**Example Problem: Photoelectric Effect**
Physics 1251
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1. **A piece of metal has a cutoff wavelength of $\lambda_{cutoff} = 450\,\text{nm}$. Consider illuminating this piece of metal with two different wavelengths of light: a $\lambda_1 = 500\,\text{nm}$ beam and a $\lambda_2 = 400\,\text{nm}$ beam. For each of the two beams, find:**

   (a) **The maximum kinetic energy of ejected electrons.**

   We can only eject an electron if the illuminating beams are energetic enough. Because $E = hf = hc/\lambda$, we can only eject an electron if $\lambda_{illuminating} \leq \lambda_{cutoff}$. So, for $\lambda_1$ ,which is *greater* than $\lambda_{cutoff}$, we eject no electrons.

   However, for $\lambda_2$, which is *less* than $\lambda_{cutoff}$, we do eject electrons. The energy of these electrons is given by:

   $$E = hf - \phi = hc/\lambda - \phi$$

   That is, we take the incoming energy of the photon ($hc/\lambda$), subtract off the energy required to bind the electron ($\phi$, the work function) and we are left with the kinetic energy of the electron.

   $$E = hf - \phi = \frac{hc}{\lambda_2} - \frac{hc}{\lambda_{cutoff}} = hc\Big(\frac{1}{\lambda_2} - \frac{1}{\lambda_{cutoff}}\Big)$$
   $$= (6.626 \times 10^{-34}\,\text{J} \cdot \text{s})(3 \times 10^{8}\,\text{m/s})\Big(\frac{1}{400 \times 10^{-9}} - \frac{1}{450 \times 10^{-9}}\Big)$$
   $$\Longrightarrow \boxed{E_2 = 5.21 \times 10^{-20}\,\text{J} = 0.344\,\text{eV}}$$

   (b) **What is their speed?**

   Because $\lambda_1$ never ejects electrons, it does not make sense to speak of their speed. For $\lambda_2$, we will simply solve using our kinetic energy formula from 1250:

   $$E = \frac{1}{2}mv^2 \Longrightarrow v = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2 \cdot 5.21 \times 10^{-20}\,\text{J}}{9.11 \times 10^{-31}\,\text{kg}}} \Longrightarrow \boxed{v_2 = 3.48 \times 10^{5}\,\text{m/s}}$$

   (c) **What is their de Broglie wavelength?**
   Again, because $\lambda_1$ never ejects electrons, it does not make sense to speak of their de Broglie wavelength. For $\lambda_2$, we can apply the de Broglie wavelength formula

   $$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34}\,\text{J} \cdot \text{s}}{9.11 \times 10^{-31}\,\text{kg} \cdot 3.48 \times 10^{5}\,\text{m/s}} \Longrightarrow \boxed{\lambda_{2,dB} = 2.09 \times 10^{-9}\,\text{m} = 2.09\,\text{nm}}$$

2. **An electron, a proton, and a photon each have a wavelength of $0.24\,\text{nm}$. For each one, find the momentum, the energy, and, where relevant, the accelerating voltage needed to achieve that wavelength:**
   The momentum calculation is the same for all three particles. We employ the de Broglie relation:

   $$p = \frac{h}{\lambda} \Longrightarrow p_p = p_e = \frac{6.626 \times 10^{-34}\,\text{J} \cdot \text{s}}{0.24 \times 10^{-9}\,\text{m}} \Longrightarrow \boxed{p_p = p_e = p_\gamma = 2.76 \times 10^{-24}\,\text{kg m/s}}$$

   Now, let's specialize to the massive particles first. For a massive particle, we can apply that $p = mv$ and $E = \frac{1}{2}mv^2$, to find: $E = \frac{1}{2}\frac{p^2}{m}$. So:

   $$E_e = \frac{1}{2}\frac{p_e^2}{m_e} = \frac{1}{2} \cdot \frac{(2.76 \times 10^{-24}\,\text{kg m/s})^2}{9.11 \times 10^{-31}\,\text{kg}} \Longrightarrow \boxed{E_e = 4.044 \times 10^{-18}\,\text{J} = 25.24\,\text{eV}}$$
   $$E_p = \frac{1}{2}\frac{p_p^2}{m_p} = \frac{1}{2} \cdot \frac{(2.76 \times 10^{-24}\,\text{kg m/s})^2}{1.67 \times 10^{-27}\,\text{kg}} \Longrightarrow \boxed{E_p = 2.206 \times 10^{-21}\,\text{J} = 0.014\,\text{eV}}$$

