Appendix A: The Independent Chi-Squared Test from Lachin (1977) Applied to Cochran's *Q* Test

Lachin (1997) shows that the sample size for an independent chi-squared test with equal sample sizes per treatment(s) and control can be obtained by the following. The necessary components, with *m* treatments and *c* outcomes, are:

$$\mathbf{Q}' = (Q_1\ Q_2\ \dots\ Q_m); \tag{A1.a}$$

$$\mathbf{A}' = (1/p_1\ \ 1/p_2\ \dots\ 1/p_c); \tag{A1.b}$$

$$\mathbf{D} = [\mathbf{D}_1|\mathbf{D}_2|\dots|\mathbf{D}_c];\ \mathbf{D}_k' = (\delta_{1k}^2\ \delta_{2k}^2\ \dots\ \delta_{mk}^2); \tag{A1.c}$$

$$\mathbf{F} = [\mathbf{F}_1|\mathbf{F}_2|\dots|\mathbf{F}_c];\ \mathbf{F}_{\boldsymbol{k}} = d_k(p_k)^{-1/2};\ d_k' = (\delta_{1k}\ \delta_{2k}\ \dots\ \delta_{mk}); \tag{A1.d}$$

where p_k is the marginal probability for outcome *k*, δ_{jk} is the proposed proportion for treatment *j* and outcome *k*, and Q_j is the reciprocal of the sample size for a given treatment.

For Cochran's *Q*, the number of outcomes is $c = 2$, such that $\mathbf{A}'$, **D**, and **F** each have 2 columns. Also, δ_{jk} will represent the cells of the independent cross-tabulation of the *m* treatments and 2 outcomes, which correspond to the marginal probabilities for each treatment as represented, for example in the case of 3 treatments, by the marginal notation in Table 8. Further, each Q_j is simply $1/m$ since all treatments are equally represented at each experimental unit. In what follows, we also use 1 in the subscripts for an affirmative response and 0 for a negative response. Thus, for the case of Cochran's *Q*, these matrices are:

$$\mathbf{Q}' = (1/m\ \ 1/m\ \dots\ 1/m); \tag{A2.a}$$

$$\mathbf{A}' = (1/p_1\ \ 1/p_0); \tag{A2.b}$$

$$\mathbf{D} = [\mathbf{D}_1|\mathbf{D}_0];\ \mathbf{D}_1' = (p_{1+\dots+}^2\ p_{+1+\dots+}^2\ \dots\ p_{+\dots+1}^2);$$

$$\mathbf{D}_0' = ((1-p_{1+\dots+})^2\ (1-p_{+1+\dots+})^2\ \dots\ (1-p_{+\dots+1})^2); \tag{A2.c}$$

$$\mathbf{F} = [\mathbf{F}_1|\mathbf{F}_0];\ \mathbf{F}_{\boldsymbol{k}} = d_k(p_k)^{-1/2};\ d_1' = (p_{1+\dots+}\ p_{+1+\dots+}\ \dots\ p_{+\dots+1});$$

$$d_0' = \big((1 - p_{1+\cdots+})\ (1 - p_{+1+\cdots+})\ \ldots\ (1 - p_{+\cdots+1})\big). \qquad \text{(A2. d)}$$

Following Lachin (1977), we can then find the appropriate sample size by dividing

$$\tau = \mathbf{Q}'\mathbf{DA} - \mathbf{Q}'\mathbf{FF}'\mathbf{Q}; \qquad \text{(A3)}$$

into the non-centrality parameter $\lambda_{\alpha,\beta}$ from a non-central chi-squared distribution for a desired α and β and degrees of freedom equal to $(m - 1) \times (c - 1)$, though the latter will simply be $m - 1$ given there are only 2 outcomes for Cochran's Q. That is, the independent sample size $n_\perp$ is given by

$$n_\perp = \frac{\lambda_{\alpha,\beta}}{\tau}. \qquad \text{(A4)}$$

To find the sample size for dependent clusters in the case of Cochran's Q, the next step is to multiply the sample size for the independent sample $n_\perp$ by the weight w_Q (equation 12), as shown in equation 11:

$$n = w_Q n_\perp. \qquad (11)$$

We show the derivation of w_Q next in Appendix B.

Appendix B: Algebraic Equivalency of $1-\kappa_m$ from Donner and Li (1990) and w_Q

Donner and Li (1990:831) state that the sample size *N* for the case of *m* matched treatments and a desired power and significance level can be found by multiplying the standard sample size formula for comparing *m* independent proportions from Lachin (1977) by $1-\kappa_m$, where

$$\kappa_m = 1 - \sum_{i=1}^{N} \frac{A_i(m-A_i)}{m(m-1)NP(1-P)}, \tag{B1}$$

In this equation, *m* is the number treatments. A_i represents the number of affirmative responses for experimental unit $i, i = 1, \ldots, n$, such that $A_i \in \{0,1,2,\ldots,m\}$. Finally, $P = \sum_{i=1}^{N} \frac{A_i}{mN}$, or the proportion of affirmative responses across all observations. Since the multiplicative weight is given by $1-\kappa_m$, we let $w_Q = 1-\kappa_m$ and rewrite (B1) by subtracting 1 and multiplying by -1:

$$w_Q = 1-\kappa_m = \sum_{i=1}^{N} \frac{A_i(m-A_i)}{m(m-1)NP(1-P)}. \tag{B2}$$

Next, we move N to the numerator and substitute for P using the equivalency above.

$$w_Q = \frac{\sum_{i=1}^{N}[A_i(m-A_i)]/N}{m(m-1)\frac{\sum_{i=1}^{N}A_i}{mN}\left(1-\frac{\sum_{i=1}^{N}A_i}{mN}\right)}. \tag{B3}$$

Given that we seek N, we need to re-express the equation without N. We accomplish this by re-expressing the equation as a function of the proportion of successes for each treatment (p_j) and summing over the *m* treatments, rather than the number of successes for each unit (A_i) summed over the number of units (N).

