing

$$G_F : H_1(F) \times H_1(F) \rightarrow \mathbb{Z}.$$ 

Its quadratic form specializes to the Trotter form when $F$ is a Seifert surface and to the Goeritz form when $F$ is the black (or white) surface of a checkerboard coloring.

Let $N$ be a tubular neighborhood of $F$, and set $\tilde{F} = \partial N$.

Then $\tilde{F} \rightarrow F$ is a double cover ($\tilde{F}$ is connected iff $F$ is not oriented).
Let $\tau: H_1(F) \to H_1(\tilde{F})$ be the transfer map. If $\alpha$ is a simple closed curve on $F$, then $\tau \alpha$ is the push-off of $\alpha$ in both directions.

**Definition**

1. The Gordon-Litherland pairing $\mathcal{G}_F: H_1(F) \times H_1(F) \to \mathbb{Z}$ is defined by setting $\mathcal{G}_F(\alpha, \beta) = \text{lk}(\tau \alpha, \beta)$.
2. The Euler number of $F$ is given by $e(F) = -\text{lk}(K, K')$, where $K'$ is a push-off of $K$ missing $F$.

**Remark.** If $F$ is oriented, then $\mathcal{G}_F$ coincides with the symmetrized Seifert pairing $V + V^T$ and $e(F) = 0$.

**Theorem (Gordon-Litherland (1978, Invent Math))**

(i) $\mathcal{G}_F$ is a symmetric bilinear pairing on $H_1(F)$.
(ii) $\sigma(K) = \text{sig}(\mathcal{G}_F) + e(F)/2$. 
Checkerboard coloring and incidence number

The GL pairing leads to a simple algorithm for computing the knot signature $\sigma(K)$ from a checkerboard coloring.

Given a checkerboard coloring, let $F$ be the spanning surface from the black regions. It is a union of disks and half-twisted bands.

Enumerate the white disks $X_0, \ldots, X_n$, they give a system of generators for $H_1(F)$. For each crossing $c$, set $\eta_c = \pm 1$ as below.

\[
\begin{array}{c}
\eta = 1 \\
\eta = -1
\end{array}
\]
For $i, j = 0, \ldots, n$, let

$$g_{ij} = \begin{cases} -\sum \eta_c & \text{if } i \neq j, \\ -\sum_{k \neq i} g_{ik} & \text{if } i = j. \end{cases}$$

The first sum is taken over all crossings $c$ incident to both $X_i$ and $X_j$.

The Goeritz matrix is given by $G = (g_{ij})_{i,j=1}^n$. It represents the GL pairing $\mathcal{G}_F$ on $H_1(F)$ with basis $\partial X_1, \ldots, \partial X_n$. 
There is also a simple formula for the correction term:

\[ e(F) = -2\mu(K), \]

where

\[ \mu(K) = \sum_{c \text{ type II}} \eta_c. \]

Here, type is determined by:

- Type I: \( \eta = -\varepsilon = -1 \)
- Type II: \( \eta = \varepsilon = -1 \)
Let $\Sigma$ be a compact, connected, oriented surface.

We extend the GL pairing to knots in $\Sigma \times I$ and use it to define signatures and determinants.

With more effort, the same results can be proved for links in $\Sigma \times I$. 
Asymmetric linking in $\Sigma \times I$

Given disjoint knots $J, K$ in $\Sigma \times I$, define $\text{lk}_\Sigma(J, K)$ to be the intersection of $J$ with a 2-chain $B$ with $\partial B = K + c$ for some 1-cycle in $\Sigma \times \{1\}$.

Then $\text{lk}_\Sigma(J, K)$ counts the number of times $J$ goes over $K$ with sign, where “above” is determined by the positive $I$ direction in $\Sigma \times I$.
Gordon-Litherland pairing in $\Sigma \times I$

Let $p : \Sigma \times I \to \Sigma$ denote the projection map.

Let $F \subset \Sigma \times I$ be a spanning surface for a knot $K \subset \Sigma \times I$.

Define the GL pairing $\mathcal{G}_F : H_1(F) \times H_1(F) \to \mathbb{Z}$ by setting

$$\mathcal{G}_F(\alpha, \beta) = \text{lk}_\Sigma(\tau \alpha, \beta) - p_\star[\alpha] \cdot p_\star[\beta],$$

where $\tau \alpha$ is again the push-off of $\alpha$ in both directions and $p_\star[\alpha] \cdot p_\star[\beta]$ is the algebraic intersection in $H_1(\Sigma)$.

