The geometric content of Tait’s conjectures

Ohio State CKVK* seminar

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Historical background: Tait’s conjectures, Fox’s question

Tait’s conjectures (1898)

Let $D$ and $D'$ be reduced alternating diagrams of a prime knot $L$. (Prime implies $\not\exists T_1 T_2$; reduced means $\not\exists \tau \tau'$. Then:

1. $D$ and $D'$ minimize crossings: $|\bigotimes_D| = |\bigotimes_{D'}| = c(L)$.
2. $D$ and $D'$ have the same writhe: $w(D) = w(D') = |\bigotimes_{D'} - |\bigotimes_D|$. 
3. $D$ and $D'$ are related by flype moves:

Question (Fox, $\sim$ 1960)

What is an alternating knot?

Tait’s conjectures all remained open until the 1985 discovery of the **Jones polynomial**. Fox’s question remained open until 2017.
Historical background: Proofs of Tait’s conjectures

In 1987, Kauffman, Murasugi, and Thistlethwaite independently proved (1) using the Jones polynomial, whose degree span is $|\mathcal{X}|_D$, e.g. $V_\bigcirc(t) = t + t^3 - t^4$. Using the knot signature $\sigma(L)$, (1) implies (2).

In 1993, Menasco-Thistlethwaite proved (3), using geometric techniques and the Jones polynomial. Note: (3) implies (2) and part of (1).

They asked if purely geometric proofs exist. The first came in 2017....

Tait’s conjectures (1898)

Given reduced alternating diagrams $D, D'$ of a prime knot $L$:

(1) $D$ and $D'$ minimize crossings: $|\mathcal{X}|_D = |\mathcal{X}|_{D'} = c(L)$.

(2) $D$ and $D'$ have the same writhe: $w(D) = w(D') = |\mathcal{X}|_{D'} - |\mathcal{X}|_{D'}$.

(3) $D$ and $D'$ are related by flype moves:

![Diagrams showing flype moves](attachment:image.png)
Historical background: geometric proofs

**Question (Fox, ~ 1960)**

What is an alternating knot?

**Theorem (Greene; Howie, 2017)**

A knot \( L \subset S^3 \) is alternating \( \text{IFF} \) it has spanning surfaces \( F_+ \) and \( F_- \) s.t.:

- **Howie**: \( 2(\beta_1(F_+) + \beta_1(F_-)) = s(F_+) - s(F_-) \).
- **Greene**: \( F_+ \) is positive-definite and \( F_- \) is negative-definite.

Using lattice flows, Greene applied his characterization to prove:

**Theorem (Greene, 2017)**

Any reduced alternating diagrams \( D, D' \) of the same knot satisfy \( |\mathcal{X}|_D = |\mathcal{X}|_{D'} \) and \( w(D) = w(D') \).

I will describe the first entirely geometric proof of the flyping theorem. This also implies the theorem above. Related problems remain open.
Outline

- Spanning surfaces $F$
  - Knot diagrams and chessboard surfaces
  - Complexity $\beta_1(F)$ and slope $s(F)$
  - Gordon-Litherland pairing $\langle \cdot, \cdot \rangle$ and signature $\sigma(F)$.
  - Greene’s characterization

- (Generalized) plumbing and re-plumbing
  - Essential surfaces
  - Flyping and re-plumbing
  - Crossing ball structures
  - Re-plumbing definite surfaces

- Geometric proof of the flyping theorem

- Related problems
Spanning surfaces

**Conventions:** Let $D, D' \subset S^2$ be reduced alternating diagrams of a prime alternating knot $L \subset S^3$; $\nu L, \nu F$, and $\nu S^2$ denote closed regular neighborhoods.

**Definition:** A spanning surface is a properly embedded surface $F \subset S^3 \setminus \nu L$ such that $\partial F$ intersects each meridian on $\partial \nu L$ transversally in one point, and $F$ is compact and connected, but not necessarily orientable.

**Definition:** $\beta_1(F) = \text{rank } H_1(F)$.

**Observation**

*If $\alpha$ consists of properly embedded disjoint arcs in a spanning surface $F$ and $F' = F \setminus \nu \alpha$ is a disk, then $\beta_1(F) = |\alpha|$.*
Chessboard surfaces

Color the regions of $S^2 \setminus D$ black and white in chessboard fashion and construct spanning surfaces $B$ and $W$ for $L$ like this:

$B$ and $W$ are called the **chessboard** surfaces from $D$. They intersect in *vertical arcs* which project to the the crossings of $D$: 
Denote projection $p : \nu F \to F$. Given any oriented simple closed curve (s.c.c.) $\gamma \subset F$, denote $\tilde{\gamma} = \partial(p^{-1}(\gamma))$, and orient $\tilde{\gamma}$ following $\gamma$.

