# Configurations in sets big and small

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# Plan

- Set-up
- History
- Avoidance
- Existence
- Abundance

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# Structures in sets: some general questions

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# What is big? What is small?

# Quantification of size

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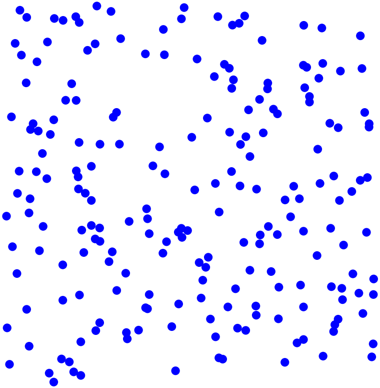
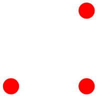
- dimension(s)
- density

# What is a configuration?

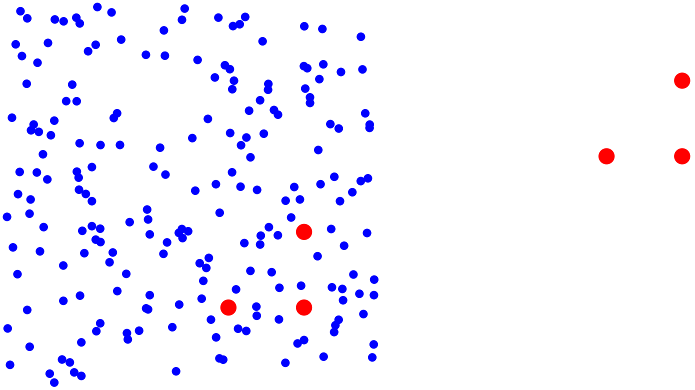
- In principle, any prescribed set.
- Could be
  - **geometric**, such as specially arranged points on a line, vertices of an equilateral triangle, or
  - **Algebro-analytic**, for example solutions of a polynomial equation.

# A set and a configuration

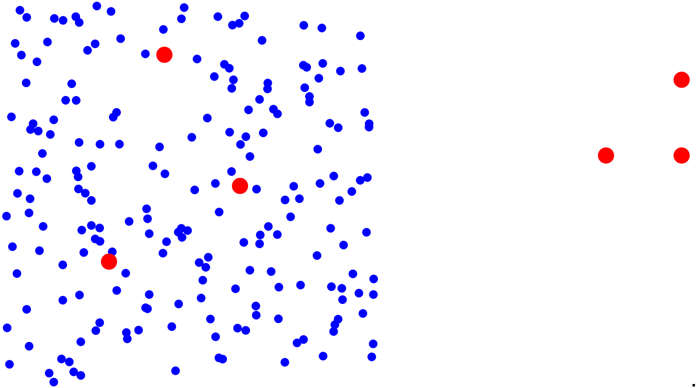




# A congruent copy of the triangle



# An affine copy of the triangle



# An example: Progressions in the integers

**Theorem (Szemerédi 1975)**

If $E \subseteq \mathbb{N}$ with positive asymptotic density, i.e.,

$$\limsup_{N \to \infty} \frac{\#(E \cap [1, N])}{N} = \delta > 0,$$

then $E$ contains an arithmetic progression of length $k$, for any $k \geq 3$.

# Example (ctd): Progressions in large sets of zero density

- (Salem and Spencer 1942, Behrend 1946) There are large sets $E_N \subseteq \{1, 2, \cdots, N\}$,

$$\#(E_N) > N^{1-c/\sqrt{\log N}}$$

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# Example (ctd): Progressions in large sets of zero density

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that contain no three-term arithmetic progression.

- But there are other large sets of zero density that have long progressions!

  - random sets (Kohayakawa, Łuczak and Rödl 1996),

  - primes (Green and Tao 2008), ...

# Plan

- Nomenclature
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# Szemerédi-type problems in the continuum

## A prototypical question

Given

- a class of subsets $\mathcal{E}$ in $\mathbb{R}^n$ whose members are "large", and
- a choice of geometric configurations $\mathcal{F}$

must every set $E \in \mathcal{E}$ contain one of the prescribed configurations $F \in \mathcal{F}$?

