Falconer type theorems for simplices

Eyvindur Ari Palsson

Department of Mathematics
Virginia Tech

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Harmonic Analysis And Geometry Of Fractal Sets
The Ohio State University
Distinct distances

- Distances: $1, 1, \sqrt{2}, \sqrt{5}, \sqrt{5}, \sqrt{8}$
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- Distinct distances: $1, \sqrt{2}, \sqrt{5}, \sqrt{8}$
- How many distinct distances are there in general?
Greatest lower bounds on a $\sqrt{N} \times \sqrt{N}$ lattice?

Asymptotically, up to constants, $N^{\sqrt{\log(N)}}$ distinct distances.
Greatest lower bounds on a $\sqrt{N} \times \sqrt{N}$ lattice?

- Asymptotically, up to constants, $\frac{N}{\sqrt{\log(N)}}$ distinct distances.
Erdős distinct distance problem (1946)

- What is the least number of distinct distances determined by $N$ points in the plane?

- Conjecture $\frac{N}{\sqrt{\log(N)}}$. 
What is the least number of distinct distances determined by $N$ points in the plane?

Conjecture $\frac{N}{\sqrt{\log(N)}}$.

If $x_1, \ldots, x_N$ are distinct points in the plane then what is the smallest possible cardinality of the distance set

$$\{|x_i - x_j| : 1 \leq i < j \leq N\}$$
Progress on the conjecture in the plane

- Have $\frac{N}{\sqrt{\log(N)}}$ for the lattice. Is it possible to have fewer distinct distances?

- $\frac{N}{\log(N)}$ (Guth, Katz 2010)
- $N^{0.864...}$ (Katz, Tardos 2004)
- $N^{0.8634...}$ (Tardos 2003)
- $N^{0.8571}$ (Solymosi, Toth 2001)
- $N^{0.8}$ (Szekely 1993)
- $\frac{N^{0.8}}{\log(N)}$ (Chung, Szemeredi, Trotter 1992)
- $N^{0.7143...}$ (Chung 1984)
- $N^{0.66...}$ (Moser 1952)
- $N^{0.5}$ (Erdős 1946)
How large does $\dim_H(E)$, for $E \subset [0, 1]^d$, need to be to ensure that the distance set

$$\Delta(E) = \{|x - y| : x, y \in E\}$$

has positive one-dimensional Lebesgue measure?
Falconer distance problem (1985)

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- Falconer’s conjecture $\dim_{\mathcal{H}}(E) > \frac{d}{2}$.
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- Falconer’s conjecture $\dim_H(E) > \frac{d}{2}$.

- Falconer obtained $\dim_H(E) > \frac{d}{2} + \frac{1}{2}$ in 1985.

- Current best result $\dim_H(E) > \frac{d}{2} + \frac{1}{3}$ due to Thomas Wolff ($d = 2$ in 1999) and Burak Erdoğan ($d > 2$ in 2005).
Distances

\[ \Phi(x_1, x_2) = |x_1 - x_2| \]
Higher order patterns: Simplexes

\[ \Phi(x_1, x_2, x_3) = (|x_1 - x_2|, |x_1 - x_3|, |x_2 - x_3|) \]
The $k + 1$ point simplex

Let

$$
\Phi : (\mathbb{R}^d)^{k+1} \rightarrow \mathbb{R}^{\binom{k+1}{2}}
$$

be defined by

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\Phi(x_1, \ldots, x_{k+1}) = (|x_2 - x_1|, |x_3 - x_1|, |x_3 - x_2|, \ldots, |x_{k+1} - x_k|).
$$

and given $E \subset [0, 1]^d$, define

$$
\Delta_\Phi(E) = \{ \Phi(x_1, \ldots, x_{k+1}) : x_j \in E \}.
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and given $E \subset [0, 1]^d$, define

$$\Delta_\Phi(E) = \{\Phi(x_1, \ldots, x_{k+1}) : x_j \in E\}.$$

How large does $dim_H(E)$ need to be to ensure that the $\binom{k+1}{2}$-dimensional Lebesgue measure of $\Delta_\Phi(E)$ is positive?
An incidence theorem is sufficient

