Furstenberg's intersection conjecture and the L^q norm of convolutions

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Base p expansions

Let $p \in \mathbb{N}_{\geq 2}$. Every point $x \in [0,1)$ has an expansion to base p:

$$x = 0.x_1 x_2 \ldots = \sum_{n=1}^{\infty} x_n p^{-n}, \quad x_i \in \{0,1,\ldots,p-1\}.$$

Basic facts:

1. All but countably many (rational) points have a unique expansion; the remaining ones have two expansions.
2. A point is rational if and only if the expansion is eventually periodic.
3. Expansions in bases p^n and p^k are “almost the same” (look at base p in blocks of length n and k).

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Multiplication by p

Definition

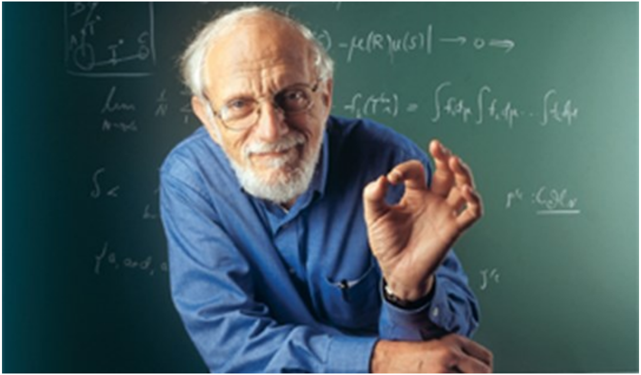
For $p \in \mathbb{N}_{\geq 2}$, let

$$T_p = px \bmod 1$$

be multiplication by p on the circle.

Symbolically, $T_p x$ corresponds to shifting the p-ary expansion x: there is a factor map, which is one-to-one outside of the countably many points with two p-ary expansions.

Multiplying by 2 and by 3: the founding father



Some of the areas that Furstenberg initiated

1. Ergodic theoretic methods in combinatorics (ergodic proof of Szemerédi's Theorem,...).
2. Products of random matrices, non-commutative ergodic theory (simplicity of Lyapunov exponents, ...).
3. Unique ergodicity of horocycle flow, toral maps, ...
4. Disjointness of dynamical systems.
5. $\times 2$, $\times 3$, rigidity of higher order actions.
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Expansions in different bases

Principle (Furstenberg)

Expansions in bases 2 and 3 have no common structure.
More generally, this holds for bases p and q which are not powers of a common integer or, equivalently, $\log p / \log q$ is irrational.

Remark

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Invariant sets

Definition

A set $A \subset [0,1)$ is T_p-invariant if $T_p(A) \subset A$. That is, shifting the p-ary expansion of a point in A gives another point in A.

- If p and q are coprime, then $\{1/q, \ldots, (q-1)/q\}$ is T_p-invariant.
- $[0,1)$ is T_p-invariant.
- Let $D \subset \{0, 1, \ldots, p-1\}$. The set $A = A_{p,D}$ of points whose base p-expansion has only digits from D is T_p-invariant. We call it a p-Cantor set. Example: the middle-thirds Cantor set.
- There is a wild abundance of invariant sets and no classification or description is possible.

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Invariant sets and shared structure

Principle (Furstenberg, slightly more concrete version)

If A, B are closed and invariant under T_2, T_3 respectively, then A and B have no common structure.

Theorem (Furstenberg (1967))

If A is jointly invariant under T_2 and T_3, then A is either finite or the whole circle $[0, 1)$.

Remarks

- *The theorem is a weak confirmation of the principle since the set A and itself certainly have a lot of common structure!*
- *One should think of finite sets and the whole circle as sets "without structure".*

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A corollary in terms of orbits

Observation

- *If x is rational, then the orbit $\{T_2^n T_3^m x\}_{n,m=1}^{\infty}$ is finite.*
- *If x is irrational, then the orbit $\{T_2^n T_3^m x\}_{n,m=1}^{\infty}$ is infinite (and its closure is invariant under T_2 and T_3).*

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"The" $\times 2$, $\times 3$ Furstenberg conjecture

Definition

A Borel probability measure μ on $[0,1)$ is T_p-invariant if

$$\mu(B) = \mu(T_p^{-1}B) \quad \text{for all Borel sets } B.$$

Conjecture (Furstenberg 1967)

If μ is T_2 and T_3 invariant, then μ is a convex combination of Lebesgue measure and an atomic measure supported on rationals.

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How to quantify "shared structure"

1. Furstenberg's Theorem says that non-trivial T_2 and T_3 invariant sets do not have too much shared structure in the most basic sense: they cannot be equal.
2. How can we quantify shared structure in finer/more quantitative ways? The sets we are interested in are fractal: they are uncountable but of zero Lebesgue measure, and have some form of (sub)-self-similarity.
3. Geometry helps quantify common structure. If two sets $A, B \subset \mathbb{R}$ have no shared structure then the intersection $A \cap B$ should be "as small as possible" (perhaps even after distorting A and/or B in some way).

