Configurations in sets big and small

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Harmonic Analysis and Geometry of Fractal Sets
Ohio State University, Columbus, Ohio
February 3, 2017
Plan

- Set-up
- History
- Avoidance
- Existence
- Abundance
Plan

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Does a thin set contain a prescribed configuration?
Structures in sets: some general questions

- Does a thin set contain a prescribed configuration?

- Must every large set contain one?
Words that need clarification

- Does a small/sparse set contain a prescribed configuration?

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- Does a small/sparse set contain a prescribed configuration?

- Must every large set contain one?
What is big? What is small?
Quantification of size

The size of a set can be specified in terms of

- a measure (e.g. counting measure in $\mathbb{Z}^d$, Lebesgue measure in $\mathbb{R}^d$)
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- dimension(s)
Quantification of size

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- a measure (e.g. counting measure in $\mathbb{Z}^d$, Lebesgue measure in $\mathbb{R}^d$)
- dimension(s)
- density
What is a configuration?

- In principle, any prescribed set.

- Could be
  
  - **geometric**, such as specially arranged points on a line, vertices of an equilaterial triangle, or
  
  - **Algebro-analytic**, for example solutions of a polynomial equation.
A set and a configuration
A congruent copy of the triangle
An affine copy of the triangle
An example: Progressions in the integers

Theorem (Szemerédi 1975)

If \( E \subseteq \mathbb{N} \) with positive asymptotic density, i.e.,

\[
\limsup_{N \to \infty} \frac{\#(E \cap [1, N])}{N} = \delta > 0,
\]

then \( E \) contains an arithmetic progression of length \( k \), for any \( k \geq 3 \).
Example (ctd): Progressions in large sets of zero density

(Salem and Spencer 1942, Behrend 1946) There are large sets $E_N \subseteq \{1, 2, \ldots, N\}$,

$$\#(E_N) > N^{1-c/\sqrt{\log N}}$$

that contain no three-term arithmetic progression.
Example (ctd): Progressions in large sets of zero density

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$$\#(E_N) > N^{1-c/\sqrt{\log N}}$$

that contain no three-term arithmetic progression.

- But there are other large sets of zero density that have long progressions!
  - random sets (Kohayakawa, Łuczak and Rödl 1996),
  - primes (Green and Tao 2008), ...
Plan

- Nomenclature
- History
- Avoidance
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A prototypical question

Given

- a class of subsets $\mathcal{E}$ in $\mathbb{R}^n$ whose members are “large”, and
- a choice of geometric configurations $\mathcal{F}$

must every set $E \in \mathcal{E}$ contain one of the prescribed configurations $F \in \mathcal{F}$?
Szemerédi-type problems in the continuum

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must every set $E \in \mathcal{E}$ contain one of the prescribed configurations $F \in \mathcal{F}$?

For instance,

- given a fixed $F \subseteq \mathbb{R}$, is there a geometrically similar copy of $F$ in every set of positive Lebesgue measure? Here $\mathcal{E} =$ sets of positive Lebesgue measure, $\mathcal{F} =$ similar copies of $F$.

- If yes, call $F$ universal.
Finite sets are universal

Theorem (Steinhaus 1920)

Given

- any finite set $F \subset \mathbb{R}$, and
- any set $E \subseteq \mathbb{R}$ of positive Lebesgue measure,

there exists $x \in \mathbb{R}$ and $t \neq 0$ such that $x + tF \subseteq E$. 
Steinhaus’s theorem - a special case

- Suppose $F = \{-1, 0, 1\}$, and $E$ has positive Lebesgue measure.

Lebesgue density theorem $\Rightarrow$ almost every $x \in E$ is a density point.

What this means is that $\exists x \in E$ such that $\lim_{r \to 0} \frac{|E \cap (x-r, x+r)|}{2r} = 1$.

If $E$ has no affine copy of $F$, then for every $t > 0$, either $x - t/2 \in E$ or $x + t/2 \in E$, so $|E \cap (x-r, x+r)| \leq 2r^2 = r$ for all $r$, contradicting Lebesgue density!
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  \]

- If $E$ has no affine copy of $F$, then for every $t > 0$, either $x - t \notin E$ or $x + t \notin E$, so
  \[
  |E \cap (x - r, x + r)| \leq \frac{2r}{2} = r \quad \forall r,
  \]
  contradicting Lebesgue density!
What about infinite sets $F$?