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For instance,

- given a fixed $F \subseteq \mathbb{R}$, is there a geometrically similar copy of $F$ in every set of positive Lebesgue measure? Here $\mathcal{E}$ = sets of positive Lebesgue measure, $\mathcal{F}$ = similar copies of $F$.
- If yes, call $F$ **universal**.

# Finite sets are universal

**Theorem (Steinhaus 1920)**

Given

- any finite set $F \subset \mathbb{R}$, and
- any set $E \subseteq \mathbb{R}$ of positive Lebesgue measure,

there exists $x \in \mathbb{R}$ and $t \neq 0$ such that $x + tF \subseteq E$.

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- If $E$ has no affine copy of $F$, then for every $t > 0$, either $x - t \notin E$ or $x + t \notin E$, so

$$|E \cap (x - r, x + r)| \leq \frac{2r}{2} = r \qquad \forall r,$$

contradicting Lebesgue density!

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## Erdős similarity problem 1974

Does there exist an infinite universal set?

- *"I hope there are no such sets"* - Erdős.
- \$100 prize!
- An earlier question:

  "If $\{x_n\}$ is an infinite sequence $\to 0$, then for every $E \subseteq \mathbb{R}$, $|E| > 0$, does $\exists x \in \mathbb{R}$ such that $x + x_n \in E$ for all sufficiently large $n$?"

  answered in the negative by Borwein and Ditor 1978.

# Erdős similarity problem : progress so far

**Conjecture restated**

Given any infinite set $F \subseteq \mathbb{R}$, there exists a set $E$ of positive measure which does not contain any nontrivial affine copy of $F$.

- Conjecture verified for
  - slowly decaying sequences $\{x_i\}$, where $x_{i+1}/x_i \to 1$ (Falconer 1984),
  - $S_1 + S_2 + S_3$, where each $S_j$ is infinite (Bourgain 1987),
  - $\{2^{-n^\alpha}\} + \{2^{-n^\alpha}\}$ for $0 < \alpha < 2$ (Kolountzakis 1997).

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- Not known even for $\{2^{-n} : n \geq 1\}$.

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# Now what?

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- Finding similar copies of infinite patterns in sets of positive Lebesgue measure seems to be hard, but ...
- Can one find other large Lebesgue-null sets that contain affine copies of all finite sets?

# Dimension: an alternative notion of size

Given $E \subseteq \mathbb{R}^n$, recall

## Definition

$$\dim_{\mathbb{H}}(E) := \sup \left\{ \alpha \in [0, n] \;\middle|\; \begin{array}{l} \exists \text{ a probability measure } \mu, \ \text{supp}(\mu) \subseteq E, \\ \displaystyle \sup_{x \in \mathbb{R}^n} \sup_{\epsilon > 0} \frac{\mu(B(x, \epsilon))}{\epsilon^{\alpha}} < \infty \end{array} \right\}.$$

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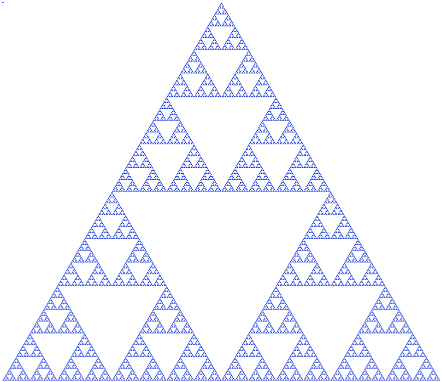
- Agrees with the standard notion of dimension for curves, surfaces etc.
- Assigns a quantitative measure of size to less regular objects, such as fractals.
- The Cantor middle-third set has Hausdorff dimension $\log 2 / \log 3$.

