Quantum error correction

Quantum error correction is vital to quantum computing. Qubits are delicate as any interaction changes them. Since qubits cannot be cloned, a different kind of redundancy than employed classically is utilized. The qubit of interest is encoded in a larger bundle of entangled qubits. As with classical computing, error detection is accomplished via a syndrome in which detectable errors reside in the nullspace to the code word.

First, we explore principles of quantum error correction with the bit flip code followed by the phase flip code. Then, we explore the Shor code which remarkably can correct any arbitrary single qubit error.

Bit flip code

Consider a generic qubit (amplitudes unknown, of course).

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

Encoding proceeds by creating a three qubit code word.

$$|\psi 00\rangle = \alpha |000\rangle + \beta |100\rangle$$

Now, apply the permutation matrix *Cnot* (controlled-not) to the first two qubits. Then, apply *Cnot* again to the second and third qubits and encoding is complete.

$$\begin{aligned} |\psi_3\rangle \equiv Cnot_{23} Cnot_{12} |\psi 00\rangle &= Cnot_{23} (\alpha |000\rangle + \beta |110\rangle) \\ &= \alpha |000\rangle + \beta |111\rangle \end{aligned}$$

If any single qubit, bit flip errors have crept in while processing our qubit, they are orthogonal to the (un-normalized) entangled state $|000\rangle + |111\rangle$. Orthogonal bit flip states are $|001\rangle + |110\rangle$ (the third qubit flipped), $|010\rangle + |101\rangle$ (the second qubit slipped), and $|100\rangle + |011\rangle$ (the first qubit flipped).

Syndrome (single qubit, bit flip) error detection can be accomplished via projection measurements. Define four projections roughly corresponding to the above (un-normalized) orthogonal, entangled states.

$$P_0 = |000\rangle \langle 000| + |111\rangle \langle 111|$$

$$\begin{aligned} P_1 &= X_1 P_0 X_1 = |100\rangle \langle 100| + |011\rangle \langle 011| \\ P_2 &= X_2 P_0 X_2 = |010\rangle \langle 010| + |101\rangle \langle 101| \\ P_3 &= X_3 P_0 X_3 = |001\rangle \langle 001| + |110\rangle \langle 110| \end{aligned}$$

These projections form the eigenstate basis for our observable.

$$E_X = 1 * P_0 + 2 * P_1 + 3 * P_2 + 4 * P_3$$

The eigenvalues (here $1, 2, 3, 4$) are the observable measurements and can be any distinct values.

Error detection proceeds with measurement. If no errors have occurred with probability one measurement reveals

$$\langle\psi_3|\, E_X \,|\psi_3\rangle = 1$$

and no operation is needed to correct any (single qubit, bit flip) error. Alternatively, suppose the first qubit has bit flipped. Then, with probability one measurement gives

$$\langle\psi_3|\, X_1 E_X X_1 \,|\psi_3\rangle = 2$$

Error correction is completed by the inverse bit flip operation applied to the first qubit, also X_1.

$$X_1 X_1 \,|\psi_3\rangle = |\psi_3\rangle$$

Bit flip error measurements on the second and third qubits are analogous with a measurement of 3 indicating a bit flip operation on the second qubit and a measurement of 4 indicating a bit flip operation on the third qubit.

$$\langle\psi_3|\, X_2 E_X X_2 \,|\psi_3\rangle = 3$$

$$\langle\psi_3|\, X_3 E_X X_3 \,|\psi_3\rangle = 4$$

An alternative measurement basis works in equivalent fashion and proves instructive for expanding capabilities of the code. This measurement basis utilizes observables $Z_1 Z_2$ and $Z_2 Z_3$. Both observables are (Hermitian, unitary) diagonal matrices with ± 1 eigenvalues. The eigenvalues for $Z_1 Z_2$ in order are $+1, +1, -1, -1, -1, -1, +1, +1$ and for $Z_2 Z_3$ are $+1, -1, -1, +1, +1, -1, -1, +1$. Measurements from $Z_1 Z_2$ followed by $Z_2 Z_3$ revealing $+1, +1$ is the same as P_0 indicating no bit flip error correction. Measurement equal to $+1$ from observable $Z_1 Z_2$ leaves the three qubit code word residing in the subspace spanned by diagonal positions $1, 2, 7, 8$ (recall measurement typically changes the state). Measurement equal to $+1$ of this state with respect to observable $Z_2 Z_3$ results in the three qubit state residing in diagonal positions $1, 8$. Hence, no bit flip of $|\psi_3\rangle$ has occurred.