The Erdős similarity problem from 1974 asks whether there exists an infinite universal set. Erdős expressed a hope that there are no such sets: "I hope there are no such sets." A $100 prize was offered for this question. An earlier question asks if \( \{x_n\} \) is an infinite sequence converging to 0, does there exist an \( E \subseteq \mathbb{R} \) with \( |E| > 0 \) such that \( \exists x \in \mathbb{R} \) such that \( x + x_n \in E \) for sufficiently large \( n \)? This was answered in the negative by Borwein and Ditor in 1978.
What about infinite sets $F$?

**Erdős similarity problem 1974**

Does there exist an infinite universal set?

"I hope there are no such sets" - Erdős.

$100$ prize!

An earlier question:

"If $\{x_n\}$ is an infinite sequence $\rightarrow 0$, then for every $E \subseteq \mathbb{R}$, $|E| > 0$, does $\exists x \in \mathbb{R}$ such that $x + x_n \in E$ for all sufficiently large $n$?"

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answered in the negative by Borwein and Ditor 1978.
Erdős similarity problem: progress so far

**Conjecture restated**

Given any infinite set \( F \subseteq \mathbb{R} \), there exists a set \( E \) of positive measure which does not contain any nontrivial affine copy of \( F \).

- Conjecture verified for
  - slowly decaying sequences \( \{x_i\} \), where \( x_{i+1}/x_i \to 1 \) (Falconer 1984),
  - \( S_1 + S_2 + S_3 \), where each \( S_j \) is infinite (Bourgain 1987),
  - \( \{2^{-n^\alpha}\} + \{2^{-n^\alpha}\} \) for \( 0 < \alpha < 2 \) (Kolountzakis 1997).
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- Not known even for $\{2^{-n} : n \geq 1\}$. 

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- **Avoidance**
- Existence
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Now what?

- Finding similar copies of infinite patterns in sets of positive Lebesgue measure seems to be hard, but ...
Now what?

- Finding similar copies of infinite patterns in sets of positive Lebesgue measure seems to be hard, but ...

- Can one find other large Lebesgue-null sets that contain affine copies of all finite sets?
Dimension: an alternative notion of size

Given $E \subseteq \mathbb{R}^n$, recall

**Definition**

$$\dim_{\mathbb{H}}(E) := \sup \left\{ \alpha \in [0, n] \left| \exists \text{ a probability measure } \mu, \text{ supp}(\mu) \subseteq E, \sup_{x \in \mathbb{R}^n} \sup_{\epsilon > 0} \frac{\mu(B(x, \epsilon))}{\epsilon^\alpha} < \infty \right. \right\}.$$ 

Agrees with the standard notion of dimension for curves, surfaces etc.
Assigns a quantitative measure of size to less regular objects, such as fractals.

The Cantor middle-third set has Hausdorff dimension $\frac{\log 2}{\log 3}$. 

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Configurations in sets
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- Assigns a quantitative measure of size to less regular objects, such as fractals.
- The Cantor middle-third set has Hausdorff dimension $\log 2 / \log 3$. 

Dimension of fractals: some examples

Sierpinski triangle

dimension = $\log_2 3$
Dimension of fractals: some examples

Graph of Brownian motion

Sample Brownian Motion Path

dimension = 3/2
Dimension: an alternative notion of size

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Dimension: an alternative notion of size

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**Definition**

$$\text{dim}_H(E) := \sup \left\{ \alpha \in [0, n] \left| \begin{array}{c} \exists \text{ a probability measure } \mu, \ \text{supp}(\mu) \subseteq E, \\ \sup_{x \in \mathbb{R}^n} \sup_{\epsilon > 0} \frac{\mu(B(x, \epsilon))}{\epsilon^\alpha} < \infty \end{array} \right. \right\}.$$ 

- Every set of positive Lebesgue measure has full Hausdorff dimension.
- Converse is not true. There are many Lebesgue-null sets of full dimension.
Revised questions

- Does there exist a Lebesgue-null subset of $\mathbb{R}$ of full Hausdorff dimension containing an affine copy of every finite configuration?

If the answer to the above is yes, must every full-dimensional Lebesgue-null set have this property?
Universality revisited

Revised questions

- Does there exist a Lebesgue-null subset of $\mathbb{R}$ of full Hausdorff dimension containing an affine copy of every finite configuration?

