Wavelet frame sets in finite vector spaces

Azita Mayeli

City University of New York

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Wavelet sets in $\mathbb{R}^d$

Wavelet.

In the classical setting, a function $\psi \in L^2(\mathbb{R}^d)$ is called **wavelet** if there is a set of $d \times d$ matrices $\mathcal{D} \subset GL(d, \mathbb{R})$ and a countable subset $T \subset \mathbb{R}^d$ such that the family

$$\{\psi_{D,t} := |\det(D)|^{1/2}\psi(Dx - t) : \ D \in \mathcal{D}, t \in T\} \tag{1}$$

forms an orthogonal basis for $L^2(\mathbb{R}^d)$.

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Wavelets were introduced by Grossmann and Morlet as wavelets in 1984 and were considered by Weiss and Coifman as atoms.

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Frame wavelet.

The function ψ is called **frame wavelet** if the family

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is a **frame** for $L^2(\mathbb{R}^d)$:

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is a **frame** for $L^2(\mathbb{R}^d)$: There exist $0 < A \leq B < \infty$ such that for all $f \in L^2(\mathbb{R}^d)$

$$A\|f\|^2 \leq \sum_{D,t} |\langle f, \psi_{D,t}\rangle|^2 \leq B\|f\|^2$$

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Consequently,

$$f = \sum_{D,t} c_{t,D}\psi_{D,t}$$

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Frame is **tight** if $A = B$.

Wavelet sets in $\mathbb{R}^d$

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$$\hat{\psi} = 1_\Omega$$

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Definition (Wavelet sets in $\mathbb{R}^d$)

A set $\Omega \subset \mathbb{R}^d$ with finite and non-zero measure is called a **wavelet set** if for $\hat{\psi} := 1_\Omega$, the function ψ is a wavelet.

Such wavelets are also called minimally supported frequency wavelets (MSF wavelets).

The wavelet sets and minimally supported frequency wavelets were introduced by Fang and Wang [1] and studied exclusively by Hernandez, Wang and Weiss. [2] [3]

[1] X. Fang, X. Wang, *Construction of Minimally-Supported-Frequencies Wavelets*, the Journal of Fourier Analysis and Application 2 (1996), no. 4, 315-327.

[2] E. Hernandez, X.-H. Wang and G. Weiss, *Smoothing minimally supported frequency wavelets. I*, J. Fourier. Anal. Appl. 2 (1996), 329–340.

[3] E. Hernandez, X.-H. Wang and G. Weiss, *Smoothing minimally supported frequency wavelets. II*, J. Fourier. Anal. Appl. 3 (1997), 23–41.

Wavelet sets in $\mathbb{R}^d$

A well-known example of a wavelet set in dimension one is the Shannon set:

$$\Omega = (-2\pi, -\pi] \cup [\pi, 2\pi)$$

$$\psi = 2\text{sinc}(2x - 1) - \text{sinc}(x)$$

ψ is known as Shannon or Littlewood-Paley wavelet.

Wavelet sets in $\mathbb{R}^d$

Few references on wavelet sets in $\mathbb{R}^d$

- [DLS] Dai, Larson, and Speegle prove the (abstract) existence of wavelet sets for any expansive matrices.

- Soardi and Weiland - provide methods for constructing explicit examples of the sets in [DLS].

- Baggett, Medina, Merill - construct of wavelet sets in $\mathbb{R}^d$ using GMRA methods for any integral dilation matrices.

- Benedetto , Benedetto, Leon - provide algorithms for construction of wavelet sets
- Dobrescu, Olafsson - construction of wavelet sets using induction

Wavelet sets in $\mathbb{R}^d$

The motivation for our work is a result by Wang which ties the existence of a wavelet set in $\mathbb{R}^d$ with the notion of **additive tiling** as well as **multiplicative tiling** and **spectral sets**.