# Dimension of fractals: some examples

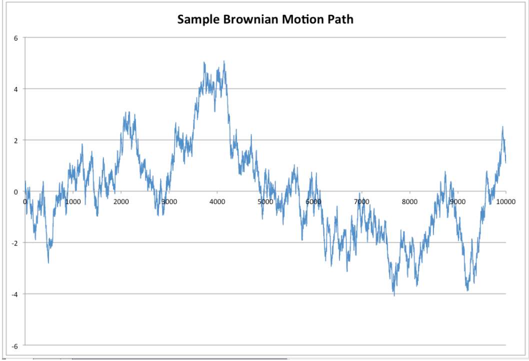
Sierpinski triangle



dimension $= \log_2 3$

# Dimension of fractals: some examples

Graph of Brownian motion




dimension $= 3/2$

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- Every set of positive Lebesgue measure has full Hausdorff dimension.
- Converse is not true. There are many Lebesgue-null sets of full dimension.

# Universality revisited

## Revised questions

- Does there exist a Lebesgue-null subset of $\mathbb{R}$ of full Hausdorff dimension containing an affine copy of every finite configuration?

# Universality revisited

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- Does there exist a Lebesgue-null subset of $\mathbb{R}$ of full Hausdorff dimension containing an affine copy of every finite configuration?
- If the answer to the above is yes, must every full-dimensional Lebesgue-null set have this property?

# Yes! and No!

**Theorem (Erdős and Kakutani 1957)**

There exists a compact Lebesgue-null set in $\mathbb{R}$ of Hausdorff dimension 1 containing similar copies of all finite subsets.

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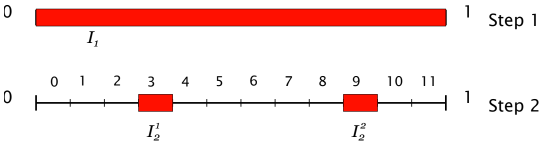
**Theorem (Keleti 1998, 2008)**

- For a given distinct triple of points $\{x, y, z\}$, there exists a compact set in $\mathbb{R}$ with Hausdorff dimension 1 which does not contain any similar copy of $\{x, y, z\}$.
- Given any countable $A \subset (1, \infty)$, there exists a compact set $E \subset \mathbb{R}$ with Hausdorff dimension 1 such that if $x < y < z$, $x, y, z \in E$ then $(z - x)/(z - y) \notin A$.

# Keleti's example: Cantor construction with memory




$\{ I_1 = J_1,\ I_2^1 = J_2,\ I_2^2 = J_3 \}$

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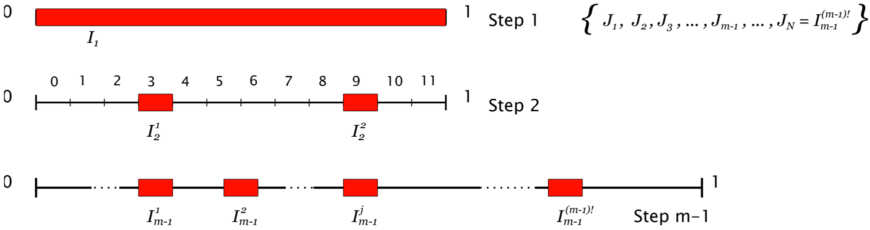

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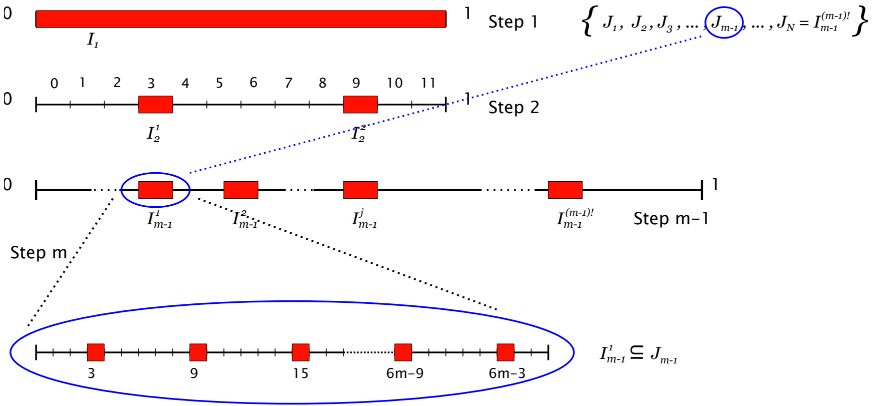

