Natural Logicism via the Logic of Orderly Pairing

by

Neil Tennant*

Department of Philosophy
The Ohio State University
Columbus, Ohio 43210
email tennant.9@osu.edu

October 7, 2008

*Earlier versions of this paper were presented to the Arché Abstraction Weekend on Tennant's Philosophy of Mathematics in St. Andrews in May 2004 and to the Midwest Workshop in Philosophy of Mathematics at Notre Dame in October 2005. The author is grateful for helpful comments from members of both audiences, especially the St. Andrews commentators Peter Milne, Ian Rumfitt, Peter Smith and Alan Weir; and from George Schumm, Timothy Smiley and Matthias Wille. Comments by two anonymous referees have also led to significant improvements.

Abstract

The aim here is to describe how to complete the constructive logicist program, in the author's book *Anti-Realism and Logic*, of deriving all the Peano-Dedekind postulates for arithmetic within a theory of natural numbers that also accounts for their applicability in counting finite collections of objects. The axioms still to be derived are those for addition and multiplication. Frege did not derive them in a fully explicit, conceptually illuminating way. Nor has any neo-Fregean done so.

These outstanding axioms need to be derived in a way fully in keeping with the spirit and the letter of Frege's logicism and his doctrine of definition. To that end this study develops a logic, in the Gentzen-Prawitz style of natural deduction, for the operation of orderly pairing. The logic is an extension of free first-order logic with identity. Orderly pairing is treated as a primitive. No notion of set is presupposed, nor any set-theoretic notion of membership. The formation of ordered pairs, and the two projection operations yielding their left and right coordinates, form a coeval family of logical notions. The challenge is to furnish them with introduction and elimination rules that capture their exact meanings, and no more.

Orderly pairing as a logical primitive is then used in order to introduce addition and multiplication in a conceptually satisfying way within a constructive logicist theory of the natural numbers. Because of its reliance, throughout, on sense-constituting rules of natural deduction, the completed account can be described as 'natural logicism'.

1 Introduction: historical background

Frege's two-volume work *Grundgesetze der Arithmetik* ([5], [6]) fell significantly short of completing his logicist program for arithmetic. This is not because of the all-too-obvious reason that his underlying theory of classes fell prey to Russell's Paradox. Although the following counterfactual enjoys no cognitive sense (since its antecedent is impossible), it nevertheless makes good rhetorical sense:

> *Even if* Frege's system of classes had *not* turned out to be inconsistent, the technical work accomplished in the two volumes of *Grundgesetze* nevertheless cannot be judged to have achieved what Frege himself demanded of a logicist account of arithmetic.

In addition, then, to the criticisms of tone, arrangement and dialectical strategy that Dummett ([4], Ch. 19) has levelled against Part III (in Vol. II) of *Grundgesetze*, one can add the criticism that Frege did not thoroughly finish off a logicist derivation and justification of the basic laws of arithmetic.

Frege's goals were twofold:

1. The account should explain the general *applicability* of arithmetic, and
2. The account should afford fully formal deductions, from its underlying logicist principles, of the basic laws of arithmetic itself.

These basic laws have standardly been taken to include *at least* the Peano-Dedekind axioms for zero and successor:

$$\forall x\, \neg sx = 0;$$

$$\forall x \forall y\, (sx = sy \rightarrow x = y);$$

and the principle of mathematical induction:

$$\forall \Phi(\forall x(\Phi x \rightarrow \Phi sx) \rightarrow (\Phi 0 \rightarrow \forall y \Phi y)).$$ [1]

Frege had set himself the goal of furnishing a logicist derivation of at least these, from the time (1884) of his earlier work *Die Grundlagen der Arithmetik* onward. But we must not lose sight of the fact that the later, more technical *Grundgesetze* (Vol. I: 1893; Vol. II: 1903) appeared after the Peano–Dedekind recursion axioms for addition:

[1] These axioms are stated without any explicit sortal restriction to the natural numbers. The quantifiers are to be understood as ranging over the natural numbers.

$$\forall x\ (x + 0 = x);$$
$$\forall x \forall y\ (x + sy = s(x + y))$$

and for multiplication:

$$\forall x\ (x \times 0 = 0);$$
$$\forall x \forall y\ (x \times sy = (x \times y) + x)$$

had been established as part of the axiomatic basis for arithmetic. Peano's axioms were published in 1889; Dedekind's *Was sind und was sollen die Zahlen?*, with its justification of definition by recursion, appeared in 1888. But Frege, in his 1893 pursuit of the goal (2), could not justifiably omit or exclude the operations of addition and multiplication from the scope of logicism's obligations with regard to goal (1).

