The reductivity of knot projections

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§ 0. Outline

P: a knot projection

$r(P)$: the reductivity of $P$

Example

$r(\infty) = 0 \quad r(\bigtriangleup) = 1 \quad r(\bowtie) = 2 \quad r(\bigtriangledown) = 3$

Theorem (S.)

$r(P) \leq 4 \quad (\forall P)$

Reductivity problem

$\exists P \text{ s.t. } r(P) = 4$
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§ 1. Knot projections
We consider knot projections which have at least one crossing.
reducible crossing

Knot projections

reducible knot projections

reduced knot projections

How reduced are we??
§ 2. Half-twisted splice
Half-twisted splice (HS)

Example

Knot (not link)
Inverse-half-twisted splice ($HS^{-1}$)

Example

$HS^{-1}$

cf. smoothing

Knot

Link
Remark

We can obtain a reducible knot projection from any knot projection by a finite number of \( HS^{-1} \).

\[
\begin{align*}
8 \text{ crossings} & \xrightarrow{HS^{-1}} 7 \text{ crossings} & \xrightarrow{HS^{-1}} \ldots & \xrightarrow{HS^{-1}} 1 \text{ crossing}
\end{align*}
\]
§ 3. Reductivity
Reductivity - how much reduced??

**Definition**  
P: a knot projection

The *reductivity* $r(P)$ of $P$ is the minimal number of $HS^{-1}$ which are needed to obtain a reducible knot projection from $P$.

**Example**

![Diagram showing the process of reducing a knot projection](image)

$r(P) = 2$
Example

\[ r\left(\begin{array}{c}
\includegraphics[width=2cm]{example1}\n\end{array}\right) = 0 \quad r\left(\begin{array}{c}
\includegraphics[width=2cm]{example2}\n\end{array}\right) = 1 \]

\[ r\left(\begin{array}{c}
\includegraphics[width=2cm]{example3}\n\end{array}\right) = 2 \quad r\left(\begin{array}{c}
\includegraphics[width=2cm]{example4}\n\end{array}\right) = 3 \]

There exist infinitely many knot projections \( P \) with \( r(P) = 0, 1, 2, \) and \( 3. \)
Reductivity is four or less

Theorem 1 (S)

\[ r(P) \leq 4 \quad (\forall P) \]

Reductivity problem

\[ \exists \, P \text{ s.t. } r(P) = 4 \]

§ 4. 2-gons & 3-gons
2-gons & 3-gons

There are two types of 2-gons:

- incoherent 2-gon
- coherent 2-gon

There are four types of 3-gons:

- type A
- type B
- type C
- type D
Example

- incoherent 2-gon
- coherent 2-gon
- type A
- type B
- type C
- type D
2-gons

**Lemma 2**

If $P$ has an incoherent 2-gon, then $r(P) \leq 1$.

If $P$ has a coherent 2-gon, then $r(P) \leq 2$. 

[Diagram showing coherent and incoherent bigons]
3-gons

Lemma 3

If $P$ has a 3-gon of type $A$, then $r(P) \leq 2$.
If $P$ has a 3-gon of type $B$, then $r(P) \leq 3$.
If $P$ has a 3-gon of type $C$, then $r(P) \leq 3$.
If $P$ has a 3-gon of type $D$, then $r(P) \leq 4$. 
Corollary 4

If \( P \) has at least one of

- incoherent 2-gon
- coherent 2-gon
- 3-gon of type A
- 3-gon of type B
- 3-gon of type C

then \( r(P) \leq 3 \).
§5. Unavoidable sets
Definition: $S$: a set consisting of parts of knot projections.

$S$ is an **unavoidable set for a knot proj.** if every knot projection has at least one of the parts in $S$.

Example:

$$\{ \begin{array}{c}
\bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc \\
\bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc
\end{array} \}$$

is an unavoidable set for a **reduced knot projection**.

prove later
Theorem (Adams–Shinjo–Tanaka)

Every reduced knot projection has a 2-gon or 3-gon.

i.e., \( \{ \bigcirc, \bigtimes \} \) is an unavoidable set for a reduced knot projection.