- In order to show that the \( \binom{k+1}{2} \)-dimensional Lebesgue measure of \( \Delta_\Phi(E) \) is positive it is sufficient to prove

\[
\nu \times \cdots \times \nu \{ (x_1, \ldots, x_{k+1}) \in E^{k+1} : |\Phi(x_1, \ldots, x_{k+1}) - \vec{t}| < \epsilon \} \lesssim \epsilon^{\binom{k+1}{2}}
\]

for any Frostman measure \( \nu \) supported on \( E \) and all \( \epsilon > 0, \vec{t} \).
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- A probability measure \(\nu\) on a compact set \(E \subset \mathbb{R}^d\) is a Frostman measure if, for any ball \(B_\delta\) of radius \(\delta\),

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\nu(B_\delta) \lesssim \delta^s
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where \(s < \text{dim}_\mathcal{H}(E)\).
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for any Frostman measure $\nu$ supported on $E$ and all $\epsilon > 0$, $\vec{t}$.

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where \( s < \text{dim}_\mathcal{H}(E) \).
Why is an incidence theorem sufficient?

- Pigeonhole principle argument
Cover $\Delta\Phi(E)$ by balls $B_{\epsilon_i}(\vec{t}_i)$, centered at $\vec{t}_i$ with radius $\epsilon_i$.

Assuming

$$\nu \times \cdots \times \nu\{ (x_1, \ldots, x_{k+1}) \in E^{k+1} : |\Phi(x_1, \ldots, x_{k+1}) - \vec{t}| < \epsilon \} \lesssim \epsilon^{\frac{k+1}{2}}$$

we get

$$1 = \nu \times \cdots \times \nu(E^{k+1})$$

$$\leq \sum_i \nu \times \cdots \times \nu\{ (x_1, \ldots, x_{k+1}) : |\Phi(x_1, \ldots, x_{k+1}) - \vec{t}_i| < \epsilon_i \}$$

$$\lesssim \sum_i \epsilon_i^{\frac{k+1}{2}}$$

which implies that the $\binom{k+1}{2}$-dimensional Lebesgue measure of $\Delta\Phi(E)$ is positive.
An analyst’s point of view

View

\[\nu \times \cdots \times \nu \{(x_1, \ldots, x_{k+1}) \in E^{k+1} : |\Phi(x_1, \ldots, x_{k+1}) - \bar{t}| < \epsilon\} \lesssim \epsilon^{(k+1)/2}\]

as

\[\int \cdots \int 1_{\{|\Phi(x_1, \ldots, x_{k+1}) - \bar{t}| < \epsilon\}} d\mu(x_1) \cdots d\mu(x_k) d\mu(x_{k+1}) \lesssim \epsilon^{(k+1)/2}\]

Similar to Falconer’s original approach
Generalized Radon transforms

- Integral operators arose in Falconer’s work.

- Specifically the spherical averaging operator, an example of a generalized Radon transform

\[ Sf(x) = \int_{|x-y|=1} f(y) d\sigma(y) \]
Generalized Radon transforms

- Integral operators arose in Falconer’s work.

- Specifically the spherical averaging operator, an example of a generalized Radon transform

$$ S f(x) = \int_{|x-y|=1} f(y) d\sigma(y) $$

- For a generalized Radon transform $\mathcal{R}$ fulfilling certain conditions one can show estimates of the type $\mathcal{R} : L^p \rightarrow L^p_\gamma$

where $L^p_\gamma$ is a Sobolev space.
Multilinear generalized Radon transforms

- The linear generalized Radon transforms of Phong and Stein are given by

\[ R f(x) = \int \Phi(x, y) f(y) d\mu_x(y) \]