One possible explanation for the widespread belief that Frege had provided a logicist derivation *and thereby* a justification of all the laws of arithmetic is that Montgomery Furth, in his Introduction to his translation of introductory portions of the first volume of *Grundgesetze*, wrote ([7], p. vi) that '[f]or Frege, the paramount part of [his] task' was to 'actually produce derivations of the standard propositions of arithmetic ... '. This leaves the reader who relies on English translations of Frege's work with the impression that Frege did indeed furnish definitions and proofs (albeit within his inconsistent system) for *all* the basic laws of arithmetic, including those governing addition and multiplication—definitions and proofs, moreover, that would have enabled Frege to claim that goal (1) had thereby been accomplished.

Michael Potter has correctly observed ([13], at p. 75), concerning the *Grundlagen*, that 'What is missing is a *justification* of the recursive definitions of addition and multiplication' [emphasis added], and goes on to say that

> ... the treatment of the *definition* of functions by recursion is in fact the least obvious part of the deduction of arithmetic from Peano's axioms: ... it is one of the most impressive achievements of Dedekind's later [1888] work. So perhaps the reason for Frege's sudden silence at this point is that he had by then seen the difficulty but not yet worked out the solution. [*Loc. cit.*; emphasis and date added.]

Potter is referring to a 'sudden silence' at a certain point in Frege's exposition within the *Grundlagen.* As Potter says (*ibid.*, at p. 79)

> ... there remained to Frege at the technical level the task, carried out fully in his *Grundgesetze* ten years later, of filling in the gaps.

One of those gaps, it is clear from Potter's discussion, he (Potter) takes to be 'the crucial step of *justifying* the recursive definitions of addition and multiplication'. [Emphasis added.] In the context of our current distinction between goal (1) of applicability and goal (2) of derivability, we should speak here on Potter's behalf of 'the crucial step of *deriving* the recursive *axioms for* addition and multiplication'. Only the latter, derivational, step of the project, but not the former, justificatory, one, was so much as addressed in *Grundgesetze*, let alone 'carried out fully'. There is room for complaint concerning how fully and explicitly this was actually done. The justificatory step, however, was not carried out *at all*. This surprising and disappointing discovery is what a close reading of *Grundgesetze* will reveal.

Richard Heck [10] argues that (at least in the case of addition) one can read into various of Frege's definitions and numbered theorems something tantamount to a formal derivation, by appeal to those definitions, of something like the Dedekind–Peano recursion axioms. Such an accomplishment, on Frege's part, would not be undermined by the subsequent discovery that his system is inconsistent. For, as Heck has also pointed out ([9], at p. 580)

> ... it has in recent years been shown that the Dedekind–Peano axioms for arithmetic can indeed be derived, within second-order logic, from Hume's Principle.

Heck's later paper [10] gives an exposition, in modern logical notation, of analogues of Frege's second-order definitions and theorems in this connection. Heck's analogues eschew Frege's own use of ordered pairs (hostage, with hindsight, to the misfortune of the inconsistency of Axiom V), and use instead what Heck calls the 2-ancestral. But the overall pattern of logical relations among the formal sentences involved is claimed to be preserved. Even so, Heck's charitably reconstructed Frege does not deal with multiplication at all.