Wavelet sets in $\mathbb{R}^d$

Definition. A set $\Omega \subset \mathbb{R}^d$ is a **spectral set** if there is a countable set $\Gamma \subset \mathbb{R}^d$ such that the exponentials

$$\mathcal{E}_\Gamma := \{e^{2\pi i \gamma \cdot x} : \ \gamma \in \Gamma\}$$

form an orthogonal basis for $L^2(\Omega)$.

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We say Ω is a **multiplicative tiling** for $\mathbb{R}^d$ if there is a countable set $\mathcal{D} \subset GL(d, \mathbb{R})$ such that $\{\alpha(\Omega) : \ \alpha \in \mathcal{D}\}$ tiles $\mathbb{R}^d$.

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$$\mathbb{R}^d = \dot{\cup}_{\alpha \in \mathcal{D}} \alpha(\Omega) \quad \text{up to measure zero}$$

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Tao ('03): spectral $\not\Longrightarrow$ tiling $(n \geq 5)$

Kolountzakis, Matolcsi ('06): tiling $\not\Longrightarrow$ spectral $(n = 4)$

Theorem (Wang[a])

[a] Y. Wang, *Wavelets, tiling, and spectral sets*, Duke Math. J. 114 (2002), no. 1, 43–57.

Given $\Omega \subset \mathbb{R}^d$ *with* $0 < |\Omega| < \infty$, Ω *is a* **wavelet set if and only if** Ω *is a* **spectral and multiplicative tiling set**, *provided that* 0 *is an element in the translation set.*

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Theorem (Wang[a])

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Inspired by Wang's characterization of wavelet sets, it is natural for us to ask for what degree one can extend the notion and concept of wavelet sets and multiplicative tiling in a finite vector space.

This is a joint work with Alex Iosevich and Chun-kit Lai.

Preliminaries

Finite vector spaces $\mathbb{F}_q^d$

q an odd prime.

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Wavelet collection.

$$\mathcal{W} := \{\psi_{a,t} : \; a \in \mathcal{A}, \; t \in T\}$$

ψ is called a generator of the wavelet collection.

Preliminaries

One of our main observations is:

Theorem

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There is no wavelet set for $\mathbb{F}_q^d$ in the traditional sense. That is:

There is no $E, T \subset \mathbb{F}_q^d$ and $\mathcal{A} \subset GL(d, \mathbb{F}_q)$ such that the set

$$\mathcal{W} = \{\delta_a \tau_t \psi : \ a \in \mathcal{A}, \ t \in T\}$$

is an orthogonal basis for $L^2(\mathbb{F}_q^d)$ when ψ with $\hat{\psi} := 1_E$.

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So, now what?

Preliminaries

Characters. $m \in \mathbb{F}_q^d$

$$\chi_m : \mathbb{F}_q^d \to \mathbb{T}; \quad \chi_m(\xi) = e^{2\pi i \frac{m\cdot\xi}{q}}$$

$m \cdot \xi := \sum_{i=1}^{d} m_i \xi_i$ (mod) q

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Fourier transform. $\psi : \mathbb{F}_q^d \to \mathbb{C}$

$$\widehat{\psi}(\xi) := q^{-d} \sum_{m \in \mathbb{F}_q^d} \psi(m)\, \chi_m(-\xi) \qquad \xi \in \mathbb{F}_q^d$$

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Inverse Fourier transform.

$$\psi(x) = \sum_{m \in \mathbb{F}_p^d} \hat{\psi}(m) \chi_m(x) \quad \forall\, x \in \mathbb{F}_p^d$$

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Let $E \subset \mathbb{F}_q^d$

$E^* := E \setminus \{0\}$

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Lemma

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Lemma

$\mathcal{W}$ is an orthogonal basis (frame) for S_0 iff $\widehat{\mathcal{W}}$ is an orthogonal basis (frame) for $L^2(Y)$.

