Stable character property of weighted ℓ^1

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$\mathcal{A}$: Banach algebra (so $\|ab\| \leq \|a\|\|b\|$).

Character

Nonzero linear $\phi : \mathcal{A} \to \mathbb{C}$ such that $\forall a, b \in \mathcal{A}$,

$$\phi(ab) = \phi(a)\phi(b).$$

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Example

- Fix $x_0 \in X$. Define $\phi : C(X) \to \mathbb{C}$ to be
$$\phi(f) = f(x_0).$$
- Fix $\theta \in \mathbb{T}$. Define $\phi : \ell^1(\mathbb{Z}) \to \mathbb{C}$ as
$$\phi(f) = \widehat{f}(\theta).$$

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Character space

$\mathrm{Char}(\mathcal{A})$ = collection of characters of $\mathcal{A}$.

Defect of a cts linear $\psi : \mathcal{A} \to \mathbb{C}$ is

$$\operatorname{def}(\psi) := \sup \{ |\psi(ab) - \psi(a)\psi(b)| : a, b \in \mathcal{A}, \|a\|, \|b\| \leq 1 \} .$$

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$\mathcal{A}$ has stable character property if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ s.t.

$$\text{if } \operatorname{def}(\psi) \leq \delta \text{ then } \operatorname{dist}(\psi, \operatorname{Mult}(\mathcal{A})) \leq \epsilon.$$

Main question

Which Banach algebras have stable character property?

Results [Johnson 85 & 88, Sidney 97, Howey 03, Choi 13, . . .]

The following Banach algebras have stable character property.

- Finite dimensional Banach algebras.
- $C_0(X)$, where X locally compact Hausdorff.
- ℓ^p with term-wise multiplication.
- $L^1(G)$, where G locally compact Abelian group.
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Goal

Construct natural examples without stable character property.

Banach algebras over semilattices

$(\mathcal{S}, \cdot)$ is a semilattice if $x \cdot y = y \cdot x$ and $x^2 = x$.

Example

- $\mathbb{N}$ with $n \cdot m = \min\{m, n\}$.
- $\mathcal{P}(\mathbb{N})$ with $A \cdot B = A \cup B$.

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For a semilattice $\mathcal{S}$,

$$\ell^1(\mathcal{S}) = \left\{ \sum_{s \in \mathcal{S}} c_s \delta_s : \sum_{s \in \mathcal{S}} |c_s| < \infty \right\}.$$

$\ell^1(\mathcal{S})$ is Banach algebra with multiplication

$$\delta_s * \delta_t = \delta_{st}.$$

Characters of $\ell^1(\mathcal{S})$

Definition

A subset $F \subseteq \mathcal{S}$ is a filter if

- $F \neq \varnothing$.
- If $x, y \in F$ then $xy \in F$.
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Easy fact

F is a filter in $\mathcal{S}$ if and only if χ_F is a character.

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Proof.

Theorem [Choi, 13]

$\ell^1(\mathcal{S})$ has stable character property.

Proof.

Suppose $\psi : \mathcal{S} \to \mathbb{C}$ satisfies

$$\mathrm{def}(\psi) := \sup \{|\psi(xy) - \psi(x)\psi(y)| : \|x\|, \|y\| \leq 1\} < \frac{1}{5}.$$

Define

$$\mathcal{S}_1 = \left\{ s \in \mathcal{S} : |\psi(s) - 1| < \frac{7}{25} \right\}.$$

Then,

- $\mathcal{S}_1$ is a filter. So $\chi_{\mathcal{S}_1}$ is a character.
- $\|\psi - \chi_{\mathcal{S}_1}\|_\infty \leq \frac{7}{5}\mathrm{def}(\psi)$.

□

Weighted semilattice algebras

Definition

- A weight is a function $w : \mathcal{S} \to [1, \infty)$ such that
$$w(xy) \leq w(x)w(y).$$
- $\ell^1_w(\mathcal{S}) := \{\sum_{s\in\mathcal{S}} c_s\delta_s : \sum_{s\in\mathcal{S}} w(s)|c_s| < \infty\}$.

- $\ell^1_w(\mathcal{S})$ is a Banach algebra.
- Filters of $\mathcal{S}$ give characters of $\ell^1_w(S)$.

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Question

For which weight w on $\mathcal{S}$, the algebra $\ell^1_w(\mathcal{S})$ has stable characters?

Our setting

Let $\psi : \mathcal{S} \to \mathbb{C}$.

- $\mathrm{def}(\psi) = \sup_{x,y \in \mathcal{S}} \frac{|\psi(xy) - \psi(x)\psi(y)|}{w(x)w(y)}$.

- $\mathrm{dist}(\psi, \mathrm{Mult}) = \inf \left\{ \sup_{x \in \mathcal{S}} \frac{|\psi(x) - \chi_F(x)|}{w(x)} : F \text{ empty or filter} \right\}$.

