Harmony and a Counterpoint

by

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Abstract

The Principle of Harmony originally stated in the author’s book Natural Logic is fine-tuned here in the way obviously required in order to bar an interesting would-be counterexample furnished by Crispin Wright, and to stave off any more of the same. The author’s earlier trio of attempts to formulate the principle had clearly hit the right note, without attaching too many strings. By underscoring a recurring motif in the earlier works, one can silence Wright’s tonkish refrain.

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1 Introduction

As the reader will no doubt be aware, the introduction rule for a logical operator λ sets out the conditions that need to be established in order to be justified in inferring, as a conclusion, a sentence with a salient occurrence of λ; whereas the corresponding elimination rule sets out the inferences one would be entitled to make by using the same sentence as a major premise.

What counts as a ‘salient occurrence’ is an interesting question. For connectives and quantifiers, the answer is simple: to be salient, the occurrence must be dominant. That is to say, the operator in question must have been applied last in forming of the sentence according to the rules of the formal grammar. Examples: In the sentence $\exists x (Fx \land Gx)$, the initial occurrence of $\exists$ is dominant, while the embedded occurrence of $\land$ is not; whereas in the sentence $\exists x Fx \land \exists x Gx$, the embedded occurrence of $\land$ is dominant, whereas the two occurrences of $\exists$ are not. It is only because one is working with Russell’s bracket-plus-infix notation that embedded (non-initial) occurrences of operators can be dominant. One of the simplifying virtues of (left) Polish notation is that the dominant operator-occurrence is always the leftmost one. In Polish notation, but using the same logical symbols, our two examples would be written, respectively, as $\exists x Fx \land Gx$ and $\land \exists x Fx \exists x Gx$.

Now, premises and conclusions of rules of inference are always sentences. So what happens when the operator in question is not a sentence-forming operator? In that case, it must be a term-forming operator—assuming, with Frege, that Term and Sentence are the only two basic categories in our categorial grammar. For a term-forming operator $\alpha$, a salient occurrence will be one that is dominant on one side of a general identity claim of the form ‘$t = \alpha(\ldots)$’. The operator $\alpha$ may or may not be variable-binding. Variable-binding term-forming operators include

- $\# x \Phi(x)$: the number of $\Phi$s;
- $\{ x \mid \Phi(x) \}$: the set of $\Phi$s;
- $\iota x \Phi(x)$: the $\Phi$.

These were treated in some detail in [7].
This study will focus on the problem of how one should explicate the notion of balance or matching or harmony between, one the one hand, the inferential conditions to which one is beholden in an introduction rule, and, on the other hand, the conditions of inferential entitlement that are set out in the corresponding elimination rule. In framing a Principle of Harmony, one is seeking to capture the aforementioned balance that should obtain—and, in the case of the usual logical operators, does obtain—between introduction and elimination rules.

In [1], a Principle of Harmony was formulated, and this formulation was subsequently refined, first in [3] and later in [5]. The reader will find below summaries of these earlier formulations of the Principle of Harmony. The purpose here is to revisit those formulations in light of an interesting example furnished by Crispin Wright.\(^1\) Wright’s tonkish example concerns only sentential connectives; but the lesson to be learned should generalize also to rules governing quantifiers, and (with the obvious necessary modifications) to rules governing term-forming operators.

2 Earlier formulations of Harmony

2.1 The formulation in *Natural Logic*

The original formulation of the Principle of Harmony ([1], at p. 74) was as follows.

Introduction and elimination rules for a logical operator \(\lambda\) must be formulated so that a sentence with \(\lambda\) dominant expresses the strongest proposition which can be inferred from the stated premises when the conditions for \(\lambda\)-introduction are satisfied; while it expresses the weakest proposition possible under the conditions described by \(\lambda\)-elimination.

By ‘proposition’ one can understand ‘logical equivalence class of sentences’. Thus when one speaks of the ‘proposition’ \(\varphi\), where \(\varphi\) is a sentence, one means the logical equivalence class to which \(\varphi\) belongs.

\(^1\)Personal communication, April 2005.
The strongest proposition with property \( P \) is that proposition \( \theta \) with property \( P \) such that any proposition \( \sigma \) with property \( P \) is deducible from \( \theta \); while the weakest proposition with property \( P \) is that proposition \( \theta \) with property \( P \) that can be deduced from any proposition \( \sigma \) with property \( P \).