Gordon-Litherland define a symmetric bilinear pairing

$$\langle \cdot, \cdot \rangle : H_1(F) \times H_1(F) \to \mathbb{Z}$$

$$\langle [\alpha], [\beta] \rangle = \text{lk}(\alpha, \tilde{\beta}).$$

The framing of a s.c.c. $\gamma \subset F$ is $\frac{1}{2} \langle [\gamma], [\gamma] \rangle$.

**Examples:** The pairings for $B$ and $W$ shown left are represented by

$$\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix},$$

that for $B$ right by

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$
Boundary slopes

The **euler number** $e(F)$ is the algebraic self-intersection number of the properly embedded surface obtained by perturbing $F$ in $B^4$. Alternatively,

$$-e(F) = \frac{1}{2} \sum_{i=1}^{m} \langle [\ell_i], [\ell_i] \rangle,$$

where $\partial F = \ell_1 \sqcup \ldots \sqcup \ell_m$. Call $s(F) = -e(F)$ the **slope** of $F$.

**Example:** $B$ and $W$ shown right have $s(W) = 0$ and $s(B) = 6$. This is because $W$ is orientable and, denoting a generator of $H_1(B)$ by $g$:

$$s(B) = \frac{1}{2} \langle [\partial B], [\partial B] \rangle = \frac{1}{2} \langle 2g, 2g \rangle = 2 \langle g, g \rangle = 6.$$
Boundary slopes and signatures

If $F$ spans a knot $L$ and $\hat{L}$ is a pushoff of $L$ in $F$, then the slope of $F$ is

$$s(F) = -\frac{1}{2} \langle [L], [L] \rangle = \text{lk} \left( L, \hat{L} \right),$$

which equals the framing of $L$ in $F$. The **signature of $F$**, denoted $\sigma(F)$, is the number of positive eigenvalues of $\langle \cdot, \cdot \rangle$ minus the number of negative eigenvalues.

Gordon-Litherland show that the quantity $\sigma(F) - \frac{1}{2}s(F)$ depends only on $L$. This is called the **knot signature**, denoted $\sigma(L)$.

**Example:** The surfaces $B$ and $W$ shown have slopes $s(B) = 6$ and $s(W) = 0$ and signatures $\sigma(B) = 1$ and $s(W) = -2$. Thus

$$\sigma \left( \bigcirc \bigcirc \right) = \begin{cases} \sigma(B) - \frac{1}{2}s(B) = 1 - 3 = -2 \\ \sigma(W) - \frac{1}{2}s(W) = -2 - 0 = -2. \end{cases}$$
Definite surfaces and Greene’s characterization

**Definition:** $F$ is **positive-definite** if $\langle \alpha, \alpha \rangle > 0$ for nonzero $\alpha \in H_1(F)$. This holds iff $\sigma(F) = \beta_1(F)$, also iff for each s.c.c. $\gamma \subset F$:

- The **framing** of $\gamma$ in $F$ is positive, or
- $\gamma$ bounds an orientable subsurface of $F$.

Greene’s characterization of alternating diagrams

A knot diagram is alternating iff its chessboard surfaces are definite surfaces of opposite signs.

Greene’s characterization of alternating links

If $B$ and $W$ are positive- and negative-definite spanning surfaces for a knot $L \subset S^3$, then $L$ has an alternating diagram $D$ whose chessboard surfaces are isotopic to $B$ and $W$.

Moreover, $D$ is **reduced** iff $\langle \alpha, \alpha \rangle \neq \pm 1$ for all $\alpha$ in $H_1(B), H_1(W)$.

Convention: The chessboard surfaces from $D$ and $D'$ are $B, W$ and $B', W'$, with $B, B'$ positive-definite and $W, W'$ negative-definite.
Recall that the knot signature \( \sigma(L) = \sigma(F) - \frac{1}{2}s(F) \) depends only on \( L \), and that \( \pm \)-definite surfaces \( F_\pm \) satisfy \( \sigma(F_\pm) = \pm \beta_1(F_\pm) \). This implies:

**Slope difference lemma**

*If \( F_\pm \), respectively, are \( \pm \)-definite spanning surfaces for \( L \), then*

\[
s(F_+) - s(F_-) = 2(\beta_1(F_+) + \beta_1(F_-)).
\]

I use the slope difference lemma and cut-and-paste arguments to prove:

**Definite intersection lemma (K)**

*If \( \alpha \) is a non-\( \partial \)-parallel arc of \( B \cap W \), then*

\[
i(\partial B, \partial W)_{\nu \partial \alpha} = 2.
\]
Two notions of essential surfaces

Definitions:

- \( F \) is **geometrically essential** if \( \exists \): 

- \( F \) is **\( \pi_1 \)-essential** if \( F \hookrightarrow S^3 \setminus L \) induces an injection of fundamental groups, and \( F \) is not a mobius band spanning the unknot.