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Hausdorff Dimension

- Best exponent for coverings of the set by balls of arbitrary (possibly different) radii:

$$\dim_H(A) = \inf \left\{ s : \inf \left\{ \sum_i r_i^s : A \subset \cup_i B(x_i, r_i) \right\} = 0 \right\}$$

- Gives a notion of “size” for sets in $\mathbb{R}^d$, varies between 0 and d, gives the right size to smooth objects, is invariant under bi-Lipschitz maps, is countably stable, assigns size $\log 2 / \log 3$ to the middle-thirds Cantor set,...
- If $A \subset \mathbb{T}$ is T_p-invariant, then $\dim_H A = h_{\text{top}}(A) / \log p$.
- If $A = A_{p,D}$ is a p-Cantor set, then $\dim_H A = \log |D| / \log p$.

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Dimensions of intersections

Question

If $A, B \subset \mathbb{R}^d$, what do we expect $\dim_H(A \cap B)$ to be "typically"?

Remark

If A, B are affine planes in $\mathbb{R}^d$ in general position, then

$$\dim(A \cap B) = \min(\dim(A) + \dim(B) - d, 0).$$

Theorem (Marstrand 1954)

If $A, B \subset \mathbb{R}$ are "nice" sets, then for almost all affine maps $f : \mathbb{R} \to \mathbb{R}$,

$$\dim_H(A \cap f(B)) \leq \min(\dim_H(A) + \dim_H(B) - 1, 0),$$

and this does not hold for any smaller number on the RHS.

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Furstenberg's intersection conjecture

Conjecture (Furstenberg 1969)

Let A, B be closed and invariant under T_p, T_q (seen as subsets of $\mathbb{R}$). Then for every affine bijection $f : \mathbb{R} \to \mathbb{R}$,

$$\dim_H(A \cap f(B)) \leq \max(\dim_H(A) + \dim_H(B) - 1, 0).$$

Previous results on Furstenberg's conjecture

Remark

Furstenberg's intersection conjecture gave rise to the study of "Furstenberg sets", containing a set of dimension α in (almost-)every direction. Finding the smallest possible dimension is such sets is a wide open problem.

Theorem (Furstenberg 1969, Wolff 2000)

The conjecture holds if $\dim_H(A) + \dim_H(B) \leq 1/2$. *More generally, one always has*

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Solution to Furstenberg's intersection conjecture

Theorem (P.S. 2016)

Furstenberg's intersection conjecture is true.

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The proof also gives new results about slices of fractals and the dimensions and abolute continuity of Bernoulli convolutions (and other self-similar measures).

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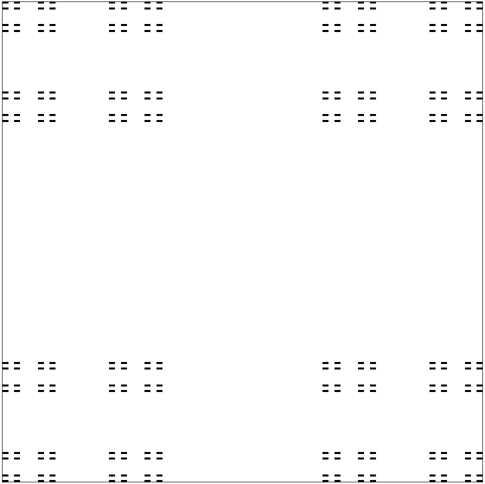
A picture!



$A \times B$.

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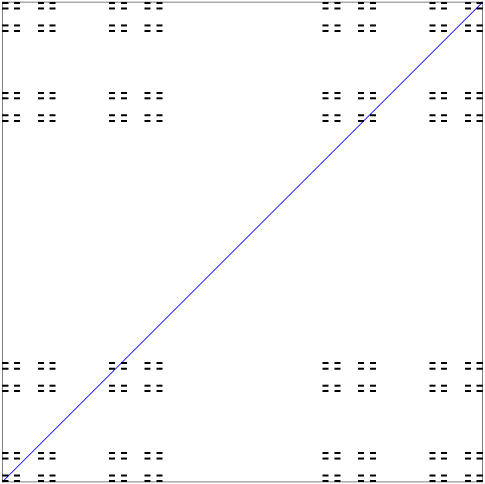
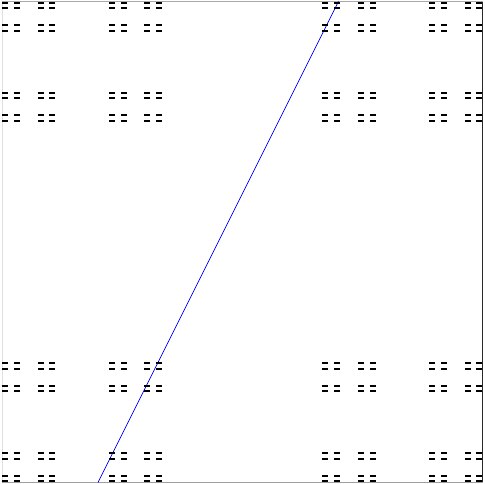


$A \times B \cap \text{diagonal} = A \cap B.$

A picture!