Strictly speaking, one should continue to speak here of logical equivalence classes of sentences, or of propositions, but any occasional laxer formulation involving reference only to sentences is unlikely to cause confusion.

2.2 The formulation in *Anti-Realism and Logic*

The first main improvement on the foregoing formulation of harmony was offered, in *Anti-Realism and Logic*, in response to an observation by Peter Schroeder-Heister, to the effect that the formulation in *Natural Logic* could not guarantee the uniqueness of the logical operator concerned. The requirements of strength of conclusion (called \((S)\)) and weakness of major premise (called \((W)\)) remained as the two halves of the harmony condition, now called \( h(\lambda_i, \lambda_e) \). (Note the lowercase ‘\( h \)’. This was essentially the condition of harmony in *Natural Logic*.) The condition \( h(\lambda_i, \lambda_e) \) was spelled out (in [3], at pp. 96–7) as follows. Note that \( \lambda \) is here assumed, for illustrative purposes, to be a binary connective.\(^2\)

\[ h(\lambda_i, \lambda_e) \] holds just in case:

\((S)\) \( A \lambda B \) is the strongest proposition that it is possible to infer as a conclusion under the conditions described by \( \lambda_i \); and

\((W)\) \( A \lambda B \) is the weakest proposition that can feature as the major premise under the conditions described by \( \lambda_e \).

It was then noted that

To prove \((S)\) one appeals to the workings of \( \lambda_e \); and to prove \((W)\) one appeals to the workings of \( \lambda_i \). The rules are thus required to interanimate to meet the requirement of harmony.

\(^2\)Minor changes of spelling and symbols have been made here.
The treatment in [5] then involved laying down a further requirement. This was called ‘Harmony’, with an uppercase ‘H’:

Given $\lambda E$, we determine $\lambda I$ as the strongest introduction rule $\lambda i$ such that $h(\lambda i, \lambda E)$; and

given [the rule] $\lambda I$, we determine [the rule] $\lambda E$ as the strongest elimination rule $\lambda e$ such that $h(\lambda I, \lambda e)$.

By requiring Harmony, the intention was to determine uniquely a simultaneous choice of introduction rule $\lambda I$ and elimination rule $\lambda E$ that would be in mutual harmony (with lowercase ‘h’). Harmony ensures a kind of Nash equilibrium between introduction and elimination rules: an ideal solution in the coordinate game of giving and receiving logically complex bundles of information.

2.3 The formulation in The Taming of The True

The earlier formulation of harmony (lowercase ‘h’), which was common to both Natural Logic and Anti-Realism and Logic, was strengthened in the following re-statement in [5], at p. 321. In this formulation, the earlier comment about how the rules would interact in the proofs of strength-of-conclusion and weakness-of-major-premise was built into the statement of harmony as an essential feature:

(S) The conclusion of $\lambda$-introduction should be the strongest proposition that can so feature; moreover one need only appeal to $\lambda$-elimination to show this; but in . . . showing this, one needs to make use of all the forms of $\lambda$-elimination that are provided

(W) The major premiss for $\lambda$-elimination should be the weakest proposition that can so feature; moreover one need only appeal to $\lambda$-introduction to show this; but in . . . showing this, one needs to make use of all the forms of $\lambda$-introduction that are provided

Suppose one wished to show that the usual introduction and elimination rules for $\rightarrow$ are in harmony. There would accordingly be two problems to solve:
1. Assume that $\sigma$ features in the way required of the conclusion $\phi \to \psi$ of $\rightarrow I$:

\[
\begin{array}{c}
\vdots \\
\phi \quad (i) \\
\sigma \quad (i)
\end{array}
\]

Show, by appeal to $\rightarrow E$, that $\phi \to \psi \vdash \sigma$.

2. Assume that $\sigma$ features in the way required of the major premiss $\phi \to \psi$ of $\rightarrow E$:

\[
\begin{array}{c}
\phi \\
\sigma
\end{array} \quad (1)
\]

\[
\begin{array}{c}
\psi \\
\sigma \quad (1)
\end{array}
\]

Show, by appeal to $\rightarrow I$, that $\sigma \vdash \phi \to \psi$.

Problem (1) is solved by the following proof:

\[
\begin{array}{c}
\phi \quad (1) \\
\phi \to \psi \quad \rightarrow E \\
\psi \quad (1) \\
\sigma
\end{array}
\]

This proof uses only $\rightarrow E$ to show that if $\sigma$ features like the conclusion $\phi \to \psi$ of $\rightarrow I$ then $\sigma$ can be deduced from $\phi \to \psi$. Thus $\phi \to \psi$ is the strongest proposition that can feature as the conclusion of $\rightarrow I$.