Remarks: The following facts are classical applications of Dehn’s Lemma:

1. If \( F \) is \( \pi_1 \)-essential, then \( F \) is geometrically essential.
2. If \( F \) is 2-sided and geometrically essential, then \( F \) is \( \pi_1 \)-essential.
Plumbing and re-plumbing

Let $V \subset S^3 \setminus F$ be a properly embedded disk s.t.

- $\partial V$ bounds a disk $U \subset F$.
- Denoting $S^3 \setminus (U \cup V) = Y_1 \sqcup Y_2$, neither $F_i = F \cap Y_i$ is a disk.

Then $V$ is a **plumbing cap** for $F$, and $U$ is its **shadow**.

Say that $F$ is obtained by (generalized) **plumbing** $F_1$ and $F_2$ along $U$, denoted $F_1 * F_2 = F$. This operation is also called **Murasugi sum**.

The operation $F \to F' = (F \setminus U) \cup V$ is called **re-plumbing**, and can also be realized via proper isotopy through the 4-ball:
Murasugi sum is a natural geometric operation

A **Seifert surface** is an oriented spanning surface.

**Theorem (Gabai 1985 [3, 4])**

Let $F_1 \ast F_2 = F$ be a Murasugi sum—i.e. (generalized) plumbing—of Seifert surfaces, $\partial F_i = L_i$, $\partial F = L$. Then:

1. $F$ is **essential** if $F_1$ and $F_2$ are essential.
2. $F$ has **minimal** genus IFF $F_1$ and $F_2$ both have minimal genus.
3. $L$ is a **fibered knot with fiber** $F$ IFF each $L_i$ is fibered with fiber $F_i$.
4. $S^3 \setminus \nu L$ has a **nice codimension 1 foliation** IFF both $S^3 \setminus \nu L_i$ do.

Property (1) also holds for arbitrary (1- and 2-sided) spanning surfaces:

**Theorem (Ozawa 2011 [15])**

Let $F_1 \ast F_2 = F$ be a Murasugi sum of spanning surfaces. If $F_1$ and $F_2$ are **$\pi_1$-essential**, then $F$ is **$\pi_1$-essential**.

Changing “$\pi_1$-essential” to “geometrically essential” makes this false...
Plumbing needn’t respect geometric essentiality

Theorem (K)

A Murasugi sum of geometrically essential surfaces need not be geometrically essential.
Irreducible plumbing caps $V$ for $F$

A plumbing cap $V$ is **acceptable** if no arc of $\partial V \cap F$ is $\partial$-parallel and no arc of $\partial V \cap \partial \nu L$ is parallel in $\partial \nu L$ to $\partial F$.

If there is a properly embedded disk $X \subset S^3 \setminus (\nu L \cup F \cup V)$ like the one shown below, then then $V$ is **reducible**; if not, then $V$ is **irreducible**.

**Lemma**

*If $F$ and $F'$ are related by a sequence of re-plumbing moves, then each move in some such a sequence follows an acceptable, irreducible cap.*
**Apparent plumbing cap theorem**

*If $V$ is an irreducible plumbing cap for $B$ in “standard position,” then $V$ is apparent in $D$, as shown top-left:*

**Sketch of proof.**

Let $V_0$ be an outermost disk of $V \setminus W$ (bottom row). If $|V \cap W| = 1$, done. Else, (top-right), and $V$ is reducible.
Apparent plumbing caps correspond to flypes.

**Proposition**

If a flype $D_0 \rightarrow D_1$ follows a plumbing cap $V$ for $B_0$, then re-plumbing $B_0$ along $V$ gives a surface isotopic to $B_1$; also, $W_0$ is isotopic to $W_1$.

**Proof.**
We have shown that apparent plumbing caps correspond to flypes and:

**Lemma:** Any re-plumbing sequence can be refined to one in which each move follows an acceptable, irreducible cap.

**Apparent plumbing cap theorem:** If $V$ is an irreducible plumbing cap for $B$ in standard position, then $V$ is apparent in $D$.

**Proposition:** If $D_0 \rightarrow D_1$ is a flype (along an apparent plumbing cap), then $W_0$ and $W_1$ are related by re-plumbing or isotopy, as are $B_0$ and $B_1$.

**Flyping re-plumbing theorem**

$D$ and $D'$ are related by flypes IFF $B$ and $B'$ are related by re-plumbing and isotopy moves, as are $W$ and $W'$.

