Radial Fourier Multipliers

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Fourier Multiplier Operators

- Fourier multiplier operators are a basic object of study in harmonic analysis.
- Given $m \in L^\infty(\mathbb{R}^d)$, we may define an operator T_m acting on Schwartz functions $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{F}[T_m f](\xi) = m(\xi)\widehat{f}(\xi).$$

- One is typically interested in the mapping properties of T_m between various function spaces.
- Most basic question to ask: For a given p, does T_m extend to a bounded operator on L^p?

L^p Mapping Properties of Multipliers

- Is there a *characterization*, i.e. a simple and useful criterion for m that determines L^p boundedness of T_m for general multipliers $m \in L^\infty$?
- If $p = 1$, T_m is bounded on L^1 if and only if m is the Fourier transform of a finite Borel measure.
- If $p = 2$, T_m is bounded on L^2 by Plancherel since $m \in L^\infty$.
- If $p \neq 1, 2$, it is widely believed that no reasonable characterization exists.
- What if we ask the same question but restrict the class of multipliers $m \in L^\infty$ to a smaller subclass, for example the subclass of bounded, *radial* functions?

The Radial Fourier Multiplier Conjecture (Simplified Version)

Let $1 < p < p_d := \frac{2d}{d+1}$ *and* $d \geq 2$*. If* $m \in L^\infty(\mathbb{R}^d)$ *is radial and supported in a compact subset of* $\{\xi : 1/2 < |\xi| < 2\}$*, then the operator* T_m *is bounded on* $L^p(\mathbb{R}^d)$ *if and only if* $K := \widehat{m} \in L^p(\mathbb{R}^d)$*. Moreover, we actually have*

$$\|T_m\|_{L^p \to L^p} \approx_p \|K\|_p \,. \tag{1}$$

- If (1) is true, we will say $\mathrm{Rad}(d, p)$ holds.

The Radial Fourier Multiplier Conjecture (Full Version)

Let $1 < p < p_d := \frac{2d}{d+1}$ *and* $d \geq 2$*. Fix an arbitrary Schwartz function* η *that is not identically* 0*. If* $m \in L^\infty(\mathbb{R}^d)$ *is radial, then*

$$\|T_m\|_{L^p \to L^p} \approx_p \sup_{t>0} t^{d/p} \|T_m[\eta(t\cdot)]\|_{L^p} \,.$$

In 2008, Garrigós and Seeger obtained a characterization of L^p_{rad} boundedness of radial Fourier multipliers, where L^p_{rad} denotes the space of radial L^p functions. The simplified version of their result states:

Theorem (Garrigós and Seeger, 2008)

Let $1 < p < p_d := \frac{2d}{d+1}$ and $d \geq 2$. If $m \in L^\infty(\mathbb{R}^d)$ is radial and supported in a compact subset of $\{\xi : 1/2 < |\xi| < 2\}$, then the operator T_m is bounded on $L^p_{rad}(\mathbb{R}^d)$ if and only if $K := \widehat{m} \in L^p(\mathbb{R}^d)$. Moreover, we actually have

$$\|T_m\|_{L^p_{rad} \to L^p_{rad}} \approx_p \|K\|_p .$$

Previous partial progress toward the Conjecture for $d \geq 4$

In 2011, Heo, Nazarov, and Seeger proved the conjecture in the partial range $1 < p < \frac{2d-2}{d+1}$ in dimensions $d \geq 4$. They actually proved the following stronger conjecture in the partial range $1 < p < \frac{2d-2}{d+1}$ and $d \geq 4$.