$A \times B \cap$ any line $= A \cap$ affine image of B.

Tools involved in the proof

1. Additive combinatorics: an inverse theorem for the L^q norm of the convolution of two finitely supported measures(Balog-Szemerédi-Gowers Theorem, Bourgain's additive part of discretized sum-product results).
2. Ergodic theory: key role played by subadditive cocycle over an irrational rotation (cocycle borrowed from Nazarov-Peres-S. 2012, uses the proof of the subadditive ergodic theorem given by Katznelson-Weiss).
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L^q norms of discrete measures

- From now on a measure is a probability measure supported on $2^{-m}\mathbb{Z} \cap [0,1) = \{j2^{-m} : 0 \le j < 2^m\}$ for some large m.
- The L^q norm of μ ($q \ge 1$) is

$$\|\mu\|_q^q = \sum_x \mu(x)^q, \quad \|\mu\|_\infty = \max_x \mu(x).$$

- $$2^{-m/q'} \le \|\mu\|_q \le 1,$$

  with a "small" L^q norm corresponding to "uniform" measures and a large L^q norm to "localized" measures.

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L^q norms of convolutions

- The convolution of μ, ν is

$$(\mu * \nu)(x) = \sum_{a+b=x} \mu(a)\nu(b).$$

(Addition modulo 1, although it makes no difference)

- Young's inequality (just convexity of $t \mapsto t^q$)

$$\|\mu * \nu\|_q \leq \|\mu\|_q \|\nu\|_1 = \|\mu\|_q.$$

- When is there (almost) equality in Young's inequality? (for $1 < q < \infty$). Two easy situations:
  1. μ is (almost) uniform.
  2. ν is (almost) an atom.
- There are less trivial examples: let A be a set that is "uniform" on some scales and "an atom" at the complementary scales. Then $\mu = \mathbf{1}_A/|A|$ satisfies $\|\mu * \mu\|_q \sim \|\mu\|_q$.

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An inverse theorem for the flattening of L^q norms

Theorem (Informal version)

Let μ, ν be measures such that

$$\|\mu * \nu\|_q \geq 2^{-\varepsilon m}\|\mu\|_q.$$

Then there are "regular" sets A, B of "large" μ, ν-measure such that in a "multiscale decomposition", on each scale either "A is almost uniform" or "B is an atom".

Trees, branching, regular sets

Definition

Suppose $m = \ell m'$ for some (large) ℓ, m'. Given a set $A \subset m\mathbb{Z} \cap [0,1)$, we consider the associated base-2^ℓ tree T_A: its vertices of level j are those dyadic intervals I of length $(2^{-\ell})^j$ that intersect A.

Definition

Given a sequence $k = (k_1, \ldots, k_{m'})$ with $k_i \in \{1, \ldots, \ell\}$, we say that A is k-regular if the following holds:
For each dyadic interval of I of length $2^{-j\ell}$ that intersects A, there are exactly k_{j+1} intervals J of length $2^{-(j+1)\ell}$ that intersect $A \cap I$.
In other words, for the tree T_A, each vertex of level j has exactly k_{j+1} children.

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The inverse theorem with more details

Theorem (P.S. 2016)

Given $\delta > 0$, there is $\varepsilon > 0$ such that the following holds for ℓ, m' large enough. Let $m = \ell m'$. If

$$\|\mu * \nu\|_q \geq 2^{-\varepsilon m}\|\mu\|_q,$$

then there are sets A, B such that:

- $\|\mu|_A\|_q \geq 2^{-\delta m}\|\mu\|_q$, $\nu(B) \geq 2^{-\delta m}\|\nu\|_1$.
- $\mu(x) \leq 2\mu(y)$ *for all* $x, y \in A$, *same for* ν *and* B.
- *A and B are k-regular and k' regular respectively for some sequences* $(k_1, \ldots, k_{m'})$, $(k'_1, \ldots, k'_{m'})$.
- *For each j,*

$$\text{Either} \quad k_j \geq 2^{(1-\delta)\ell} \quad \text{or} \quad k'_j = 1.$$

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Two main tools in the proof of the inverse theorem

Asymmetric Balog-Szemerédi-Gowers Theorem (Tao-Vu): If $A, B \subset 2^{-m}\mathbb{Z} \cap [0,1)$ are such that

$$\|\mathbf{1}_A * \mathbf{1}_B\|_2 \geq 2^{-\delta m} \|\mathbf{1}_A\|_2,$$

then there are subsets $A' \subset A$, $B' \subset B$ such that $|A'| \geq 2^{-\varepsilon}|A|$, $|B'| \geq 2^{-\varepsilon}|B|$, and

$$|A' + B'| \leq 2^{\varepsilon m}|A'|.$$

Bourgain's additive part of sum-product machinery: If $|A' + A'| \leq 2^{\varepsilon m}|A'|$, then A' contains a k-regular subset A'' such that:

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