We begin by re-expressing the numerator:

$$\sum_{i=1}^{N} \frac{[A_i(m-A_i)]}{N} \Leftrightarrow \tag{B4}$$

$$\frac{0(m-0)+\cdots+0(m-0)}{N}+\frac{1(m-1)+\cdots+1(m-1)}{N}$$

$$+\frac{2(m-2)+\cdots+2(m-2)}{N}+\cdots+\frac{m(m-m)+\cdots+m(m-m)}{N}\Leftrightarrow \qquad \text{(B5)}$$

$$0(m-0)\frac{[1+1+\cdots+1]}{N}+1(m-1)\frac{[1+1+\cdots+1]}{N}$$

$$+2(m-2)\frac{[1+1+\cdots+1]}{N}+\cdots+m(m-m)\frac{[1+1+\cdots+1]}{N}\Leftrightarrow \qquad \text{(B6)}$$

$$0(m-0)p_0+1(m-1)p_1+2(m-2)p_2+\cdots+m(m-m)p_m\Leftrightarrow \qquad \text{(B7)}$$

$$\sum_{j=0}^{m} j(m-j)p_j. \qquad \text{(B8)}$$

As can be seen in (B7), the first and last terms are eliminated since the lead coefficient is zero, such that the numerator does not require information from the completely concordant or discordant cells.

Now, we re-express $\frac{\sum_{i=1}^{N} A_i}{mN}$ in the denominator:

$$\frac{\sum_{i=1}^{N} A_i}{mN}\Leftrightarrow\frac{1}{m}\frac{\sum_{i=1}^{N} A_i}{N}\Leftrightarrow \qquad \text{(B9)}$$

$$\frac{1}{m}\left[\frac{0+\cdots+0}{N}+\frac{1+\cdots+1}{N}+\frac{2+\cdots+2}{N}+\cdots+\frac{m+\cdots+m}{N}\right]\Leftrightarrow \qquad \text{(B10)}$$

$$\frac{1}{m}\left[0\frac{1+\cdots+1}{N}+1\frac{1+\cdots+1}{N}+2\frac{1+\cdots+1}{N}+\cdots+m\frac{1+\cdots+1}{N}\right]\Leftrightarrow \qquad \text{(B11)}$$

$$\frac{1}{m}[0p_0+1p_1+2p_2+\cdots+mp_m]\Leftrightarrow \qquad \text{(B12)}$$

$$\sum_{j=0}^{m}\frac{jp_j}{m}. \qquad \text{(B13)}$$

Unlike (B7), only the first term in (B12) is eliminated. The last term corresponding to complete concordance remains, such that information from the completely concordant cell is necessary for sample size calculation.

We then substitute (B8) and (B13) into (B3) to yield our formula for w_Q in equation (12):

$$w_Q = \frac{\sum_{j=0}^{m} j(m-j)p_j}{m(m-1)\sum_{j=0}^{m}\frac{jp_j}{m}\left(1-\sum_{j=0}^{m}\frac{jp_j}{m}\right)}. \quad (12)$$

Appendix C: Table for Cochran's *Q* Test for Four Matched Proportions

Example of an empirical callback result	Treatment 1 (e.g. White) Response	Treatment 2 (e.g. Asian) Response	Treatment 3 (e.g. Latino) Response	Treatment 4 (e.g. Black) Response	Population proportion	Sample proportion	Sample cell size
All	1	1	1	1	π_{1111}	p_{1111}	n_{1111}
White & Asian & Latino	1	1	1	0	π_{1110}	p_{1110}	n_{1110}
White & Asian & Black	1	1	0	1	π_{1101}	p_{1101}	n_{1101}
White & Latino & Black	1	0	1	1	π_{1011}	p_{1011}	n_{1011}
Asian & Latino & Black	0	1	1	1	π_{0111}	p_{0111}	n_{0111}
White & Asian	1	1	0	0	π_{1100}	p_{1100}	n_{1100}
White & Latino	1	0	1	0	π_{1010}	p_{1010}	n_{1010}
White & Black	1	0	0	1	π_{1001}	p_{1001}	n_{1001}
Asian & Black	0	1	0	1	π_{0101}	p_{0101}	n_{0101}
Asian & Latino	0	1	1	0	π_{0110}	p_{0110}	n_{0110}
Latino & Black	0	0	1	1	π_{0011}	p_{0011}	n_{0011}
White only	1	0	0	0	π_{1000}	p_{1000}	n_{1000}
Asian only	0	1	0	0	π_{0100}	p_{0100}	n_{0100}
Latino only	0	0	1	0	π_{0010}	p_{0010}	n_{0010}
Black only	0	0	0	1	π_{0001}	p_{0001}	n_{0001}
None	0	0	0	0	π_{0000}	p_{0000}	n_{0000}
Marginal population proportion	π_{1+++}	π_{+1++}	π_{++1+}	π_{+++1}			
Marginal sample proportion	p_{1+++}	p_{+1++}	p_{++1+}	p_{+++1}			
Marginal sample size	n_1	n_2	n_3	n_4			