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This is equivalent to say that:

$$\widehat{\mathcal{W}} = \{\chi_{(a^{-1})(l)} 1_{a^t(E^*)} : l \in L, a \in \mathcal{A}\}$$

is (frame) orthogonal basis for $L^2(Y)$. $(Y := \mathbb{F}_q^d \setminus \{0\})$

Wavelet sets in $\mathbb{F}_q^d$

Recall that:

Theorem (Wang)

Given $\Omega \subset \mathbb{R}^d$ with $0 < |\Omega| < \infty$, Ω is a wavelet set if and only if Ω is a spectral and multiplicative tiling set, provided that 0 is in translation set.

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Question: Can we characterize the wavelet sets on $\mathbb{F}_q^d$ in the same fashion?

Wavelet sets in $\mathbb{F}_q^d$

Additive tiling. A subset $E \subseteq \mathbb{F}_q^d$ is an additive tiling set for $\mathbb{F}_q^d$ if there exists $\Lambda \subseteq \mathbb{F}_q^d$ such that

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Multiplicative tiling. Let E be a subset of $\mathbb{F}_q^d$. We say E is a multiplicative tiling set for $\mathbb{F}_q^d$ if there is a set of automorphisms $\mathcal{A} \subset \mathcal{A}ut(\mathbb{F}_q^d)$ such that E tiles $\mathbb{F}_q^d \setminus \{0\}$ multiplicatively by $\mathcal{A}$, i.e.,

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We must then have $0 \notin E$.

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That is

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$$f(a) = \sum_{l \in L} c_l \ \chi_l(a)$$

(2) Orthogonality. $\forall \ l \neq l'$

$$\hat{1}_E(l - l') = q^{-d} \sum_{a \in E} \chi_a(-(l - l')) = 0$$

We call (E, L) a spectral pair.

Wavelet sets in $\mathbb{F}_q^d$

Theorem (Multiplicative tiling + Spectral $\to$ Wavelet set)

Let $\mathcal{A} \subseteq \mathbb{A}ut(\mathbb{F}_q^d)$, $L \subset \mathbb{F}_q^d$ *and* $E \subseteq \mathbb{F}_q^d$. *Assume that*

- (E, L) *is a spectral pair, and*
- $\{a^t(E^*) : a \in \mathcal{A}\}$ *is a tiling for* Y. *(*$Y := \mathbb{F}_q^d \setminus \{0\}$*)* .

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Then

- *If $0 \notin E \implies E$ is a wavelet set.*
- *If $0 \in E \implies (E^*, L)$ is a tight frame spectral pair $\implies \widehat{\mathcal{W}}_{\mathcal{A},L}$ is a tight frame for $L^2(Y)$, i.e., E is a tight frame wavelet set .*

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- *If* $0 \in E \implies (E^*, L)$ *is a tight frame spectral pair* $\implies \widehat{\mathcal{W}}_{\mathcal{A},L}$ *is a tight frame for* $L^2(Y)$, *i.e.,* E *is a tight frame wavelet set .*

Recall that

$$\widehat{\mathcal{W}}_{\mathcal{A},L} := \{\overline{\chi_{a^{-1}(l)}}\mathbf{1}_{a^t(E^*)} : \ a \in \mathcal{A}, l \in L\}.$$

Theorem (Multiplicative tiling + Spectral $\to$ Wavelet set)

Let $\mathcal{A} \subseteq \mathbb{A}ut(\mathbb{F}_q^d)$, $L \subset \mathbb{F}_q^d$ and $E \subseteq \mathbb{F}_q^d$. Assume that

- (E, L) *is a spectral pair, and*
- $\{a^t(E^*) : a \in \mathcal{A}\}$ *is a tiling for* Y. $(Y := \mathbb{F}_q^d \setminus \{0\})$.

Then

- *If* $0 \notin E \implies E$ *is a wavelet set.*
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Recall that

$$\widehat{\mathcal{W}}_{\mathcal{A},L} := \{\overline{\chi_{a^{-1}(l)}}\mathbf{1}_{a^t(E^*)} : \ a \in \mathcal{A}, l \in L\}.$$

Note that unlike in the Euclidean case, by removing one point we lose the orthogonality.