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Reduction to "discrete"

The following are equivalent:

- $\ell^1_w(\mathcal{S})$ has stable characters.
- $\forall \epsilon > 0 \ \ \exists \delta > 0 \ \ \forall \psi : \mathcal{S} \to \{0, 1\}$,

$$\mathrm{def}(\psi) \leq \delta \ \text{ then } \ \mathrm{dist}(\psi, \mathrm{Mult}) < \epsilon.$$

Theorem [Choi, 2013]

Suppose $\mathcal{S}$ has finite *breadth*. Then for every weight, $\ell^1_w(\mathcal{S})$ has stable character property.

Theorem [Choi-G.-Pham, 2016]

Suppose $\mathcal{S}$ has infinite *breadth*. Then there exists a weight w such that $\ell^1_w(\mathcal{S})$ does not have stable character property.

Stability of Filters in Weighted Semilattices

Examples of semilattices

- $(\mathcal{P}(\Omega), \cup)$.
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Filters

A nonempty subset $\mathcal{F}$ of a semilattice $\mathcal{S}$ is a filter if for every $x, y \in \mathcal{S}$

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Let $G \subseteq \mathcal{S}$, and w a weight on $\mathcal{S}$.

- $\mathrm{def}(G) = \sup_{x,y\in\mathcal{S}} \frac{|1_G(x\cup y) - 1_G(x)1_G(y)|}{w(x)w(y)}$.
- $\mathrm{dist}(G, \mathrm{Filters}) = \inf \left\{ \sup_{x\in\mathcal{S}} \frac{|1_G(x) - 1_F(x)|}{w(x)} : F \text{ empty or filter} \right\}$.
- $(\mathcal{S}, w)$ has stable filters if $\forall \epsilon > 0,\ \exists \delta > 0$ such that if $\mathrm{def}(G) < \delta$ then $\mathrm{dist}(G, \mathrm{Filters}) < \epsilon$.

Breadth of a semilattice

- $\{A_1, \ldots, A_n\} \subseteq \mathcal{S}$ is incompressible if $\exists\ a_i \in A_i \setminus \bigcup_{j \neq i} A_j$.
- Breadth of $\mathcal{S}$ defined

$$\inf \{ n \in \mathbb{N} : \nexists \text{ incompressible subset of } \mathcal{S} \text{ with size } n+1 \}.$$

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Example

$[n] := \{1, \ldots, n\}$.

- $b(\mathcal{P}([n])) = n$.
- $b(\mathcal{S}) = 1$ where $\mathcal{S} = \{[n] : n \in \mathbb{N}\}$.

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- $b(\mathcal{S}) = 1$ where $\mathcal{S} = \{[n] : n \in \mathbb{N}\}$.

- If $b(\mathcal{S}) < \infty$ then $(\mathcal{S}, w)$ has stable filters.
- If $b(\mathcal{S}) = \infty$ then $\exists$ some weight s.t. $(\mathcal{S}, w)$ fails stable filter property [Choi-G-Pham, 2016].

Example of semilattices with infinite breadth

Let $E = \dot{\bigcup}_{n\geq 1} E_n$, where $|E_n| \to \infty$.

- Semilattice of Type I:

$$\mathcal{S}_1 := \{E_1 \cup \ldots \cup E_{n-1} \cup A : \quad n \in \mathbb{N},\ A \subseteq E_n\}.$$

- Semilattice of Type II:

$$\mathcal{S}_2 := \{A \cup E_{n+1} \cup E_{n+2} \cup \ldots : \quad n \in \mathbb{N},\ A \subseteq E_n\}.$$

- Semilattice of Type III:

$$\mathcal{S}_3 := \{E_1 \cup \ldots \cup E_{n-1} \cup A \cup E_{n+1} \cup E_{n+2} \cup \ldots : \quad n \in \mathbb{N},\ A \subseteq E_n\}.$$

Note: They have different nature.

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Structure of infinite breadth semilattices [Choi-G-Pham]

$\mathcal{S}$: infinite breadth subsemilattice of $(2^\Omega, \cup)$. Then there exists $E \subseteq \Omega$ such that $\mathcal{S} \wedge E$ contains a semilattice of type I or II or III.

Example of "non-stable" weight

Theorem (Choi-G.-Pham, 2016)

Suppose $\mathcal{S}$ has infinite breadth. Then there exists a weight w such that $\ell^1_w(\mathcal{S})$ does not have stable character property.

Idea: Let $E = \dot{\bigcup}_{n\geq 1} E_n$, where $|E_n| = n$.

$$\mathcal{S}_1 := \{E_1 \cup \ldots \cup E_{n-1} \cup A : \quad n \in \mathbb{N},\ A \subseteq E_n\}.$$

Define $\eta : \mathcal{S} \to [0, \infty)$ as

$$\eta(x) := |\{n : E_n \subseteq x\}| + |x \setminus \cup_{E_n \subseteq x} E_n|.$$

Then $(\mathcal{S}, 2^\eta)$ does not have stable character property.

Thank you!