Problem (2) is solved by the following proof:

\[
\begin{array}{c}
\phi \quad (1) \\
\sigma
\end{array} \quad (1) \quad \rightarrow I
\]

\[
\begin{array}{c}
\psi \quad (1) \\
\phi \to \psi
\end{array}
\]

This proof uses only $\rightarrow I$ to show that if $\sigma$ features like the major premiss $\phi \to \psi$ of $\rightarrow E$ then $\phi \to \psi$ can be deduced from $\sigma$. Thus $\phi \to \psi$ is the weakest proposition that can feature as the major premiss of $\rightarrow E$. 
By way of further example, the following two proof schemata show that the usual introduction and elimination rules for $\lor$ are in harmony:

\[
\begin{array}{c}
\frac{\phi}{\lor \phi} \quad \frac{\psi}{\lor \psi} \\
\sigma \quad \sigma \\
\hline
\sigma
\end{array}
\quad
\begin{array}{c}
\frac{\phi \lor \psi}{\lor I \phi} \\
\frac{\phi \lor \psi}{\lor I \psi}
\end{array}
\]

\[
\begin{array}{c}
\frac{\phi \lor \psi \lor \phi \lor \psi}{\lor E \phi \lor \psi} \\
\sigma \\
\hline
\sigma
\end{array}
\quad
\begin{array}{c}
\frac{\phi \lor \psi \lor \phi \lor \psi}{\lor E \phi \lor \psi} \\
\sigma \\
\hline
\sigma
\end{array}
\]

Note how in the second proof schema we need to employ both forms of $\lor I$ in order to construct the proof schema; the reader should consider once again the precise statement of the Principle of Harmony given above.

3 Wright’s tonkish example

The example to be disposed of here arose, according to Wright, in the context of a graduate seminar at NYU, conducted with Hartry Field in the Spring of 2005, on revision of classical logic. Wright set out his problem as follows:\(^3\)

Let $\lambda$ be a binary connective associated with rules $\lambda I$ and $\lambda E$. According to the $AR\&L$ characterisation, these are harmonious just in case

Condition (i): any binary connective $@$ for which the pattern of $\lambda I$ is valid may be shown by $\lambda E$ to be such that $A \lambda B \models A @ B$ (so $A \lambda B$ is the strongest statement justified by the $I$ premisses); and

Condition (ii): any binary connective $@$ for which the pattern of $\lambda E$ is valid may be shown by $\lambda I$ to be such that $A @ B \models A \lambda B$ (so, in effect, $\lambda E$ is the strongest $E$-rule justified by the $I$-rule.)

OK. Let $\lambda I$ be $\lor I$, and $\lambda E$ be $\land E$. So, to establish Condition (i), assume $A \models A @ B$, and $B \models A @ B$. We need to show via $\lambda E$ that $A \lambda B \models A @ B$. Assume $A \lambda B$. Then both $A$ and $B$ follow

\(^3\)Direct quote from correspondence, with minor typos corrected, extra formatting supplied, and some symbols changed in the interests of uniformity of exposition.
by \( \lambda E \). Either will then suffice for \( A@B \), by the assumption. As for Condition (ii), assume \( A@B \). By hypothesis, the pattern of \( \lambda E \) is valid for \( @ \), so we have both \( A \) and \( B \). Either will suffice via \( \lambda I \) for the proof of \( A\lambda B \). So \( A@B \models A\lambda B \).

What has gone wrong? Manifestly the \( \lambda \)-rules are disharmonious (in fact they are the rules for tonk, of course.)

4 The revised version of the Principle of Harmony

Perhaps the best way to explain what has gone wrong is to clarify how [5] had already, in effect, put it right. The following emended version of harmony is a minor variation of the theme in [5], and was communicated to Wright in response to this interesting problem. In [5], the statement of (\( S \)) (strength-of-conclusion) involved the condition that in establishing strength ‘one need only appeal to \( \lambda \)-elimination’ and the statement of (\( W \)) (weakness-of-major-premise) involved the condition that in establishing weakness ‘one need only appeal to \( \lambda \)-introduction’. These two conditions will be made more emphatic: respectively, ‘one may not make any use of \( \lambda \)-introduction’ and ‘one may not make any use of \( \lambda \)-elimination’.