- If the answer to the above is yes, must every full-dimensional Lebesgue-null set have this property?
Theorem (Erdős and Kakutani 1957)

There exists a compact Lebesgue-null set in $\mathbb{R}$ of Hausdorff dimension 1 containing similar copies of all finite subsets.
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Theorem (Keleti 1998, 2008)
For a given distinct triple of points $\{x, y, z\}$, there exists a compact set in $\mathbb{R}$ with Hausdorff dimension 1 which does not contain any similar copy of $\{x, y, z\}$.
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Theorem (Keleti 1998, 2008)
- For a given distinct triple of points $\{x, y, z\}$, there exists a compact set in $\mathbb{R}$ with Hausdorff dimension 1 which does not contain any similar copy of $\{x, y, z\}$.

- Given any countable $A \subset (1, \infty)$, there exists a compact set $E \subset \mathbb{R}$ with Hausdorff dimension 1 such that if $x < y < z$, $x, y, z \in E$ then $(z - x)/(z - y) \notin A$. 
Keleti’s example: Cantor construction with memory

\[ \{ I_1 = J_1, \; I_2' = J_2, \; I_2 = J_3 \} \]
Keleti’s example: Cantor construction with memory

0 1
0 1 2 3 4 5 6 7 8 9 10
0 1
0 1 2 3 4 5 6 7 8 9 10 11
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Step 1 \[ \{ J_1, J_2, J_3, \ldots, J_{m-1}, \ldots, J_N = I_{m-1}^{(m-1)!} \} \]

Step 2

Step m-1
Keleti’s example: Cantor construction with memory

\[ I_1 \]

\[ I_2 \]

\[ I_{m-1} \]

\[ I_m \]

\[ J_1, J_2, J_3, \ldots, J_{m-1}, \ldots, J_N = I_{m-1}^{(m-1)!} \]

\[ I_{m-1}^{(m-1)!} \leq J_{m-1} \]
Keleti's example: Cantor construction with memory

$$I_1$$

$$I_2^1$$ $$I_2^2$$

Step 1

$$\left\{ J_1, J_2, J_3, \ldots, J_{m-1}, \ldots, J_N = I_{m-1}^{(m-1)!} \right\}$$

Step 2

Step m

$$I_{m-1}^i \subseteq J_{m-1}$$

$$I_{m-1}^i \not\subseteq J_{m-1}$$
Avoidance of $x_1 < x_2 \leq x_3 < x_4$ with $x_2 - x_1 = x_4 - x_3$

Let $x_1 \in J_{m-1}$, $x_2, x_3, x_4 \notin J_{m-1}$.

At step $m$, $x_1 \in$ an interval indexed by $6\mathbb{Z} + 3$, but $x_2, x_3, x_4$-index is $6\mathbb{Z}$!
Higher dimensional configurations: different points of view

Theorem (Maga 2010)

(a) For distinct $x, y, z \in \mathbb{R}^2$, there exists a compact set in $\mathbb{R}^2$ with Hausdorff dimension 2 not containing any similar copy of $\{x, y, z\}$.

(b) There exists a compact set in $\mathbb{R}^n$ with Hausdorff dimension $n$ which does not contain any parallelogram $\{x, x+y, x+z, x+y+z\}$ with $y, z \neq 0$.

Questions (Maga 2010)

(a) If $E \subseteq \mathbb{R}^2$ is compact with $\dim_H(E) = 2$, must $E$ contain the vertices of an isosceles triangle?

(b) Given a set $E \subset \mathbb{R}^n$, how large can $\dim_H(E)$ be if $E$ does not contain a triple of points forming a particular angle $\theta$?
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Avoidance: angles, functional zeros

Angles in sets

Say $\angle \theta \in E \subset \mathbb{R}^n$ if there exist distinct points $x, y, z \in E$ such that the angle between the vectors $y - x$ and $z - x$ is $\theta$. Define

$$C(n, \theta) := \sup \{ s : \exists E \subset \mathbb{R}^n \text{ compact with } \dim_H(E) = s, \angle \theta \notin E \}$$
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- For $\theta = 0, \pi$, $C(n, \theta) = n - 1$.
- For $\theta = \frac{\pi}{2}$, $n/2 \leq C(n, \theta) \leq \lceil(n + 1)/2 \rceil$.
- Sets of dimension $1/d$ avoiding zeros of a multivariate polynomial of degree $d$ with rational coefficients (Mathé 2012)
- Large sets of special Fourier-analytic structure avoiding all $k$-variate rational linear relations (Körner 2009)
- Generalizations involving non-polynomials (Fraser-P 2016)
Avoidance of functional zeros

Theorem (Fraser-P 2016)

Given

- $\eta > 0$,
- any countable collection $\mathcal{F} = \{ f_q : \mathbb{R}^\nu \to \mathbb{R} \}$, each $f_q$ has a nonvanishing derivative of some finite order on $[0, \eta]^\nu$.