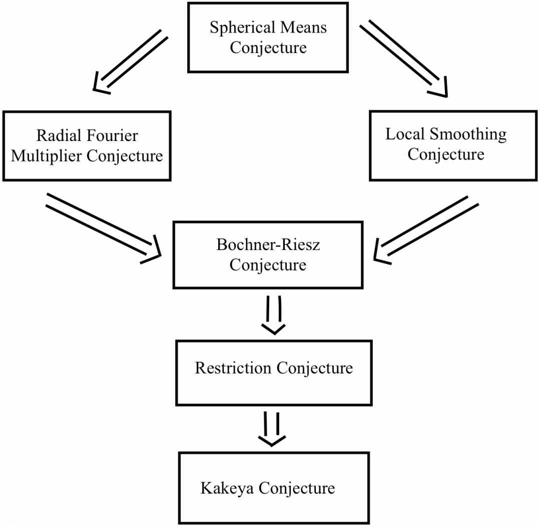
The Spherical Means Conjecture

Let $1 < p < p_d := \frac{2d}{d+1}$. Let σ_r denote the surface measure on the $(d-1)$-sphere of radius r centered at the origin. Let ψ_0 be a smooth, radial function supported in the unit ball centered at the origin whose Fourier transform vanishes to higher order (say $100d$) at the origin. Set $\psi = \psi_0 * \psi_0$. There is a constant C_p so that for every $h \in L^p(\mathbb{R}^d \times \mathbb{R}^+; dy\, r^{d-1}\, dr)$ we have

$$\left\| \int_{\mathbb{R}^d} \int_1^\infty h(y,r)\, \sigma_r * \psi(\cdot - y)\, dr\, dy \right\|_{L^p(\mathbb{R}^d)} \leq C_p \left(\int \int_{\mathbb{R}^d \times \mathbb{R}^+} |h(y,r)|^p\, dy\, r^{d-1}\, dr \right)^{1/p}. \quad (2)$$

- If (2) is true we will say $\mathrm{Sph}(d,p)$ holds.

A Tree of Conjectures in Harmonic Analysis



New Results in Three and Four Dimensions

We improve Heo, Nazarov, and Seeger's range for $\mathrm{Rad}(4, p)$ from $1 < p < 6/5$ to $1 < p < 36/29$.

Theorem 1 (C., 2016)

The Radial Fourier Multiplier Conjecture in four dimensions holds in the range $1 < p < 36/29$.

In three dimensions, we obtain a characterization in the range $1 < p < 13/12$ in terms of the $L^{p,1}$ norm of the kernel.

Theorem 2 (C., 2016)

Let $1 < p < 13/12$. Let $m \in L^\infty(\mathbb{R}^3)$ be radial and supported in a compact subset of $\{\xi : 1/2 < |\xi| < 2\}$. Then T_m is restricted strong type (p, p) if $K = \widehat{m} \in L^p(\mathbb{R}^3)$, and T_m is bounded on $L^p(\mathbb{R}^3)$ if $K \in L^{p,1}(\mathbb{R}^3)$. Moreover

$$\|T_m\|_{L^p \to L^p} \lesssim \|K\|_{L^{p,1}} .$$

- We expect $\|K\|_{L^{p,1}}$ in the second theorem could be improved to $\|K\|_{L^p}$, which would imply that the Radial Fourier Multiplier Conjecture holds in the range $1 < p < 13/12$ in $\mathbb{R}^3$.

Preliminaries to the proof of Theorem 2

- First, some motivation for what we are about to do next:
- Since the multiplier m is compactly supported away from the origin, we have $\widehat{m} =: K = K * \phi$ where ϕ is a smooth bump with $\hat{\phi}$ supported in a compact set away from the origin (which implies ϕ has a lot of cancellation).
- Moreover, since $\widehat{K}$ is supported in the double of the unit ball, K is "essentially constant" at unit scales, and so since it is radial it is essentially constant on annuli centered at the origin of thickness 1.
- Thus we should expect that we should be able to "decompose" the kernel K into functions that have a lot of cancellation and are supported on annuli of thickness ≈ 1 centered at the origin.

Preliminaries to the proof of Theorem 2

- Some machinery of Heo, Nazarov, and Seeger is needed.
- As in [HNS], the first step is to **discretize** the problem and to reduce it to proving an inequality involving sums of functions with cancellation supported on annuli of thickness ≈ 1 whose centers and radii lie in a discrete set.
- Let $\mathcal{Y} \subset \mathbb{R}^3$ be the integer lattice in $\mathbb{R}^3$ which will represent **centers** of 3-D annuli and let $\mathcal{R} \subset \mathbb{R}$ be the integers, which will represent **radii** of 3-D annuli.
- For $(y, r) \in \mathcal{Y} \times \mathcal{R}$, let $F_{y,r}$ denote the function $\psi * \sigma_r(\cdot - y)$, where ψ is a smooth compactly supported function whose Fourier transform vanishes to high order at the origin and where σ_r is the surface measure on the 2-sphere of radius r centered at the origin.
- Discretization, followed by an application of a dyadic interpolation lemma, reduces $\mathrm{Sph}(3, p)$ to proving the following inequality for every finite set $\mathcal{E} \subset \mathcal{Y} \times \mathcal{R}$ and every measurable function $c : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$ with $|c(y, r)| \leq 1$:

$$\left\| \sum_{(y,r)\in\mathcal{E}} c(y, r) F_{y,r} \right\|_p^p \lesssim_p \sum_k 2^{2k} \# \mathcal{E}_k, \tag{3}$$

where $\mathcal{E}_k = \mathcal{E} \cap (\mathcal{Y} \times [2^k, 2^{k+1}])$.

Preliminaries to the proof of Theorem 2

Density decompositions

- For dyadic numbers $u \geq 1$, we decompose $\mathcal{E}_k$ into sets $\mathcal{E}_k(u)$ of "density" u as follows.
- Set $\widehat{\mathcal{E}_k}(u) := \{(y,r) \in \mathcal{E}_k : \exists$ a ball B of radius $\leq 2^k$ containing (y,r) such that $\#(\mathcal{E}_k \cap B) \geq u(\mathrm{rad}(B))\}$.
- Set $\mathcal{E}_k(u) = \widehat{\mathcal{E}_k}(u) \setminus \bigcup_{u' > u \text{ dyadic}} \widehat{\mathcal{E}_{k'}}(u)$.
- We have

$$\mathcal{E}_k = \bigcup_{u \geq 1 \text{ dyadic}} \mathcal{E}_k(u).$$

- For a given function $c(y,r) : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$, set

$$G_{u,k} = \sum_{(y,r) \in \mathcal{E}_k(u)} c(y,r) F_{y,r},$$

$$G_u = \sum_k G_{u,k}.$$

Preliminaries to the proof of Theorem 2

L^2 bounds vs. support size

Lemma (Support size estimate, [HNS])

For all dyadic $u \geq 1$, the Lebesgue measure of the support of $G_{u,k}$ is $\lesssim u^{-1} 2^{2k} \# \mathcal{E}_k$.

- To prove the restricted strong type version of $\mathrm{Rad}(3, p)$, it actually suffices to prove inequality (3) with the additional assumption that $\mathcal{E}$ is a **product**, i.e. a set of the form $Y \times R$ where $Y \subset \mathcal{Y}$ and $R \subset \mathcal{R}$. Under this assumption, we obtain the following L^2 estimate, which is an improvement over the L^2 estimate proved in [HNS].

Lemma (Improved L^2 bound)

For all $\epsilon > 0$,

$$\|G_u\|_2^2 \lesssim_\epsilon u^{11/13+\epsilon} \sum_k 2^{2k} \# \mathcal{E}_k.$$

(Recall $G_u := \sum_k \sum_{(y,r) \in \mathcal{E}_k(u)} c(y,r) F_{y,r}$.)

Outline of proof of the L^2 estimate

- We require **scalar product estimates**:

$$
|\langle F_{y,r}, F_{y',r'}\rangle| \lesssim \frac{r\,r'}{(1+|y-y'|+|r-r'|)} \times \sum_{\pm,\pm}(1+r\pm r'\pm|y-y'|)^{-N}
$$

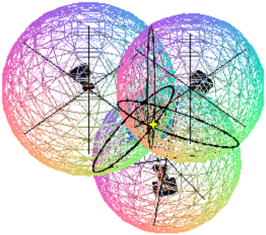
  for any $N > 0$.

- For a fixed y, r, r' the set of y' for which the second term in the product is the worst is contained in the union of two annuli centered at y, one of radius $r+r'$ and one of radius $|r-r'|$.
- This corresponds exactly to **tangencies** of annuli.
- We will use a geometric argument to control the number of tangencies between annuli.
- For our argument, it will be essential to use the product structure of the set $\mathcal{E}$.

A geometric lemma

Lemma

Fix integers m, l with $l \leq m$. Fix $t \approx 2^m$. Then the size of the intersection of three annuli in $\mathbb{R}^3$ of thickness ≈ 1 and inner radius t such that the distance between the centers of any pair is at least 2^l and no greater than $2^m/10$ is $\lesssim 2^{3(m-l)}$, provided that $l \geq m/2 + 10$.



Thank you!