In order to distinguish the ‘final’ version of harmony offered below from its predecessors, let us use (\( S' \)) and (\( W' \)) as the respective labels for strength-of-conclusion and weakness-of-major-premise. Wright’s apparent counterexample can be rendered inadmissible by laying down the following more emphatic version of the requirement for harmony in [5]. It is in spirit of the original, but now also—one hopes—in appropriately captious letter. As has been the case all along, it is framed by reference to a connective \( \lambda \). The newly emphasized conditions are in boldface.

(\( S' \)) \( A\lambda B \) is the strongest conclusion possible under the conditions described by \( \lambda I \). Moreover, in order to show this,

(i) one needs to exploit all the conditions described by \( \lambda I \);
(ii) one needs to make full use \( \lambda E \); but
(iii) one may not make any use of $\lambda I$.

($W'$) $A \lambda B$ is the weakest major premise possible under the conditions described by $\lambda E$. Moreover, in order to show this,

(iv) one needs to exploit all the conditions described by $\lambda E$;

(v) one needs to make full use $\lambda I$; but

(vi) one may not make any use of $\lambda E$.

Suppose now that we try to follow Wright’s foregoing suggestion. That is, suppose we try to stipulate $\lambda I$ as $\lor$-like and $\lambda E$ as $\land$-like:

($\lambda I$) \[
\begin{array}{c}
A \\
\hline
A \lambda B
\end{array} \quad \begin{array}{c}
B \\
\hline
A \lambda B
\end{array}
\]

($\lambda E$) \[
\begin{array}{c}
A \lambda B \\
\hline
A
\end{array} \quad \begin{array}{c}
A \lambda B \\
\hline
B
\end{array}
\]

It will be shown that this stipulation does not satisfy the joint harmony requirement ($S'$) and ($W'$).

In order to establish ($S'$) we would need to show, inter alia, that given the inferences $A/C$ and $B/C$ one could form a proof $\Pi$ (say) of $C$ from $A \lambda B$, but would (i) need both those inferences to do so, and (ii) need to make full use of $\lambda E$.

In order to establish ($W'$) we would need to show, inter alia, that given the inferences $C/A$ and $C/B$ one could form a proof $\Sigma$ (say) of $A \lambda B$ from $C$, but would (i) need both those inferences to do so, and (ii) need to make full use of $\lambda I$.

Candidates for the sought proof $\Pi$ (for ($S'$)) might be thought to be

\[
\begin{array}{c}
A \lambda B \\
\hline
\begin{array}{c}
A \\
\hline
A \lambda B
\end{array} \\
\begin{array}{c}
B \\
\hline
C
\end{array}
\end{array}
\]

but neither of these proofs exploits both the inference $A/C$ and the inference $B/C$. So these candidate proofs violate requirement (i) in ($S'$) to the effect
that one needs to exploit all the conditions described by \( \lambda I \). Moreover, each candidate proof uses only ‘one half’ of \( \lambda E \), not full \( \lambda E \). So these proofs also violate requirement (ii) in \((S')\) to the effect that one must use all of \( \lambda E \).

Candidates for the sought proof \( \Sigma \) (for \((W')\)) might be thought to be

\[
\begin{array}{c}
\frac{\frac{C}{A}}{A\lambda B} & \frac{\frac{C}{B}}{A\lambda B}
\end{array}
\]

but neither of these proofs exploits both the inference \( C/A \) and the inference \( C/B \). So these candidate proofs violate requirement (i) in \((W')\) to the effect that one needs to exploit all the conditions described by \( \lambda E \). Moreover, each candidate proof uses only ‘one half’ of \( \lambda I \), not full \( \lambda I \). So these proofs also violate requirement (ii) in \((W')\) to the effect that one must use all of \( \lambda I \).

Suppose rather that one were to stipulate a perverse choice of ‘halves’ of the preceding \( I \)- and \( E \)-rules for \( \lambda \). Thus suppose that the rules \( \lambda I \) and \( \lambda E \) were now taken to be, respectively,

\[
\begin{array}{c}
\frac{A}{A\lambda B} \quad \frac{A\lambda B}{B}
\end{array}
\]

In order to establish \((S')\) we would need to show, \( \textit{inter alia} \), that given the inference \( A/C \) one could form a proof \( \Pi \) (say) of \( C \) from \( A\lambda B \), but without (by (iii)) making any use of \( \lambda I \).