If $Z_1 Z_2$ reveals -1, the evolved state resides in a subspace defined by diagonal positions $3, 4, 5, 6$. If this is followed by measurement of $Z_2 Z_3$ equal to $+1$, then the evolved state resides in a subspace defined by diagonal positions $4, 5$. This corresponds to P_1 and bit flip error is corrected by X_1. Alternatively, if -1 from $Z_1 Z_2$ is followed by -1 again from $Z_2 Z_3$, then the evolved state resides in a subspace defined by diagonal positions $3, 6$. This corresponds to P_2 and is bit flip error is corrected by X_2. Finally, successive measurements equal to $+1, -1$ result in an evolved state residing in a subspace defined by diagonal positions $2, 7$. This corresponds to P_3 and bit flip error is corrected with X_3.

Phase flip code

The bit flip code does not detect phase flips (amplitude sign changes). However, a similar code detects and corrects phase flips.

Since X is the bit flip operator, Z is the phase flip operator, and $HXH = Z$ (also $HZH = X$), phase flip syndrome error detection and correction works analogous to the bit flip code after applyiing H (the Hadamard or qubit splitter operator). In particular, we encode $|\psi_3\rangle$ as $HHH\,|\psi_3\rangle$ and likewise the observable is

$$E_Z = HHHE_XHHH$$

$$= 1{*}HHHP_0HHH + 2{*}HHHP_1HHH + 3{*}HHHP_2HHH + 4{*}HHHP_4HHH$$

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Single qubit phase flip errors are detected with this syndrome in analogous manner to the bit flip code. Detected phase flip errors associated with qubit i are corrected by inverse operations $HHHZ_i$.

Shor code

The foregoing codes are effective at detecting and correcting either a bit flip error and phase flip error but not both on the same qubit. Fortunately, the Shor code can handle such errors and much more. In fact, the Shor code can correct for any arbitrary single qubit error. The Shor code employs a much longer code word — nine qubits ($2^9 = 512$ elements). The qubit is first encoded with the phase flip code $|0\rangle \to |+++\rangle$ and $|1\rangle \to |---\rangle$. Then, each of these qubits is entangled via the bit flip code $|+\rangle \to \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $|-\rangle \to \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$

$$|\psi_9\rangle = \alpha\,|0_L\rangle + \beta\,|1_L\rangle$$

where $|0_L\rangle \equiv \frac{1}{2\sqrt{2}}\left[(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)\right]$ and $|1_L\rangle \equiv \frac{1}{2\sqrt{2}}\left[(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\right]$.

Specifically, encoding can be implemented as follows. Create three blocks of three qubits. The first block is $|\psi 00\rangle$ while the second and third blocks are both $|000\rangle$. Apply H_1 to each block, then apply $Cnot_{12}$ followed by $Cnot_{23}$ to each block.

$$Cnot_{23}Cnot_{12}H_1\,|\psi 00\rangle = \frac{\alpha+\beta}{\sqrt{2}}\,|000\rangle + \frac{\alpha-\beta}{\sqrt{2}}\,|111\rangle$$

$$Cnot_{23}Cnot_{12}H_1\,|000\rangle = \frac{1}{\sqrt{2}}\,(|000\rangle + |111\rangle)$$

Combine the three blocks then apply another set of block-wise permutation matrices, $BCnot$, that utilize the leading three qubit block as control (when also the block omitted is $|000\rangle$) and the second three-qubit block as the target in the same manner as $Cnot$.[1]

$$|\psi_9\rangle = BCnot_{31}BCnot_{21}*$$

[1] The permutation matrix described by $BCnot_{31}BCnot_{21}$ is an identify matrix except elements $8, 8$ and $456, 456$ are zero with ones in positions $8, 456$ and $456, 8$ as well as elements $57, 57$ and $505, 505$ are zero with ones is positions $57, 505$ and $505, 57$. Of course, this is a unitary operator.

$$\left(\frac{\alpha+\beta}{\sqrt{2}}\left|000\right\rangle + \frac{\alpha-\beta}{\sqrt{2}}\left|111\right\rangle\right)\left[\frac{1}{\sqrt{2}}\left(\left|000\right\rangle + \left|111\right\rangle\right)\right]\left[\frac{1}{\sqrt{2}}\left(\left|000\right\rangle + \left|111\right\rangle\right)\right]$$

$$= \frac{\alpha}{2\sqrt{2}}\left[\left(\left|000\right\rangle + \left|111\right\rangle\right)\left(\left|000\right\rangle + \left|111\right\rangle\right)\left(\left|000\right\rangle + \left|111\right\rangle\right)\right]$$

$$+\frac{\beta}{2\sqrt{2}}\left[\left(\left|000\right\rangle - \left|111\right\rangle\right)\left(\left|000\right\rangle - \left|111\right\rangle\right)\left(\left|000\right\rangle - \left|111\right\rangle\right)\right]$$

Briefly, the Shor code employs ten observables.

$$Z_1Z_2, Z_2Z_3, Z_3Z_4, Z_4Z_5, Z_5Z_6, Z_6Z_7, Z_7Z_8, Z_8Z_9$$

$$X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9$$

The first eight are bit flip error detectors as described above for the bit flip code in which adjacent measurements indicate which (if any) qubits are bit flipped. Syndrome measurements and corrections are indicated below.