**Lemma**

The GL pairing $\mathcal{G}_F : H_1(F) \times H_1(F) \to \mathbb{Z}$ is symmetric.

As before, $\text{sig}(\mathcal{G}_F)$ can be combined with a correction term to give a signature invariant for knots in thickened surfaces.
**Definition**

An $S^*$-equivalence of spanning surfaces consists of:

(a) ambient isotopy,
(b) attaching (or removing) a 1-handle,
(c) attaching (or removing) a small half-twisted band.

**Facts.** 1. Every classical knot admits a spanning surface.
2. Any two spanning surfaces for a classical knot are $S^*$-equivalent.
3. Neither is true for knots in thickened surfaces.
**Lemma**

If $F_1$ and $F_2$ are $S^*$-equivalent spanning surfaces in $\Sigma \times I$, then

$$\text{sig}(G_{F_1}) + \frac{1}{2} e(F_1) = \text{sig}(G_{F_2}) + \frac{1}{2} e(F_2).$$

Note that if $K'$ is the push-off of $K \subset \Sigma \times I$ which misses $F$, then $e(F) = -\text{lk}_\Sigma(K, K')$.

**Corollary**

Suppose $F \subset \Sigma \times I$ is a spanning surface for $K \subset \Sigma \times I$. Then

$$\sigma(K, F) = \text{sig}(G_F) + \frac{1}{2} e(F)$$

depends only on the $S^*$-equivalence class of $F$.

**Remark.** If $F$ is oriented, then $e(F) = 0$ and $\sigma(K, F) = \text{sig}(G_F)$ agrees with the signature of $K$ defined using the Seifert form $\Theta$. 
Determinant and nullity

One can also use this approach to define determinant and nullity invariants by taking

\[ \det(K, F) = |\det(G_F)| \quad \text{and} \quad n(K, F) = \text{nul}(G_F). \]

Again, \(|\det(G_F)|\) and \(\text{nul}(G_F)\) depend only on the \(S^*\)-equivalence class of \(F\).

**Figure:** An alternating knot with dual checkerboard colorings.
Let $F$ be the black surface on left and $F^*$ the *dual surface*.

Take basis $\alpha, \beta$ for $H_1(F)$, then $\mathcal{G}_F = \begin{bmatrix} -3 & -1 \\ -1 & -1 \end{bmatrix}$ and $e(F) = 4$. Thus $\sigma(K, F) = -2 + 4/2 = 0$ and $\det(K, F) = 2$.

Take basis $\alpha, \beta, \gamma$ for $H_1(F^*)$, then $\mathcal{G}_{F^*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $e(F^*) = -2$. Thus $\sigma(K, F^*) = 3 + -2/2 = 2$ and $\det(K, F) = 1$. 
Existence of spanning surfaces $\Sigma \times I$

**Fact.** For classical knots, spanning surfaces always exist and are unique up to $S^*$-equivalence ([GL, 1978], [Yasuhara, 2014 JKTR]).

For knots in $\Sigma \times I$ with $\Sigma \neq S^2$, the situation is more complicated. Firstly, existence is not guaranteed.

**Proposition**

If $K \subset \Sigma \times I$ is a knot in a thickened surface, then TFAE:

(i) $K$ is the boundary of a spanning surface $F \subset \Sigma \times I$,
(ii) the homology class $[K] = 0$ in $H_1(\Sigma \times I, \mathbb{Z}_2)$.

If either (i) or (ii) hold, then it is easy to see that $K$ admits a diagram on $\Sigma$ which is *checkerboard colorable*. 
Uniqueness of spanning surfaces $\Sigma \times I$

Given a knot $K \subset \Sigma \times I$ with coloring $\xi$, let $F$ be the black surface and $F^*$ the dual surface.

**Lemma**

Suppose $K \subset \Sigma \times I$ is a checkerboard colorable knot and $g(\Sigma) > 0$.

(i) If $F_1$ and $F_2$ are $S^*$-equivalent spanning surfaces, then $[F_1] = [F_2]$ in $H_2(\Sigma, K; \mathbb{Z}_2)$.

(ii) Any spanning surface is $S^*$-equivalent to either $F$ or the dual surface $F^*$.

**Remark.** $F$ and $F^*$ are not $S^*$-equivalent unless $\Sigma = S^2$. Thus, signatures, determinants, and nullities take on two possible values.
Virtual knots were introduced by Kauffman [1999, Eur J Comb] as virtual knot diagrams up to generalized Reidemeister moves.