Proof of AST' s theorem

\[ P: \text{a reduced knot projection} \]

\[ C_n: \text{the number of } n\text{-gons of } P \]

Euler' s characteristic

\[ \sum_k \frac{k C_k}{4} \]

\[ \sum_k \frac{k C_k}{2} \]

\[ \sum_k C_k \]

\[ v - e + f = 2 \]

\[ 2C_2 + C_3 = 8 + C_5 + 2C_6 + 3C_7 + \cdots \]

\[ C_2 \geq 0 \text{ or } C_3 > 0 \]

AST' s formula
Proof of Theorem 1

If $P$ is reducible, then $r(P) = 0$.
(by definition)

If $P$ is reduced, $P$ has a $2$-gon or $3$-gon.
(by AST's theorem)

If $P$ has a $2$-gon, then $r(P) \leq 2$.
(by Lemma 2)

If $P$ has a $3$-gon, then $r(P) \leq 4$.
(by Lemma 3)
Further unavoidable set

Lemma 5

\{ ♦, ♦♦, ⋅⋅, ⋅□, ⋅♦, ♦♦ \}

is an unavoidable set for a reduced knot projection.
Proof of Lemma 5

Use the “discharging method” from graph theory (four-color theorem)!

P: a reduced knot projection

Assume P does not have any part in \{\text{\includegraphics[width=0.2\textwidth]{knots.png}}\}. Then, ...
Give “charge” \((4-n)\) to each \(n\)-gon.

\[
\begin{array}{cccc}
3\text{-gon} & 4\text{-gon} & 5\text{-gon} & 6\text{-gon} \\
1 & 0 & -1 & -2 \\
\end{array}
\]

Then the total charge is...

\[
c_3 - c_5 - 2c_6 - 3c_7 - \cdots
\]

\[= 8\]

\(c_n\): the number of \(n\)-gons

\[2c_2 + c_3 = 8 + c_5 + 2c_6 + 3c_7 + \cdots\]
“Discharging” at every 3-gon to the neighbor six regions by $\frac{1}{6}$. 
After discharging...

Contradicts that the total charge is 8.

Hence \( \{ \text{\text{-}, } \text{\text{-}, } \text{\text{-}, } \text{\text{-}, } \text{\text{-}, } \text{\text{-}} \} \) is an un-avoidable set for a reduced knot proj.
§6. 4-gons & 5-gons
There are 13 types of 4-gons:
Lemma 6

If a knot projection $P$ has one of

\[ 2a \quad 2b \quad 3a \quad 4a \]

then $r(P) \leq 3$. 

$H_2 \leftarrow 2$
Unavoidable set for $P$ with $r(P)=4$

Theorem 7 (Onoda-S)

\[
\{ \begin{array}{c}
\begin{array}{cccc}
\bullet & 1b & 1b & 1b \\
2c & 4b & 1a & 1a \\
1a & 1b & 1b & 1b \\
3b & 3b & 3c & 3c \\
\end{array}
\end{array}
\} \]

is an unavoidable set for a knot projection with reductivity four.

There are 56 types of 5-gons:
Unavoidable set for \( P \) with \( r(P) = 4 \)

Theorem 8 (Kashiwabara-S)

is an unavoidable set for a knot projection with reducitivity four.

§7. 2-gons & 3-gons again
Question

Is \( \{ \text{incoherent 2-gon}, \text{coherent 2-gon}, 3\text{-gon of type A}, 3\text{-gon of type B}, 3\text{-gon of type C} \} \)

an unavoidable set for a reduced knot projection?

(If so, the reductivity problem is to be solved negatively, i.e., \( r(P) \leq 3 \) for any \( P \).)
NO!

This does not have \( \text{A} \), \( \text{B} \), or \( \text{C} \).

and has only \( \text{D} \).

However, the reductivity is not four!
Reductivity problem

\[ \exists P \text{ s.t. } r(P) = 4 \]
Thank you for watching!