Z_1Z_2	Z_2Z_3	Z_3Z_4	Z_4Z_5	Z_5Z_6	Z_6Z_7	Z_7Z_8	Z_8Z_9	*action*
$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	I
-1	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	X_1
-1	-1	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	X_2
$+1$	-1	-1	$+1$	$+1$	$+1$	$+1$	$+1$	X_3
$+1$	$+1$	-1	-1	$+1$	$+1$	$+1$	$+1$	X_4
$+1$	$+1$	$+1$	-1	-1	$+1$	$+1$	$+1$	X_5
$+1$	$+1$	$+1$	$+1$	-1	-1	$+1$	$+1$	X_6
$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1	$+1$	X_7
$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1	X_8
$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	-1	X_9

As the encoding suggests, phase flips occur in blocks of three. As both observables, $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$, are Hermitian, unitary operators, they have ±1 eigenvalues. A measurement equal to -1 from $X_1X_2X_3X_4X_5X_6$ followed by $+1$ from $X_4X_5X_6X_7X_8X_9$ indicates a phase flip on a qubit in the first block. This is corrected by $Z_1Z_2Z_3$. Similarly, a measurement equal to $-1, -1$ indicates a phase flip on a qubit in the second block. This is corrected by $Z_4Z_5Z_6$. Thirdly, a measurement equal to $+1, -1$ indicates a phase flip on a qubit in the third block. This is corrected by $Z_7Z_8Z_9$.

Consider a random error generated by an un-normalized operator that acts on qubit i.

$$E_i = c_0I_i + c_1X_i + c_2Z_i + c_3X_iZ_i$$

This includes simple bit flips, phase flips, combinations,[2] or any arbitrary angle applied to a single qubit. The Shor code recovers from any single qubit error generated from E_i.

[2] $c_3X_iZ_i$ is equivalent to the fourth Pauli operator Y_i when $c_3 = \sqrt{-1}$.

Example

Suppose $E_1 = \frac{1}{\sqrt{2}} I_1 + X_1 Z_1$ is applied to $|\psi_9\rangle$ during encoding of $|\psi\rangle$. With probability $\frac{2}{3}$ observable $Z_1 Z_2$ produces a measurement equal to -1. This evolves $E_1 |\psi_9\rangle$ (following normalization by $\sqrt{\langle\psi_9| E_1 E_1 |\psi_9\rangle}$) into a subspace containing $X_1 X_2 |\psi_9\rangle$. A measurement of this evolved state by observable $Z_2 Z_3$ produces a result equal to $+1$. Hence, the evolved state resides in a subspace containing $X_1 |\psi_9\rangle$. The bit flip of the first qubit is reversed by applying the inverse operation, also X_1. However, this does not assure that $|\psi_9\rangle$ has been recoved.

Since the Shor code is effective at correcting error on a single qubit, for brevity, we bypass checking for bit flips on other qubits and move on to phase flips. The evolved state is measured with respect to $X_1 X_2 X_3 X_4 X_5 X_6$ which produces -1. Hence, the state has evolved to reside in a subspace including phase flips on the first two three-qubit blocks. Measurement of this evolved state by observable $X_4 X_5 X_6 X_7 X_8 X_9$ is $+1$ indicating the phase flip occurred in the first three-qubit block. This is reversed by the inverse operation $Z_1 Z_2 Z_3$. Remarkably, this sequence of projections coupled with appropriate inverse operations corrects the error on the first qubit.

Alternatively, if the first measurement utilizing observable $Z_1 Z_2$ is $+1$ rather than -1 (with probability $\frac{1}{3}$), the above sequence of projections corrects the first qubit error. This procedure works as follows. The first measurement changes the state but the subspace in which it resides no longer includes a bit flip of the first qubit. The remaining measurements in sequence, $Z_2 Z_3$, $X_1 X_2 X_3 X_4 X_5 X_6$, and $X_4 X_5 X_6 X_7 X_8 X_9$ each produce $+1$ results. Hence, no inverse operations are applied but the series of projections corrects the error on the first qubit — $|\psi_9\rangle$ is recovered.