**Proof.**

The proposition gives one direction. For the converse, the lemma and theorem give re-plumbing sequences $B = B_0 \rightarrow \cdots \rightarrow B_m = B'$ and $W = W_m \rightarrow \cdots \rightarrow W_n = W'$ along apparent plumbing caps. The proposition then gives a flyping sequence

$$D_{B,W} = D_0 \rightarrow \cdots \rightarrow D_{m_{B',W}} \rightarrow \cdots \rightarrow D_{m+n_{B',W'}}.$$
Logical interlude: how to prove the flyping theorem

**Flyping re-plumbing theorem (shown)**

\( D \) and \( D' \) are related by flypes if and only if \( B \) and \( B' \) are related by isotopy and re-plumbing moves, as are \( W \) and \( W' \).

**Definite re-plumbing theorem (still need to show)**

Any essential positive- (resp. negative-) definite surface spanning \( L \) is related to \( B \) (resp. \( W \)) by isotopy and re-plumbing moves.

**Flyping theorem (will then follow)**

All reduced alternating diagrams of \( L \) are related by flypes.

**Proof of the flyping theorem (assuming definite re-plumbing theorem).**

Let \( D, D' \) be reduced alternating diagrams of a prime knot \( L \) with respective chessboard surfaces \( B, W \) and \( B', W' \), where \( B, B' \) are positive-definite. The definite re-plumbing theorem implies that \( B \) and \( B' \) are related by re-plumbing and isotopy moves, as are \( W \) and \( W' \). Thus, by the flyping re-plumbing theorem, \( D \) and \( D' \) are related by flypes.
Construct a tiny closed crossing ball $C_t$ at each crossing point $c_t$ of $D$, and denote $C = \bigsqcup_{t=1}^{n} C_t$. Adjust $D$ to embed $L$ in $(S^2 \setminus \text{int}(C)) \cup \partial C$.

Denote the two balls of $S^3 \setminus (S^2 \cup C \cup \nu L)$ by $H_{\pm}$, with $\partial H_+ = S_+$ and $\partial H_- = S_-$.  

To prove the definite re-plumbing theorem, we will put an arbitrary essential positive-definite surface $F$ in a “standard position” and consider innermost disks, etc.
Example: a spanning surface in the crossing ball setting
If $F$ is in standard position, then:

- Each component of $F \cap C$ is a **crossing band** or a **saddle disk**:

- Each crossing band in $F$ is disjoint from $S_+$:

Define the **complexity** of $F$ to be

$$||F|| = \#(\text{crossings without crossing bands}) + \#(\text{saddle disks})$$
Definite re-plumbing theorem

Any essential positive-definite spanning surface $F$ for $L$ is related to $B$ by isotopy and re-plumbing moves.

Sketch of proof.

Isotop $F$ into standard position with $||F||$ minimized. Modify an innermost circle of $F \cap S_+$ to get an annulus $A \subset S^2$. Cut $A$ it into rectangles $A_i$. Each prism $\pi^{-1}(A_i)$ intersects $F$ as shown left. As shown, there is a re-plumbing move which decreases $||F||$. Repeat this process until $||F|| = 0$, whence $F$ is isotopic to $B$. 

□
Flyping re-plumbing theorem (shown)

\[ D \text{ and } D' \text{ are related by flypes if and only if } B \text{ and } B' \text{ are related by isotopy and re-plumbing moves, as are } W \text{ and } W'. \]

Definite re-plumbing theorem (shown)

Any essential positive- (resp. negative-) definite surface spanning \( L \) is related to \( B \) (resp. \( W \)) by isotopy and re-plumbing moves.

Flyping theorem (shown)

All reduced alternating diagrams of \( L \) are related by flypes.

The flyping theorem immediately gives a new proof of the same part of Tait’s conjectures that Greene proved:

Theorem

Any two reduced alternating diagrams of the same knot have the same crossing number and writhe.

Yet, it does not follow that any reduced alternating diagram minimizes crossings. All existing proofs of this fact use the Jones polynomial.
### Geometric proofs: open problems

**Theorem**

*Any two reduced alternating diagrams of the same knot have the same crossing number and writhe.*

**Open problem**

*Give an entirely geometric proof that any reduced alternating knot diagram realizes the underlying knot’s crossing number.*

**Open problem**

*Give an entirely geometric proof that any reduced alternating tangle diagram realizes the underlying tangle’s crossing number.*

**Open problem**

*Give an entirely geometric proof that any adequate knot diagram realizes the underlying knot’s crossing number.*
Open problem

*Give an entirely geometric proof that any (reduced alternating knot / reduced alternating tangle / adequate) diagram minimizes crossings.*

One approach to these problems is to translate statements about diagrams to statements about chessboard surfaces, a la:

Howie’s characterization of alternating knots

*A knot in $S^3$ is alternating if and only if it has spanning surfaces $F_\pm$ which satisfy

$$2(\beta_1(F_+) + \beta_1(F_-)) = s(F_+) - s(F_-).$$

Alternatively, using the flyping theorem, one can extend any alternating knot to a spatial graph in a way that captures all symmetries, and all alternating diagrams, of the knot. Crossings become geometric objects:


Thank you!