- By analogy the \( k \)-linear generalized Radon transforms are defined to be

\[ R_k(f_1, \ldots, f_k)(x_{k+1}) = \int f_1(x_1) \cdots f_k(x_k) d\mu_{x_{k+1}}(x_1, \ldots, x_k) \]

\[ \Phi(x_1, \ldots, x_k, x_{k+1}) = \vec{t} \]
An estimate for multilinear generalized Radon transforms

Theorem (Grafakos, Greenleaf, Iosevich, P)

Define

\[ T_\mu(f_1, \ldots, f_k)(x) = \int f_1(x - u_1) \ldots f_k(x - u_k) d\mu(u_1, \ldots, u_k) \]

where \( x, u_j \in \mathbb{R}^d \) and \( \mu \) is a nonnegative Borel measure. Suppose for some \( \gamma > 0 \)

\[ |\hat{\mu}(-\xi, \xi, 0 \ldots, 0)| \lesssim |\xi|^{-\gamma} \]

Then for all \( \gamma_1, \gamma_2 > 0 \) such that \( \gamma = \gamma_1 + \gamma_2 \) we obtain the following estimate on nonnegative functions

\[ T_\mu : L^2_{-\gamma_1}(\mathbb{R}^d) \times L^2_{-\gamma_2}(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \times \cdots \times L^\infty(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d) \]
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Theorem (Grafakos, Greenleaf, Iosevich, P)

Define $\Phi(x_1, \ldots, x_{k+1}) = (|x_i - x_j|)_{i > j}$. Suppose $1 \leq k \leq d$ and

$$\dim_H(E) > d - \frac{d - 1}{2k}$$

Then the $\binom{k+1}{2}$-dimensional Lebesgue measure of $\Delta_{\Phi}(E)$ is positive.

- Greenleaf and Iosevich established in 2012 the condition $\dim_H(E) > \frac{7}{4}$ in the case $d = k = 2$. 
Further improvements

Theorem (Greenleaf, Iosevich, Liu, P)

Define $\Phi(x_1, \ldots, x_{k+1}) = (|x_i - x_j|)_{i>j}$. Suppose $2 \leq k \leq d$ and

$$\dim_{\mathcal{H}}(E) > d - \frac{d - 1}{k + 1}.$$ 

Then the $\binom{k+1}{2}$-dimensional Lebesgue measure of $\Delta_{\Phi}(E)$ is positive. If $d = k = 2$, the same conclusion holds if $\dim_{\mathcal{H}}(E) > \frac{8}{5}$.

- Here the threshold is $d - \frac{d-1}{k+1}$ compared to $d - \frac{d-1}{2k}$ in the first approach.
Proof ingredients

- Define a measure $\nu(t)$ on $\Delta_\Phi(E)$ by the relation

$$\int f(t) d\nu(t) = \int \cdots \int f(|x_1 - x_2|, \ldots, |x_k - x_{k+1}|) d\mu(x_1) \cdots d\mu(x_{k+1})$$

where the entries of the $\binom{k+1}{2}$-vector $t$ are the distances $t_{ij}$ from $x_i$ to $x_j$, $1 \leq i < j \leq k + 1$, and $\mu$ is a Frostman measure supported on $E$. 

Finite point configurations in thin sets  Further improvements
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where the entries of the $\binom{k+1}{2}$-vector $t$ are the distances $t_{ij}$ from $x_i$ to $x_j$, $1 \leq i < j \leq k + 1$, and $\mu$ is a Frostman measure supported on $E$.

- Try to establish the bound $\int \nu^2(t) \, dt \lesssim 1$. 

Finite point configurations in thin sets

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- Try to establish the bound $\int \nu^2(t) dt \lesssim 1$.

- Idea: $\int \nu^2(t) dt = \iint_{s=t} \nu(t)\nu(s) dt ds$
A group-theoretic point of view

- Leads you to consider \((x_1, \ldots, x_{k+1})\) and \((y_1, \ldots, y_{k+1})\) that give rise to the same simplex, in other words
  \[|x_i - x_j| = |y_i - y_j|\] for all \(1 \leq i < j \leq k + 1\).