Let us for the present confine ourselves to addition. There is warrant for the complaint that the actual derivations that Frege gave (and analogues of which Heck expounds) of the 'recursive definition' of addition do not amount to the *justification* that the Fregean should be demanding of a satisfactory logicist account of addition. Frege failed to complete the *philosophical* and *foundational* logicist project that he set himself. He failed to furnish a definition of addition that would explain the *applicability* of that operation. For the operation applies to natural numbers as *cardinal* numbers—numbers that are themselves applied as measures of size of finite collections. A prerequisite for a conceptually illuminating definition of addition on (finite) cardinal numbers is that the definition should apply also to infinite cardinals—or, at the very least, be smoothly generalizable to them (should we wish to engage in such generalization). Yet is it clear that the Dedekindian definition of addition that Frege provided cannot be generalized so as to deal with *infinite* cardinals. For, on that definition, the result of adding n to m is the result one obtains by starting with m and applying the successor operation n times. *If* (and this is a big 'if') this approach could be generalized so as to deal with Frege's number *Endlos* (or what we would today call ω) in place of m, we would obtain *ordinal* addition: for a natural number n, one would have the *ordinal sum* $\omega + n$ as the result of such adding of n to ω, rather than the *cardinal sum*, which is of course ω itself. This failure, on Frege's part, to show us how to add together *cardinal* numbers rather than *ordinal* numbers is all the more significant given his own insistence on the importance of infinite cardinals—for it is only in the infinite case that the conceptual distinction between cardinals and ordinals reveals itself in extension. In the *Grundlagen* (p. 97) Frege had praised Cantor [3] for introducing infinite numbers, writing (pp. 97–8)

> I heartily share his contempt for the view that in principle only finite Numbers ought to be admitted as actual. ... our concept of Number has from the outset covered infinite numbers as well ...

Frege was also acutely aware (*loc. cit.*) of the distinction between cardinal and ordinal numbers in the infinite case:

> ... my terminology diverges to some extent from [Cantor's]. For my Number he uses "power", while his concept[fn] of Number [i.e. *ordinal* number—NT] has reference to arrangement in an order. Finite Numbers, certainly, emerge as independent nevertheless of sequence in series, but not so transfinite Numbers. But now in ordinary use the word "Number" and the question "how many?" have no reference to arrangement in a fixed order. CANTOR's Number gives rather the answer to the question: "the how-manyeth member in the succession is the last member?" *So that it seems to me that my terminology accords better with ordinary usage.* [Emphasis added.] If we extend the meaning of a word, we should take care that, so far as possible, no general proposition is invalidated in the process, *especially one so fundamental as that which asserts of Number its independence of*

> *sequence in series.* [Emphasis added.] For us, because our concept of Number has from the outset covered infinite numbers as well, no extension of its meaning has been necessary at all.

So Frege bequeathed to the logicist tradition a gap that by his own lights ought to be filled. It has three dimensions. To fill it, one must give a conceptually illuminating account of addition of *cardinal* numbers, which will (i) confer the correct sense on any additive numerical term; (ii) apply indifferently to both finite and infinite numbers; and (iii) afford derivations, from suitable first principles, of the Peano-Dedekind recursion equations for addition on the natural numbers. Ditto for multiplication.

It would be fatal for a genuine neo-logicism if this gap could not be filled. It is not enough to assume that the Fregean neo-logicist—let alone Frege—could rest content with having generated the natural number sequence in a satisfyingly logicist fashion, but with having thereafter finished the job simply by reprising Dedekind's account of definition (of addition and multiplication) by recursion on the natural numbers (albeit by means of definitions of addition and multiplication in terms of zero and successor). For such a move would not account for the *applicability* of the operations of addition and multiplication. Definition by recursion lives entirely on the pure mathematical side; it affords no account whatsoever of what, conceptually, addition and multiplication amount to as operations on (finite) *cardinal* numbers. The logicist wants to show what information or logical constraints concerning numbers, and the collections they number, the operations of addition and multiplication afford. Yet definition by recursion does not (by itself) explain, for example, how and why it is that if there are n Fs and m Gs, and nothing is both F and G, then the sum $n + m$ turns out to be (of necessity) the number of things that are either F or G.

Dummett, [4] p. 51, makes this point against Dedekind in what he intends as a comparison favorable to Frege. He says that Frege 'defines the sum of two *natural* numbers' [emphasis added] '... in effect, as the number of members of the union of two disjoint classes.' But Dummett provides no reference to where, exactly, Frege's alleged definition of the sum of two natural numbers is to be found.