Because these are massive particles, they can be brought to this energy by an accelerating potential, given by $E = q\Delta V$:

$$|\Delta V_e| = \frac{E_e}{q_e} = \frac{4.044 \times 10^{-18}\,\mathrm{J}}{1.602 \times 10^{-19}\,\mathrm{C}} \Longrightarrow \boxed{V_e = 25.24\,\mathrm{V}}$$
$$|\Delta V_p| = \frac{E_p}{q_p} = \frac{2.206 \times 10^{-21}\,\mathrm{J}}{1.602 \times 10^{-19}\,\mathrm{C}} \Longrightarrow \boxed{V_e = 0.013\,\mathrm{V}}$$

For the photon, the *massless* particle, the calculation is more straightforward:

$$E_\gamma = hf = \frac{hc}{\lambda} = \frac{(6.626 \times 10^{-34}\,\mathrm{J \cdot s})(3 \times 10^8\,\mathrm{m/s})}{0.24 \times 10^{-9}\,\mathrm{m}} \Longrightarrow \boxed{E_\gamma = 8.28 \times 10^{-16}\,\mathrm{J} = 5.68\,\mathrm{keV}}$$

(Which is termed a "soft x-ray" in the astrophysics community, by the way.)

3. **What is the de Broglie wavelength of an electron that has $2.0\,\mathrm{keV}$ of kinetic energy? What about an electron with $200\,\mathrm{keV}$ of kinetic energy? The second one requires relativity–why?**

For the 2 keV electron, we can apply:

$$\lambda = \frac{h}{p} = \frac{h}{m \cdot v} = \frac{h}{m \cdot \sqrt{\frac{2E}{m}}} = \frac{h}{\sqrt{2Em}} = \frac{6.626 \times 10^{-34}\,\mathrm{J \cdot s}}{\sqrt{2 \cdot (3.2 \times 10^{-16}\,\mathrm{J}) \cdot (9.11 \times 10^{-31}\,\mathrm{kg})}}$$
$$\Longrightarrow \boxed{\lambda_{2\,keV} = 2.74 \times 10^{-11}\,\mathrm{m}}$$

We must be more careful for the 200 keV electron, because its velocity is, to first order:

$$v = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2 \cdot 3.204 \times 10^{-14}\,\mathrm{J}}{9.11 \times 10^{-31}\,\mathrm{kg}}} \approx 2.65 \times 10^8\,\mathrm{m/s}$$

which is very near the speed of light.
So, to be relativistically correct, we must apply the relativistic kinetic energy formula to get $p$:

$$E = \sqrt{(mc^2)^2 + p^2c^2} - mc^2 \Longrightarrow p = \frac{1}{c}\sqrt{(E + mc^2)^2 - (mc^2)^2}$$
$$p = \frac{1}{c}\sqrt{[3.204 \times 10^{-14}\,\mathrm{J} + (9.11 \times 10^{-31}\,\mathrm{kg}) \cdot (3 \times 10^8\,\mathrm{m/s})^2]^2 - [(9.11 \times 10^{-31}\,\mathrm{kg})(3 \times 10^8\,\mathrm{m/s})^2]^2}$$
$$\Longrightarrow p = 2.64 \times 10^{-22}\,\mathrm{kg\,m/s}$$

Now, we can plug this correct momenta into the de Broglie equation

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34}\,\mathrm{J \cdot s}}{2.64 \times 10^{-22}\,\mathrm{kg\,m/s}} \Longrightarrow \boxed{\lambda_{200\,keV} = 2.51 \times 10^{-12}\,\mathrm{m}}$$