- Observe that for \(x_i \neq x_j\), \(|x_i - x_j| = |y_i - y_j|\) if and only if
  \(x_i - x_j = gy_i - gy_j\) for some \(g \in \text{O}(d)\), the orthogonal group.
Using the group-theoretic point of view it follows essentially that

\[ \int \nu^2(t) \, dt \leq c \int \mu^{2(k+1)} \left\{ (x_1, \ldots, x_{k+1}, y_1, \ldots, y_{k+1}) : x_i - gy_i = x_j - gy_j, \ 1 \leq i < j \leq k + 1 \right\} \, dx \, dy \, dg \]

where \( dg \) denotes the Haar measure on \( \mathbb{O}(d) \).
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where $dg$ denotes the Haar measure on $\mathbb{O}(d)$.

Define a measure $\nu_g$ on $E - gE$ by the relation

$$\int f(z) d\nu_g(z) := \int \int f(u - gv) d\mu(u) d\mu(v).$$
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Define a measure $\nu_g$ on $E - gE$ by the relation

$$\int f(z) d\nu_g(z) := \int \int f(u - gv) \, d\mu(u) \, d\mu(v).$$

Then can write the inequality above as

$$\int \nu^2(t) \, dt \lesssim \int \int \nu_g^{k+1}(x) \, dx \, dg.$$
Case $k = 1$

From the definition of $\nu_g$ one obtains

$$\hat{\nu}_g(\xi) = \hat{\mu}(\xi)\hat{\mu}(g\xi).$$
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- Thus using Plancharel we get when $k = 1$

\[
\int \nu^2(t) \, dt \lesssim \int \int \nu_g^{1+1}(x) \, dx \, dg = \int |\hat{\mu}(\xi)|^2 \left\{ \int |\hat{\mu}(g\xi)|^2 \, dg \right\} \, d\xi.
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Case $k = 1$

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- Thus using Plancharel we get when $k = 1$

$$\int \nu^2(t) \, dt \lesssim \int \int \nu^{1+1}_g(x) \, dx \, dg = \int |\hat{\mu}(\xi)|^2 \left\{ \int |\hat{\mu}(g\xi)|^2 \, dg \right\} \, d\xi.$$ 

- This in turn is a constant multiple of the classical Mattila integral which has so far been the main tool in the study of the Falconer distance problem

$$\int \left( \int_{S^{d-1}} \left| \hat{\mu}(t\omega) \right|^2 d\omega \right)^2 t^{d-1} \, dt.$$
Case $k > 1$

> Recall

\[ \int \nu^2(t) \, dt \lesssim \int \int \nu_g^{k+1}(x) \, dx \, dg. \]
Case $k > 1$

- Recall
  \[ \int \nu^2(t) \, dt \lesssim \int \int \nu^{k+1}_g(x) \, dx \, dg. \]

- Idea is to use
  \[ \int \int \nu^{k+1}_g(x) \, dx \, dg \leq \| \nu_g \|^{-1}_\infty \int \int \nu^2_g(x) \, dx \, dg \]

  on Littlewood-Paley pieces, reduce to the Mattila integral and then use classical boundedness results for it.
Most recent improvement

Theorem (Greenleaf, Iosevich, Liu, P)

Define $\Phi(x_1, \ldots, x_{k+1}) = (|x_i - x_j|)_{i > j}$. Suppose $1 \leq k \leq d$ and

$$\dim_{\mathcal{H}}(E) > \frac{d + k}{2}$$

Then the $\left(\begin{array}{c} k+1 \\ 2 \end{array}\right)$-dimensional Lebesgue measure of $\Delta_\Phi(E)$ is positive.