In order to establish \((W')\) we would need to show, \( \textit{inter alia} \), that given the inference \( C/B \) one could form a proof \( \Sigma \) (say) of \( A\lambda B \) from \( C \), but without (by (vi)) making any use of \( \lambda E \).

It might be thought that a candidate proof for \( \Pi \) would be
\[
\begin{array}{c}
A \lambda B \quad (\lambda E) \\
\quad B \quad (\lambda I) \\
\quad B \lambda C \quad (\lambda E) \\
\quad C
\end{array}
\]

but this violates requirement (iii), since it uses \(\lambda I\).

It might be thought that a candidate proof for \(\Sigma\) would be

\[
\begin{array}{c}
C \quad (\lambda I) \\
A \lambda B \quad (\lambda E) \\
A \quad (\lambda I) \\
A \lambda B
\end{array}
\]

but this violates requirement (vi), since it uses \(\lambda E\).

5 On the requirements of full use of conditions and rules

Salvatore Florio has produced an interesting case which illustrates the need to insist on ‘full use’ of conditions and rules when establishing the statements \((S')\) and \((W')\) for harmony.

Consider the obviously non-harmonious rules

\[
(\lambda I) \quad \frac{A}{A \lambda B} \quad (\lambda E) \quad \frac{A \lambda B}{C}
\]

where the rule \((\lambda E)\) is like the Absurdity Rule, in that it allows the conclusion \(C\) to be any sentence one pleases.

First we prove \((S')\), but without heeding fully the requirements that have been laid down on such a proof.

Assume that \(D\) features in the way required of the conclusion \(A \lambda B\) of \(\lambda\)-introduction:
We are required to show that $A \lambda B \vdash D$. The following proof suffices:

\[
\begin{array}{c}
A \quad B \\
\hline 
D
\end{array}
\]

Next we prove $(W')$, again without heeding fully the requirements that have been laid down on such a proof.

Assume that $D$ features in the way required of $A \lambda B$ as the major premiss for $\lambda$-elimination:

\[
\frac{D}{C}
\]

We are required to show that $D \vdash A \lambda B$. The following proof suffices:

\[
\frac{D}{A \lambda B}
\]

These swift proofs, however, as already intimated, are unsatisfactory. The respective reasons are as follows.

The proof of $(S')$ does not avail itself at all of the assumption made at the outset (that $D$ features in the way required of the conclusion $A \lambda B$ of $\lambda$-introduction). The technical infraction on the part of this attempted proof of $(S')$ is its violation of condition (i): the proof does not ‘exploit all the conditions described by $\lambda I'$. (In fact, one can see from the very statement of $(\lambda E)$ that $A \lambda B$ is the strongest proposition tout court, and not just the strongest proposition that can feature as the conclusion of $\lambda$-introduction.)

The proof of $(W')$ is defective also. It does not use the rule $(\lambda I)$, thereby violating condition (v).
6 Conclusion

Wright’s example brings out nicely why it is that stipulations concerning licit methods of proof (of strength-of-conclusion and of weakness-of-major-premise) must be built into the formulation of harmony. The author is moderately confident that the new formulation above, with its emphases on prohibited resources, will withstand any further attempted counterexamples. This new version of harmony (lowercase ‘h’) should, of course, still be coupled with the uniqueness condition that was called Harmony (uppercase ‘H’).

In closing, it should be stressed that the formulation of a Principle of Harmony is not just a technical exercise in proof theory of limited (or no) value to the philosophy of logic and language. On the contrary: armed with a satisfactory account of harmony, the naturalizing anti-realist can venture an interesting account of how logical operators could have found their way into an evolving language. Harmony is a transcendental precondition for the very possibility of logically structured communication. A would-be logical operator that does not display harmony (such as, for example, Prior’s infamous operator ‘tonk’) could not possibly be retained within an evolving language after making a first debut. Because the ‘deductive reasoning’ that it would afford would go so haywire, it would have been rapidly selected against. Only those operators would have survived that were governed by harmoniously matched introduction rules (expressing obligations on the part of assertors) and elimination rules (expressing the entitlements of their listeners). For only they could have usefully enriched the medium by means of which social beings can informatively communicate. Only those operators would have been able to make their way through the selective filter for the growing medium.\footnote{For a more detailed development of these ideas, the reader is referred to [2]; [3], ch. 9; [4]; and [6].}
References