It might be thought, on Dummett's behalf, that this 'definition' of addition is provided in *Grundegesetze* Vol. II §33. This section can be ruled out, however, as a source of a satisfactory definition of addition for the logicist. The rather contorted gloss with which Frege opens his *Zerlegung* in that section shows that he is well aware that he is not really offering a definition that would meet his own adequacy criteria:

,,Die Summe von zwei Anzahlen ist durch diese bestimmt", in diesem Ausdrucke is der Gedanke des Satzes unserer Hauptüberschrift am leichtesten zu erkennen, und darum mag er angeführt sein, obwohl der bestimmte Artikel beim Subject die Aussage der Bestimmtheit eigentlich vorwegnimmt und obwohl das Wort ,,Summe" hier anders gebraucht ist, als wie wir es später bei den Zahlen gebrauchen werden. Wir nennen hier nämlich [$\#x(Fx \vee Gx)$] Summe von [$\#xFx$] und [$\#xGx$], wenn kein Gegenstand zugleich unter den [F-] und unter den [G-] Begriff fällt.

"The sum of two numbers is determined by them": this expression provides the easiest way to recognize the thought rendered by the formal sentence of our main heading, and for that reason this expression may be offered—despite the fact that the definite article in the subject actually anticipates the claim of determination and the fact that the word "sum" is used here differently from the way we shall later be using it in connection with the (real) numbers. That is, we call [$\#x(Fx \vee Gx)$] the sum of [$\#xFx$] and [$\#xGx$], if no object falls both under the concept [F] and under the concept [G].

[For Frege, *Anzahl* means cardinal number, *Zahl* means real number.]

Observe, first, that no defined symbol for addition is hereby introduced, and the 'definition' does not find its way into the later *Tafel der Definitionen* in the subsequent appendices. It [i.e., the *Hauptüberschrift* of Vol. II, §33, at p. 44], after being proved as the result labelled '(469' (see Vol. II, p. 58 *supra*) is listed, rather, in the *Tafel der wichtigeren Lehrsätze* (*ibid.*, p. 274 *infra*), which follows immediately after the *Tafel der Definitionen.*

Secondly, if one were to try to turn (469 into a formal definition, one would at best obtain something along the lines (in modern notation) of

$\#xF_1(x) + \#xG_1(x) = \#xF_2(x) + \#xG_2(x)$
whenever $\#xF_1(x) = \#xF_2(x)$ and $\#xG_1(x) = \#xG_2(x)$
and nothing is both F_1 and G_1
and nothing is both F_2 and G_2.

This of course would not do as a definition of addition, because one would not be providing a sense in sufficient generality for expressions of the form $t + u$.

Thirdly, consider the second sentence of Vol. II §33, ‘Wir nennen … fällt.’ It might be thought that this would furnish an explicit definition of + as follows:

$$\#xFx + \#xGx =_{df} \#x(Fx \vee Gx) \text{ if } \neg\exists x(Fx \wedge Gx).$$

But this, too, is not a proper definition by Frege’s criteria. In Vol. II §65 he writes

> …das Additionszeichen ist nur erklärt, wenn die Bedeutung jeder möglichen Zeichenverbindung von der Form »$a{+}b$« bestimmt ist, welche Bedeutungsvolle Eigennamen man auch für »a« und für »b« einsetzen möge.

> …the sign for addition is explained only when the denotation of every possible sign-combination of the form »$a + b$« is determined, no matter what meaningful proper names one might substitute for »a« and for »b«.

The ‘Wir nennen’ is at best a (metalinguistic) gloss concerning what one might say under rather particular circumstances (of disjointness of the concepts involved). It is not something that leads to a proper explicit definition.

Certainly, if one *had* a proper explicit definition of +, one would expect to be able to derive, as a theorem,

$$\text{If } \neg\exists x(Fx \wedge Gx), \text{ then } \#xFx + \#xGx = \#x(Fx \vee Gx)$$

along with a similar theorem corresponding to (469 :

> If $\#xF_1(x) = \#xF_2(x)$ and $\#xG_1(x) = \#xG_2(x)$
> and nothing is both F_1 and G_1
> and nothing is both F_2 and G_2,
> then $\#xF_1(x) + \#xG_1(x) = \#xF_2(x) + \#xG_2(x)$.

We are still left empty-handed, though, in so far as we want a definition of + that would not only afford derivations from ‘logical’ first principles of the recursion axioms for + (and likewise for ×) but also accomplish the goal of providing an account of the applicability of these operations to (finite) cardinal numbers. So it does seem, on balance, as though Vol. II §33 does not provide the kind of textual evidence one would like to see for Dummett’s claim about addition.[2]

[2] These considerations also counsel against attempting an explicit higher-order or set-theoretic definition of addition (say) that directly exploits the desired recursion equations.