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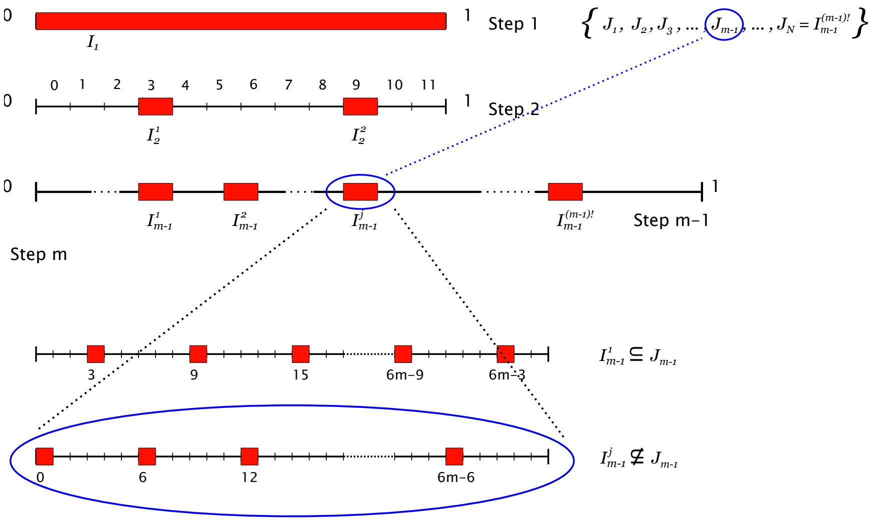
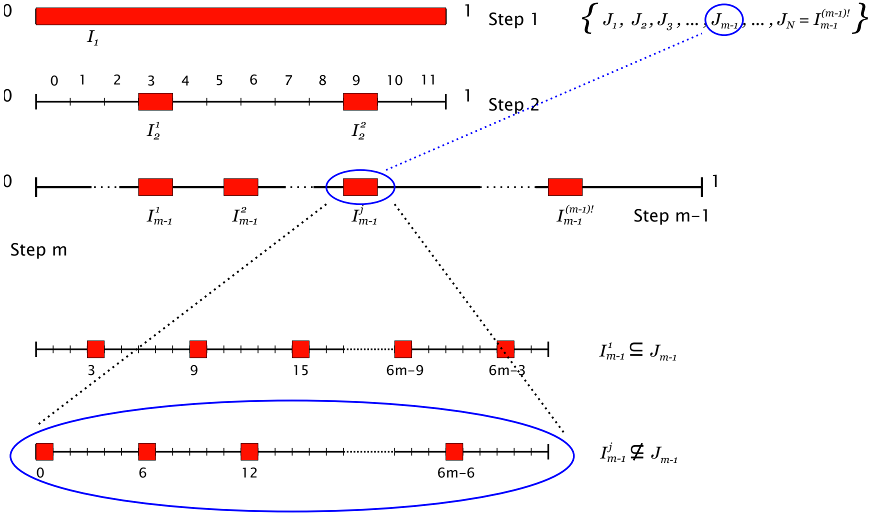


# Avoidance of $x_1 < x_2 \le x_3 < x_4$ with $x_2 - x_1 = x_4 - x_3$

Let $x_1 \in J_{m-1}$, $x_2, x_3, x_4 \notin J_{m-1}$.




At step $m$, $x_1 \in$ an interval indexed by $6\mathbb{Z} + 3$, but $x_2, x_3, x_4$-index is $6\mathbb{Z}$!