Then there exists a set $E \subseteq [0, \eta]$ of full Minkowski dimension and Hausdorff dimension at least $1/(\nu - 1)$ such that $f_q(x_1, \cdots, x_\nu) \neq 0$ for each $q$ and any $\nu$-tuple of distinct points $x_1, \cdots, x_\nu \in E$. 
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**Theorem (Fraser-P 2016)**

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Variants available when

- $\mathcal{F}$ consists of vector-valued functions,
- $\mathcal{F}$ possibly uncountable, containing functions with a common linearization.
Plan

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The notion of Fourier dimension

Definition

The Fourier dimension of a set $E \subseteq \mathbb{R}^n$ is defined as

$$\dim_F(E) = \sup \left\{ \beta \in [0, n] \left| \exists \text{ a probability measure } \mu, \text{ supp}(\mu) \subseteq E, \sup_{\xi \in \mathbb{R}^n} |\hat{\mu}(\xi)| (1 + |\xi|)^{\frac{\beta}{2}} < \infty \right. \right\}.$$
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- $\dim_{F}(E) \leq \dim_{H}(E)$ for all $E \subseteq \mathbb{R}^n$.
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- $\dim_{\mathcal{F}}(E) \leq \dim_{\mathcal{H}}(E)$ for all $E \subseteq \mathbb{R}^n$.
- Inequality can be strict; for instance, $\dim_{\mathcal{H}}(E) = \frac{\log 2}{\log 3}$ and $\dim_{\mathcal{F}}(E) = 0$ for the Cantor middle-third set.
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- Inequality can be strict; for instance, $\dim_H(E) = \frac{\log 2}{\log 3}$ and $\dim_F(E) = 0$ for the Cantor middle-third set.
- Sets for which equality holds are called *Salem sets*. 
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- $\dim_F(E) \leq \dim_H(E)$ for all $E \subseteq \mathbb{R}^n$.
- Inequality can be strict; for instance, $\dim_H(E) = \frac{\log 2}{\log 3}$ and $\dim_F(E) = 0$ for the Cantor middle-third set.
- Sets for which equality holds are called Salem sets.
- Deterministic constructions for Salem sets exist, but are rare. They are however ubiquitous among random sets (Salem, Kahane).
Theorem (Shmerkin 2016)

For every $t \in (0, 1]$, there exists a Salem set of dimension $t$ that avoids progressions.
Large Fourier dimension $\implies$ configurations? Take 2

**Theorem (Łaba-P 2009)**

Suppose $E \subseteq [0, 1]$ is a closed set which supports a probability measure $\mu$ with the following properties:

(a) (Ball condition)

$$\mu([x - \epsilon, x + \epsilon]) \leq C_1 \epsilon^{\alpha} \text{ for all } 0 < \epsilon \leq 1,$$

(b) (Fourier decay condition)

$$|\hat{\mu}(\xi)| \leq C_2 |\xi|^{-\frac{\beta}{2}} \text{ for all } \xi \neq 0,$$

where $0 < \alpha < 1$ and $\frac{2}{3} < \beta \leq 1$. If $\alpha > 1 - \epsilon_0(C_1, C_2, \beta)$, then $E$ contains a 3-term arithmetic progression.

- Many higher dimensional generalizations (Chan, Henriot, Łaba, P)
Where have all the points gone?

**Metaprinciple:** If a set is large, then the class of certain configurations lying in this space must be large as well, in terms of a natural measure defined on the class of such configurations.
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**Steinhaus’s theorem**

Every set $E$ of real numbers of positive measure must have a difference set $E - E = \{x - y : x, y \in E\}$ that contains a nontrivial interval centred at the origin.
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**Falconer’s conjecture**

If $E \subseteq \mathbb{R}^n$, $\dim_{\text{H}}(E) > \frac{n}{2}$, then the set of distances between pairs of points in $E$ must have positive Lebesgue measure.
Large configuration spaces

For $E \subseteq \mathbb{R}^2$, let $T_2(E) = E^3 / \sim$, where

$$(a, b, c) \sim (a', b', c')$$

if and only if $\triangle abc$ and $\triangle a'b'c'$ are congruent.