It would appear that *Gg.* Vol. II §33 is a nod from Frege in the direction of Cantor's 1895 treatment of addition and multiplication of powers in [2], §3, 'Die Addition und Multiplikation von Mächtigkeiten', at p. 485. Cantor contented himself with a definition of these operations that applied, as it were, *de rebus*, without providing a sense for every possible expression of the form $t+u$, where t and u are terms which, if they denote at all, denote numbers. This approach is characteristic of the mathematician who is concerned only to characterize the extension of an operation or of a predicate, within an abstract ontology to which intellectual access is already assumed. If one takes the numbers (or, in Cantor's case, the 'powers') as already given, then the task, understandably, is simply to give an extensionally correct characterization of the 'action' of the operation of addition on those numbers. Thus Cantor could write (*loc. cit.*)

> Die Vereinigung zweier Mengen M und N, die keine gemeinschaftlichen Elemente haben, wurde in §1, (2) mit (M, N) bezeichnet. Wir nennen sie die „*Vereinigungsmenge von M und N*".
>
> Sind M', N' zwei andere Mengen ohne gemeinschaftliche Elemente, und ist $M \sim M'$, $N \sim N'$, so sahen wir, daß auch
>
> $$(M, N) \sim (M', N').$$
>
> Daraus folgt, daß die Kardinalzahl von (M, N) nur von den Kardinalzahlen $\overline{\overline{M}} = \mathfrak{a}$ und $\overline{\overline{N}} = \mathfrak{b}$ abhängt.
>
> Dies führt zur Definition der Summe von $\mathfrak{a}$ und $\mathfrak{b}$, indem wir setzen
>
> $$\mathfrak{a} + \mathfrak{b} = (\overline{\overline{M, N}}). \qquad (1)$$

One might, for example, attempt to define the relation $\mathrm{SUM}(x, y, z)$ (which says that $z = x + y$) as the intersection of all relations $F(u, v, w)$ such that

1. $F(u, 0, u)$, and
2. If $F(u, v, w)$ and v' is the successor of v, then $F(u, v', w')$, where w' is the successor of w.

The drawback of such a proposal is that it involves either overly powerful set-theoretic commitments, or the resources of second-order logic. The definitions of addition and multiplication to be offered here, by contrast, involve the more modest resources of orderly pairing, and remain at first order. Moreover, the recursion equations, which will be derivable from the definitions, do not need to be 'built in', the way they are with the foregoing definition of SUM. Instead, the recursion equations arise as *fruitful* (albeit logically necessary) byproducts of the definitions of these operations in terms of orderly pairing, whose form is chosen with an eye to meeting the Fregean requirement of 'illumination of applicability'.

Jourdain's translation (at p. 91) is as follows:

> The union of two aggregates M and N which have no common elements, was denoted in §1, (2) by (M,N). We call it the "union-aggregate (*Vereinigungsmenge*) of M and N."
>
> If M′, N′ are two other aggregates without common elements, and if M$\sim$ M′, N$\sim$ N′, we saw that we have
>
> $$(\mathrm{M},\mathrm{N}) \sim (\mathrm{M}',\mathrm{N}').$$
>
> Hence the cardinal number of (M,N) only depends upon the cardinal numbers $\overline{\overline{\mathrm{M}}} = \mathfrak{a}$ und $\overline{\overline{\mathrm{N}}} = \mathfrak{b}$.
>
> This leads to the definition of the sum of $\mathfrak{a}$ and $\mathfrak{b}$. We put
>
> $$(1) \qquad \mathfrak{a} + \mathfrak{b} = (\overline{\overline{\mathrm{M},\mathrm{N}}}).$$

Certainly if one compares Frege's Vol. II §33 with this Cantorian definition of addition, the affinity is unmistakeable. But whereas Cantor succeeded in the merely mathematical task of stipulating what cardinal number one would obtain by adding two cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$—a success merely *in extenso*—the Fregean logicist surely cannot rest content with that. He cannot allow himself the compromising loss of generality that is involved in having to consider *disjoint* sets M and N in the foregoing Cantorian characterization of addition. The Fregean logicist has the more demanding (and self-imposed) task of providing a sense for *any* expression of the form $t + u$ when both t and u are terms ostensibly denoting cardinal numbers. Thus both t and u may indifferently (and independently) be numerals; parameters in natural deductions; or abstractive terms of the form $\#xF(x)$, where $F(x)$ itself may be *any* predicate of the language. And that language, note, may contain predicates in whose extensions fall physical objects. The language is not necessarily that more limited, wholly mathematical one for which Cantor plied his account of addition. Thus the Fregean logicist, in pursuit of goal (1) of applicability, would need to furnish a definition of addition that would confer the required senses upon terms such as