Theorem (Wavelet set $+\{0\} \rightarrow$ Multiplicative tiling + Spectral)

The system

$$\widehat{\mathcal{W}}_{\mathcal{A},L} := \{\overline{\chi_{a^{-1}(l)}} 1_{a^t(E^*)} : \quad a \in \mathcal{A}, l \in L\}$$

is an orthogonal basis for $L^2(Y)$ and $0 \in L$, then

- (E^*, L) *is a spectral pair, and*
- $\{a^t(E^*) : a \in \mathcal{A}\}$ *is a tiling for $\mathbb{F}_q^d \setminus \{0\}$.*

Construction of tight frame wavelet sets E in $\mathbb{F}_q^d$

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Two things are to considered about $E \subset \mathbb{F}_q^d$:

- we want E is a spectral set.
- we want E^* tiles $\mathbb{F}_q^d \setminus \{0\}$ multiplicatively.

Existence of multiplicative tiling

Circles in $\mathbb{F}_q^2$. For $r \in \mathbb{F}_q$, the circle of radius r is defined by

$$S_r := \left\{(x,y) \in \mathbb{F}_q^2 : \; x^2 + y^2 = r \text{ (mod } q).\right\}$$

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$$S_r := \{(x, y) \in \mathbb{F}_q^2 : \; x^2 + y^2 = r \text{ (mod } q).\}$$

If $q \equiv 3$ (mod 4), then

$$\#S_r = \begin{cases} 1, & \text{if } \; r = 0; \\ q + 1, & \text{if } r \neq 0 \end{cases}$$

Existence of multiplicative tiling

Lemma

Suppose that $q \equiv 3 \pmod 4$. *Then there exists an orthogonal matrix*

$$R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

such that $a^2 + b^2 = 1 \pmod q$, $R^{q+1} = I$ *and*

$$S_r = \{\mathbf{e}, R\mathbf{e}, R^2\mathbf{e}, ..., R^q\mathbf{e}\} \quad \forall r \neq 0,\ \mathbf{e} \in S_r$$

Existence of multiplicative tiling

Theorem

There exists multiplicative tiling set E in $\mathbb{F}_q^d$ for $q \equiv 3$ (mod 4).

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$d = 2$: take E such that

$$\sharp(E \cap S_r) = 1 \quad \forall r \neq 0$$

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Let $\mathcal{A} := \{I, R, R^2, ..., R^q\}$ where R is the orthogonal matrix in the previous slide. Then

$$\mathbb{F}_q^2 \setminus \{0\} = E \cup R(E) \cup \ldots \cup R^q(E)$$

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$$\mathbb{F}_q^2 \setminus \{0\} = E \cup R(E) \cup \ldots \cup R^q(E)$$

$d > 2$: take $\widetilde{E} := E \times \mathbb{F}_q^{d-2}$ and $\widetilde{R} := \begin{pmatrix} R & O \\ O & I \end{pmatrix}$.

Then

$$\mathbb{F}_q^d \setminus \{0\} = \widetilde{E} \cup \widetilde{R}(\widetilde{E}) \cup \ldots \cup \widetilde{R}^q(\widetilde{E})$$

We are done! □

Spectral sets in $\mathbb{F}_q^2$

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A non-empty subset E is spectral if and only if E tiles $\mathbb{F}_q^2$ additively.

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For the spectral part, we use the following observation on the Fuglede Conjecture on $\mathbb{F}_q^2$.

Theorem (A. Iosevich, J. Pakianathan and A.M.)

A non-empty subset E is spectral if and only if E tiles $\mathbb{F}_q^2$ additively.

Therefore we are looking for a set E which is an additive tiling.

Additive tiling sets in $\mathbb{F}_q^2$

Theorem (A. Iosevich, J. Pakianathan and A.M.)

Let E be a set that tiles $\mathbb{F}_q^2$ by translation. Then $\sharp E = 1, q$ or q^2.