**Theorem (Greenleaf and Iosevich 2010)**

Let $E \subseteq \mathbb{R}^2$ be a compact set with $\dim H(E) > \frac{7}{4}$. Then $T_2(E)$ has positive 3-dimensional measure.

Result says that triangles are abundant in sets of large Hausdorff dimension. However it does not (and in light of Maga's theorem cannot) guarantee the existence of (a similar copy of) a specific triangle.

Higher dimensional extensions: Erdogan, Hart, Grafakos, Greenleaf, Iosevich, Taylor, Liu, Mourgoglu, Palsson, P, ...

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Let $E \subseteq \mathbb{R}^2$ be a compact set with $\dim_{\mathbb{H}}(E) > \frac{7}{4}$. Then $T_2(E)$ has positive 3-dimensional measure.

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Large configuration spaces

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- Higher dimensional extensions: Erdogan, Hart, Grafakos, Greenleaf, Iosevich, Taylor, Liu, Mourgoglu, Palsson, P, ...
Plan

- Setup
- History
- Avoidance
- Existence
- Abundance
A different direction: Euclidean Ramsey theory

Suppose that

- \( E \subseteq \mathbb{R}^n \) has positive upper density (with respect to Lebesgue measure), i.e.,
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  \limsup_{R \to \infty} \sup_{x \in \mathbb{R}^n} \frac{|E \cap B(x; R)|}{R^n} > 0.
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- \( F \) is a non-degenerate \((k - 1)\)-dimensional simplex (i.e., a set of \( k \) points in general position).
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**Theorem (Bourgain 1986)**

If $k \leq n$, then $E$ contains a translated and rotated copy of $\lambda F$ for all $\lambda$ sufficiently large.
A set $E \subseteq \mathbb{R}^2$ of positive upper density can reproduce all sufficiently large distances.
All sufficiently large copies of configurations (ctd)

- A set $E \subseteq \mathbb{R}^2$ of positive upper density can reproduce all sufficiently large distances.

- Given three distinct and non-collinear points $a, b, c \in \mathbb{R}^3$ a set $E \subseteq \mathbb{R}^3$ of positive upper density contains translated and rotated copies of $\triangle abc$ rescaled by $\lambda$, for all $\lambda$ large enough.
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Bourgain’s result applies to triangles in $\mathbb{R}^3$ but not to triangles in $\mathbb{R}^2$. 

(Graham 1994) The result is false for any non-spherical set in $\mathbb{R}^n$.
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- (Graham 1994) The result is false for any non-spherical set in $\mathbb{R}^n$!
- Fix any $1 < p < \infty$, $p \neq 2$.
- Then $\exists n_p < \infty$ such that $\forall n \geq n_p$,
- any set $A \subseteq \mathbb{R}^n$ of positive upper density contains a 3-term AP of every sufficiently large gap length, measured in $\ell^p$ norm.
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any set $A \subseteq \mathbb{R}^n$ of positive upper density contains a 3-term AP of every sufficiently large gap length, measured in $\ell^p$ norm.

In other words, there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, there exist $x, x + y, x + 2y \in A$ such that $||y||_p = (\sum_{i=1}^d |y_i|^p)^{1/p} = \lambda$. 
Configuration counting functions, such as

\[ \Lambda(f) = \int\int f(x)f(x+y)f(x+2y) \, dy \, dx. \]

If \( f = 1_A \), then \( \Lambda(f) \) measures the number of 3 AP-s in \( A \). If \( A \) is a sparse set supporting a measure, finer definitions of \( \Lambda \) are necessary.
Proving existence and/or abundance: Methodology

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- Estimation of \( \Lambda \):
  - Fourier-analytic methods, oscillatory integral estimates,
  - tools from additive combinatorics, such as Gowers norms,
  - tools from time-frequency analysis, e.g. bounds for multilinear singular integral operators such as the bilinear Hilbert transform.
Questions

- Same result for corners by Durcik, Kovač and Rimanić, ... other examples and/or counterexamples?
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- To what extent do configuration results rely on the underlying topology?
Thank you!