$$\#x(x \text{ captained England in cricket}) + \#x(x \text{ has a Philosophy degree}),$$

and

$$\#x(x \text{ was an Oxfordshire cricketer}) + \#x(x \text{ played chess for Oxfordshire}).$$

The accomplishments of Mike Brearley ensure that the former of these additive terms denotes a number at least one greater than

$$\#x(x \text{ captained England in cricket or } x \text{ has a Philosophy degree});$$

and if Paul Grice's well-known autobiographical claim is true, the latter of these additive terms denotes a number *exactly* one greater than

$$\#x(x \text{ was an Oxfordshire cricketer or } x \text{ played chess for Oxfordshire}).$$

If M is chosen to be the set of all England cricket captains and N is chosen to be the set of all Philosophy degree-holders, then Cantor's definitional method will not be able to tell us what $\overline{\overline{M}} + \overline{\overline{N}}$ is—since M and N are not disjoint. But if one takes the cardinality $\mathfrak{a}$ of the set of all England cricket captains and the cardinality $\mathfrak{b}$ of the set of all Philosophy degree-holders, then Cantor's definitional method does provide us with a route to the sum $\mathfrak{a}+\mathfrak{b}$, albeit perforce circuitously. The route is this: choose some set M with cardinality $\mathfrak{a}$ and some set N *disjoint from* M with cardinality $\mathfrak{b}$; then for $\mathfrak{a}+\mathfrak{b}$ take (in modern notation) $\overline{\overline{M \cup N}}$. Obviously in making *these* choices of M and N one will have to avoid taking for M the set of all England cricket captains and for N the set of all Philosophy degree-holders!

The circuitousness of Cantor's route to sums is neither here nor there when the sole preoccupation is that of fixing the action of the operation of addition within a domain of abstract objects (the cardinal numbers) presumed already given. Any *inter res* route from $\mathfrak{a}$ and $\mathfrak{b}$ to $\mathfrak{a}+\mathfrak{b}$ will do. But the Fregean logicist wants in addition (if one will excuse the pun) to assign a sense to *any* expression of the form $\#xF(x) + \#xG(x)$, regardless of how the extensions of the predicates $F(x)$ and $G(x)$ might fall. Each such linguistic expression must receive a sense; and in assigning such senses, one expects the sign '+' of addition to receive the same sense wherever it occurs. That, for the Fregean logicist, is the more exigent task of 'defining addition'.

The recent resurgence of interest in logicism has been based on the observation that for the purposes of a Fregean derivation of the axioms for zero and successor, and of the principle of mathematical induction, Frege's ultimately inconsistent theory of classes is not needed. It suffices to assume Hume's Principle—one of the theorems derived in Frege's system—along with other logical principles collectively consistent with it. Frege had derived Hume's Principle in the first volume of *Grundgesetze*, and all his subsequent work in pursuit of goal (2) used no more than Hume's Principle. So, a neo-logicist such as Wright contends, all that logicism needs is a defence of Hume's Principle as analytic, and the philosophical task of logicism will have been accomplished. Arithmetic will have been derived from analytic—that is to say, logical—principles.

Quite so. But that holds only in so far as the basic axioms of arithmetic really have been derived within the logicist's refurbished, and now consistent, theory, from *conceptually illuminating definitions*. The current literature on neo-logicism contains no hint as to how a logicist might define addition and multiplication in a conceptually illuminating way, and then use those definitions to provide a derivation of the recursion axioms for addition and multiplication.

Nor is there any guidance on this score from Frege himself. This is in stark contrast to his legacy of deductive trail-blazing within the easier territory of zero, successor and mathematical induction. He gave conceptually illuminating definitions, and amply suggestive deductive signpostings in the *Grundlagen*,[3] and followed them up with the captious hardscaping of the *Grundgesetze*. But that was only for zero and successor.