Alternatively, virtual knots can be represented as knots in thickened surfaces up to stable equivalence Carter, Kamada, Saito [2002, JKTR].

**Stabilization** is the addition of a handle to $\Sigma$ disjoint from $K$, and destablization is the removal of a handle.

A knot $K \subset \Sigma \times I$ is said to be **minimal** if it is not isotopic to one that admits a destabilization.

*Kuperberg* showed that for a virtual knot, any minimal representative $K \subset \Sigma \times I$ is unique up to diffeomorphism of $\Sigma \times I$. 
Detecting the virtual genus

**Definition**

The *virtual genus* of a virtual knot is the genus $g(\Sigma)$ of a minimal representative $K \subset \Sigma \times I$.

**Definition**

A knot $K \subset \Sigma \times I$ is said to be *cellularly embedded* if $\Sigma \setminus p(K)$ is a union of disks, where $p: \Sigma \times I \to \Sigma$.

**Theorem**

Suppose $K \subset \Sigma \times I$ is cellularly embedded and checkerboard colorable with coloring $\xi$. If $\det(K, F) \neq 0$ and $\det(K, F^*) \neq 0$, then $K$ is a minimal representative for its virtual knot.
Chromatic duality

Let $F' = F \#_\tau \Sigma$ be obtained by tubing $F$ to a parallel copy of $\Sigma$. Then $F'$ is $S^*$-equivalent to the dual surface $F^*$ with $e(F') = e(F)$.

**Theorem**

Let $F \subset \Sigma \times I$ be a spanning surface such that the map $H_1(F) \to H_1(\Sigma \times I)$ is surjective. Set $\mathcal{H} = \text{Ker}(H_1(F) \to H_1(\Sigma \times I))$. Then $\text{sig}(\mathcal{G}_{F'}) = \text{sig}(\mathcal{G}_F|_\mathcal{H})$, the restriction of $\mathcal{G}_F$ to $\mathcal{H}$.

A similar statement holds for knot determinant and nullity.

This result is useful, as it allows computation of both sets of invariants from the same surface $F$.

**Remark.** If $K$ is cellularly embedded and checkerboard colorable, then $F$ and its dual $F^*$ satisfy the hypothesis of the theorem.
Example

Then $G_F = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, so $\sigma(K, F) = 2$ and $\det(K, F) = 4$.

Since $\mathcal{H} = 0$, it is trivial that $\sigma(K, F') = 0$ and $\det(K, F') = 1$. 
Chromatic duality

Using Goeritz matrices, Im, Lee, and Lee defined signature, determinant, and nullity invariants for checkerboard colorable virtual knots [2010 JKTR].

**Corollary**

If \( K \subset \Sigma \times I \) is checkerboard colored with coloring \( \xi \) with black surface \( F \) and dual surface \( F^* \).

Then the signature from the GL pairing and the Goeritz matrices are dually equivalent. In particular,

\[
\sigma(K, F) = \sigma^{\text{ILL}}_{\xi^*}(K) \quad \text{and} \quad \sigma(K, F^*) = \sigma^{\text{ILL}}_{\xi}(K).
\]

A similar statement holds for knot determinant and nullity.
**Fact.** Alternating virtual knots are all checkerboard colorable. A diagram is alternating iff every crossing has the same incidence number.

**Convention.** All crossings have incidence $\eta_c = -1$.

**Theorem**

If $K$ is an alternating diagram on a surface $\Sigma$ with black and white spanning surfaces $B$ and $W$. Then the Gordon-Litherland pairing $G_B$ and $G_W$ are definite and of opposite sign.

**Remark.** With the above convention, $G_B$ will be negative definite and $G_W$ will be positive definite. Notice that $\det(K, B) \neq 0 \neq \det(K, W)$.

**Corollary**

If $K$ is an alternating virtual knot diagram, then $K$ has minimal genus.
Theorem

A checkerboard colorable knot $K$ in a thickened surface $\Sigma \times I$ is alternating iff it admits positive and negative definite spanning surfaces.

Remark. This extends the results of Greene and Howie and gives a topological characterization of alternating virtual knots.
B–, Micah Chrisman, and Homayun Karimi
*Gordon-Litherland pairing and signatures of virtual knots*
in preparation (2020)

B– and Homayun Karimi
*A characterization of alternating links in thickened surfaces*
arXiv/2010.14030

Homayun Karimi
*Alternating virtual knots*
Thank you for your attention!