# Higher dimensional configurations: different points of view

## Theorem (Maga 2010)

(a) For distinct $x, y, z \in \mathbb{R}^2$, there exists a compact set in $\mathbb{R}^2$ with Hausdorff dimension 2 not containing any similar copy of $\{x, y, z\}$.

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(a) If $E \subseteq \mathbb{R}^2$ is compact with $\dim_{\mathbb{H}}(E) = 2$, must $E$ contain the vertices of an isosceles triangle?

(b) Given a set $E \subset \mathbb{R}^n$, how large can $\dim_{\mathbb{H}}(E)$ be if $E$ does not contain a triple of points forming a particular angle $\theta$?

# Avoidance: angles, functional zeros

## Angles in sets

Say $\angle\theta \in E \subset \mathbb{R}^n$ if there exist distinct points $x, y, z \in E$ such that the angle between the vectors $y - x$ and $z - x$ is $\theta$. Define

$$C(n,\theta) := \sup\{s : \exists E \subseteq \mathbb{R}^n \text{ compact with } \dim_{\mathbb{H}}(E) = s,\ \angle\theta \notin E\}$$

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- For $\theta = 0, \pi$, $C(n,\theta) = n - 1$.
- For $\theta = \frac{\pi}{2}$, $n/2 \leq C(n,\theta) \leq \lfloor (n+1)/2 \rfloor$.
- Sets of dimension $1/d$ avoiding zeros of a multivariate polynomial of degree $d$ with rational coefficients (Mathé 2012)
- Large sets of special Fourier-analytic structure avoiding *all* $k$-variate rational linear relations (Körner 2009)
- Generalizations involving non-polynomials (Fraser-P 2016)

# Avoidance of functional zeros

**Theorem (Fraser-P 2016)**

Given

- $\eta > 0$,
- any countable collection $\mathcal{F} = \{f_q : \mathbb{R}^v \to \mathbb{R}\}$, each $f_q$ has a nonvanishing derivative of some finite order on $[0, \eta]^v$.

Then there exists a set $E \subseteq [0, \eta]$ of full Minkowski dimension and Hausdorff dimension at least $1/(v-1)$ such that $f_q(x_1, \cdots, x_q) \neq 0$ for each $q$ and any $v$-tuple of distinct points $x_1, \cdots, x_v \in E$.

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Variants available when

- $\mathcal{F}$ consists of vector-valued functions,
- $\mathcal{F}$ possibly uncountable, containing functions with a common linearization.

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# The notion of Fourier dimension

## Definition

The Fourier dimension of a set $E \subseteq \mathbb{R}^n$ is defined as

$$\dim_{\mathbb{F}}(E) = \sup \left\{ \beta \in [0, n] \,\middle|\, \begin{array}{l} \exists \text{ a probability measure } \mu, \ \text{supp}(\mu) \subseteq E, \\ \sup_{\xi \in \mathbb{R}^n} \left|\widehat{\mu}(\xi)\right| \left(1 + |\xi|\right)^{\frac{\beta}{2}} < \infty \end{array} \right\}.$$

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- Sets for which equality holds are called *Salem sets*.
- Deterministic constructions for Salem sets exist, but are rare. They are however ubiquitous among random sets (Salem, Kahane).

# Large Fourier dimension $\implies$ configurations? Take 1

**Theorem (Shmerkin 2016)**

For every $t \in (0, 1]$, there exists a Salem set of dimension $t$ that avoids progressions.

# Large Fourier dimension $\implies$ configurations? Take 2

**Theorem (Łaba-P 2009)**

Suppose $E \subseteq [0, 1]$ is a closed set which supports a probability measure $\mu$ with the following properties:

(a) (Ball condition)

$$\mu([x - \epsilon, x + \epsilon]) \leq C_1 \epsilon^{\alpha} \text{ for all } 0 < \epsilon \leq 1,$$

(b) (Fourier decay condition)

$$|\widehat{\mu}(\xi)| \leq C_2 |\xi|^{-\frac{\beta}{2}} \quad \text{for all } \xi \neq 0,$$

where $0 < \alpha < 1$ and $\frac{2}{3} < \beta \leq 1$. If $\alpha > 1 - \epsilon_0(C_1, C_2, \beta)$, then $E$ contains a 3-term arithmetic progression.