- Erdoğan, Hart and Iosevich established in 2012 the condition $\dim_{\mathcal{H}}(E) > \frac{d+k+1}{2}$.
Proof ingredients for $k = 2$

- Need to prove an incidence theorem

$$\Lambda_2(\mu) = \int \int \int \sigma_{t_{12}}^\varepsilon (x-y) \sigma_{t_{13}}^\varepsilon (x-z) \sigma_{t_{23}}^\varepsilon (y-z) d\mu(x) d\mu(y) d\mu(z) \lesssim 1$$
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- Given complex numbers $\alpha, \beta, \gamma$, define

$$\Lambda_{\alpha,\beta,\gamma}^\varepsilon(\mu) = \int \int \int \sigma_{t_{12}}^{\varepsilon,\alpha} (x-y) \sigma_{t_{13}}^{\varepsilon,\beta} (x-z) \sigma_{t_{23}}^{\varepsilon,\gamma} (y-z) d\mu(x)d\mu(y)d\mu(z)$$

where

$$\sigma_{t}^{\varepsilon,z}(x) := \frac{2^{\frac{d-z}{2}}}{\Gamma(z/2)} (\sigma_{t}^\varepsilon * | -d+z)(x)$$
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- Given complex numbers $\alpha, \beta, \gamma$, define

\[ \Lambda_{\alpha,\beta,\gamma}^{\alpha,\beta,\gamma}(\mu) = \int \int \int \sigma_{t12}^{\epsilon,\alpha}(x-y) \sigma_{t13}^{\epsilon,\beta}(x-z) \sigma_{t23}^{\epsilon,\gamma}(y-z) d\mu(x)d\mu(y)d\mu(z) \]

where

\[ \sigma_{t}^{\epsilon,\alpha}(x) := \frac{2^{d-z}}{\Gamma(z/2)} (\sigma_t^{\epsilon} * | \cdot |^{-d+z})(x) \]

- When $Re(\alpha) = 1 + \frac{\delta}{2}$

\[ |\Lambda_{\alpha,\beta,\gamma}^{\alpha,\beta,\gamma}(\mu)| \leq C \int \int \int |\sigma_{t13}^{\epsilon,\beta}|(x-z)|\sigma_{t23}^{\epsilon,\gamma}|(y-z)d\mu(x)d\mu(y)d\mu(z) \]

and the problem reduces to working with chains.
Suppose $d \geq 2$ and $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$. Then for any $k \geq 1$ there exists an open interval $I$, such that for any $\{t_i\}_{i=1}^k \subset I$ there exists a $k$-chain in $E$ with gaps $\{t_i\}_{i=1}^k$.

**Corollary (Bennett, Iosevich, Taylor)**

Suppose $d \geq 2$ and $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$. Then the $k$-dimensional Lebesgue measure of the following set is positive

$$\{(|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_k - x_{k+1}|) : x_j \in E\}.$$
Theorem (Bennett, Iosevich, Taylor)

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Sharpness?

<table>
<thead>
<tr>
<th></th>
<th>Theorem</th>
<th>Counterexample</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance in $\mathbb{R}^d$</td>
<td>$\frac{d}{2} + \frac{1}{3}$</td>
<td>$\frac{d}{2}$</td>
</tr>
<tr>
<td>Triangle in $\mathbb{R}^2$</td>
<td>$\frac{8}{5}$</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
<td>Simplex in $\mathbb{R}^d$</td>
<td>$\min \left{ \frac{dk+1}{k+1}, \frac{d+k}{2} \right}$</td>
<td>$\frac{d(k+1)}{d+2}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Incidence Theorem</th>
<th>Technique limit</th>
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<tbody>
<tr>
<td>Distance in $\mathbb{R}^d$</td>
<td>$\frac{d}{2} + \frac{1}{2}$</td>
<td>$\frac{d}{2} + \frac{1}{2}$</td>
</tr>
<tr>
<td>Triangle in $\mathbb{R}^d$</td>
<td>$\min \left{ \frac{3d}{4} + \frac{1}{4}, \frac{d}{2} + 1 \right}$</td>
<td>$\frac{d}{2} + \frac{1}{2}$</td>
</tr>
</tbody>
</table>

† Joint work with Kevin Ford, Steven J. Miller, Steven Senger and SMALL students.