If $\sharp E = q$, then E is a graph, i.e.

$$E = \{x\mathbf{e}_1 + f(x)\mathbf{e}_2 : \; x \in \mathbb{F}_q\}$$

for some basis $\mathbf{e}_1, \mathbf{e}_2$ in $\mathbb{F}_q^2$ and function $f : \mathbb{F}_q \to \mathbb{F}_q$.

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for some basis $\mathbf{e}_1, \mathbf{e}_2$ in $\mathbb{F}_q^2$ and function $f : \mathbb{F}_q \to \mathbb{F}_q$.

Therefore, the non-trivial tiling set that we are looking for has q elements, and it is graph of a function.

Magic Lemma

One of main observation in the construction of a wavelet set is the following result:

Lemma (Magic Lemma)

There exists $\mathbf{k} \in \mathbb{F}_q$, $0 < \mathbf{k} \leq \frac{q-1}{2}$ *such that* $1 + \mathbf{k}^2$ *is a quadratic non-residue (mod q) and we have*

$$\mathit{QNR} = \{(1 + \mathbf{k}^2)x^2 : (q+1)/2 \leq x \leq q-1\}.$$

Construction of tight frame wavelet sets

Theorem

Assume that q is an odd prime and $q \equiv 3 \pmod 4$. Then (non-trivial) tight frame wavelet set exists in $\mathbb{F}_q^d$.

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Assume that q is an odd prime and $q \equiv 3$ (mod 4). Then (non-trivial) tight frame wavelet set exists in $\mathbb{F}_q^d$.

Proof.

Step 1. tight frame wavelet set E exists in $\mathbb{F}_q^2$.

We want E is a spectral set for $d = 2$

E non-trivial and tiling $\Longrightarrow \sharp E = q \;\Longrightarrow E$ must be graph of a function.

Take E such that

- $\vec{0} = (0,0) \in E$
- $(x,0) \in E$ where $0 < x \leq \frac{q-1}{2}$
- For $\mathbf{k}$ in Magic Lemma, let $(x, \mathbf{k}x) \in E$ where $\dfrac{q+1}{2} \leq x < q$.

E is obviously graph of a function, thus a tiling set and a spectral set. □

Proof cont'd.

We want E^* is multiplicative tiling for $\mathbb{F}_q^2$.

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Construction of tight frame wavelet sets

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Proof cont'd.

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By the above construction we have $\sharp(E \cap S_r) = 1$.

Take $\mathcal{A} = \{R^j, 0 \leq j \leq q\}$.

Then E^* is a multiplicative setting with respect to $\mathcal{A}$.

Construction of tight frame wavelet sets

Proof cont'd.

We want E^* is multiplicative tiling for $\mathbb{F}_q^2$.

By the above construction we have $\sharp(E \cap S_r) = 1$.

Take $\mathcal{A} = \{R^j, 0 \le j \le q\}$.

Then E^* is a multiplicative setting with respect to $\mathcal{A}$.

Step 2. Lifting example to $\mathbb{F}_q^d$

$\widetilde{E} := E \times \mathbb{F}_q^{d-2}$ is a spectral set and $\widetilde{E}^* := E^* \times \mathbb{F}_q^{d-2}$ is a multiplicative tiling for $\mathbb{F}_q^d \setminus \{0\}$. □

Construction of tight frame wavelet sets

Example

For $q = 7$, $\mathbf{k} \in \{2, 3\}$.

If we pick $\mathbf{k} = 2$

$$E_2 = \{(0,0), (1,0), (2,0), (3,0), (4,1), (5,3), (6,5)\}$$

Questions to study:

- Does there exist tight frame wavelet sets when $q \equiv 1 \pmod 4$?
- To what extent is it possible to generalize the results to $\mathbb{F}_q^d$, for $q = p^\alpha$ and $q = p^\alpha r$, $(p, r) = 1$?

Thanks!