- Many higher dimensional generalizations (Chan, Henriot, Łaba, P)

# Where have all the points gone?

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Every set $E$ of real numbers of positive measure must have a difference set $E - E = \{x - y : x, y \in E\}$ that contains a nontrivial interval centred at the origin.

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## Falconer's conjecture

If $E \subseteq \mathbb{R}^n$, $\dim_{\mathbb{H}}(E) > \frac{n}{2}$, then the set of distances between pairs of points in $E$ must have positive Lebesgue measure.

# Large configuration spaces

For $E \subseteq \mathbb{R}^2$, let $T_2(E) = E^3/\sim$, where

$$(a, b, c) \sim (a', b', c') \text{ if and only if } \triangle abc \text{ and } \triangle a'b'c' \text{ are congruent.}$$

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$$(a, b, c) \sim (a', b', c') \text{ if and only if } \triangle abc \text{ and } \triangle a'b'c' \text{ are congruent.}$$

**Theorem (Greenleaf and Iosevich 2010)**

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- Higher dimensional extensions: Erdogan, Hart, Grafakos, Greenleaf, Iosevich, Taylor, Liu, Mourgoglu, Palsson, P, ...

# Plan

- Setup
- History
- Avoidance
- Existence
- **Abundance**

# A different direction: Euclidean Ramsey theory

Suppose that

- $E \subseteq \mathbb{R}^n$ has positive upper density (with respect to Lebesgue measure), i.e.,
$$\limsup_{R\to\infty} \sup_{x\in\mathbb{R}^n} \frac{|E \cap B(x;R)|}{R^n} > 0.$$
- $F$ is a non-degenerate $(k-1)$-dimensional simplex (i.e., a set of $k$ points in general position).

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**Theorem (Bourgain 1986)**

If $k \leq n$, then $E$ contains a translated and rotated copy of $\lambda F$ for all $\lambda$ sufficiently large.

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- Bourgain's result applies to triangles in $\mathbb{R}^3$ but not to triangles in $\mathbb{R}^2$.
- The result is false for three collinear equispaced points in $\mathbb{R}^2$.
- (Graham 1994) The result is false for any **non-spherical** set in $\mathbb{R}^n$!

# Norms and configurations?

**Cook, Magyar, P 2016**

- Fix any $1 < p < \infty$, $p \neq 2$.
- Then $\exists n_p < \infty$ such that $\forall n \geq n_p$,
- any set $A \subseteq \mathbb{R}^n$ of positive upper density contains a 3-term AP of every sufficiently large gap length, measured in $\ell^p$ norm.

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In other words, $\exists \lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, $\exists x, x+y, x+2y \in A$ such that $||y||_p = (\sum_{i=1}^d |y_i|^p)^{1/p} = \lambda$.

# Proving existence and/or abundance: Methodology

- Configuration counting functions, such as

$$\Lambda(f) = \iint f(x)f(x+y)f(x+2y)\,dy\,dx.$$

If $f = 1_A$, then $\Lambda(f)$ measures the number of 3 AP-s in $A$. If $A$ is a sparse set supporting a measure, finer definitions of $\Lambda$ are necessary.

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- Estimation of $\Lambda$:
  - Fourier-analytic methods, oscillatory integral estimates,
  - tools from additive combinatorics, such as Gowers norms,
  - tools from time-frequency analysis, e.g. bounds for multilinear singular integral operators such as the bilinear Hilbert transform.

# Questions

- Same result for corners by Durcik, Kovač and Rimanić, ... other examples and/or counterexamples?

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- To what extent do configuration results rely on the underlying topology?

# Thank you!