With addition and multiplication, matters are starkly different. The logicist searching within the two volumes of the *Grundgesetze* for any conceptual illumination of addition and multiplication will be disappointed. Frege appears not to attach any significance at all to the problem of how to characterize addition and multiplication on the whole numbers *as cardinal numbers*, and how on such a basis to justify the basic arithmetical laws governing them. Instead, he treats his reader (Vol. II, §65) to a remarkably unconvincing argument for the view that it would be wrong-headed to attempt to do so! He maintains that a study of arithmetical operations like addition and multiplication *restricted to the natural numbers* would sin against one of his canons of definition: that the definition of any function, such as $+$, has to provide a sense for any term of the form $a + b$, whatever *bedeutungsvolle Eigennamen* might be put in place of a and b. Thus he is setting out to accomplish the apparently more ambitious task of characterizing addition (he makes no mention of multiplication) as a function on at least the real numbers, construed as ratios of magnitudes. He gets as far as establishing the commutativity and associativity of an addition-like operation defined for his so-called *Positivalklassen*. But that still falls short of

[3] It was these signpostings that Wright sought to make into a slightly more detailed logical itinerary in his monograph [20], which stimulated the recent revival of interest in logicism. Like Frege in the *Grundlagen*, Wright did not concern himself with addition, let alone with multiplication. But a fully detailed map of the logical terrain as it concerned only zero and successor had of course already been provided by Frege in Vol. I of *Grundgesetze*. Wright had not read the *Grundgesetze* (personal communication), because it was in German and its notation was 'rebarbative'.

deriving even the Peano–Dedekind recursion axioms for addition alone, on the basis of a conceptually illuminating definition of addition. And besides, Frege himself (Vol. II, §157) regarded the natural numbers (*Anzahlen*) as ontologically distinct from the positive whole numbers (*positive ganze Zahlen*) that are the multiples of the unit real within a system of real numbers. So even if he succeeds in deriving laws (such as commutativity and associativity) for addition and multiplication on the latter kind of number, that would not yet vouchsafe the corresponding laws on the former kind.

This brings to a close the philosophical stage-setting for the logical and foundational contribution to follow. The logical methods to be employed centrally involve the notion of an ordered pair, taken as primitive and *sui generis*. It is by means of the logical notion of orderly pairing that the logicist's gap can be filled.

2 Orderly pairing

An ordered pair is a 'thing of the form' $\langle t, u\rangle$, in which the order of the things paired *matters*. It is this ordering which first gives us the first 'member', and then gives us the second 'member'. So the ordered pair $\langle t, u\rangle$ has t as its first 'member' and u as its second 'member'. The word 'member' is used in scare quotes in order to flag the fact that it is not to be assumed that we are dealing with membership in the sense of set theory. The ordered pair $\langle t, u\rangle$ will be identical to the ordered pair $\langle u, t\rangle$ if, but only if, $t = u$.

In order to avoid any possible future confusion between talk of members of ordered pairs and talk of set-theoretic members of *sets*, reference henceforth will be to the (first and second) *coordinates* of an ordered pair.

We are used to thinking of ordered pairs within the confines of modern set theory. The definition we use today, due to Kuratowski [12] at pp. 170–1, is

$$\langle t, u\rangle =_{df} \{\{t\}, \{t, u\}\}.$$

Against the background of set theory, the Kuratowski definition of ordered pair meets the following essential condition of adequacy: it allows unambiguous specification of both the *first* and the *second* coordinate of any given ordered pair. With the Kuratowski pair-set, which is the set-theoretic surrogate for the ordered pair, we can define the ordered pair's first coordinate as

> that thing that is an element of each member of the pair-set.

The ordered pair's second coordinate can then be defined as follows:

> if the pair-set's union has just one member, then that member; otherwise, that member of the pair-set's union that is not the first coordinate.

By contrast with Kuratowski's set-theoretic reduction of ordered pairs, this study is concerned to deal directly with orderly pairing as an operation *sui generis*. The values of that operation, the ordered pairs themselves, are likewise individuals *sui generis*.

Given the history of set theory and of proof theory, there was never an appropriate juncture at which it would have been natural to pursue the question of introduction and elimination rules for both orderly pairing and the projection of coordinates. For by the time Gentzen first produced his systems of natural deduction in 1936, set theorists had already settled on Kuratowski's definition of ordered pair, on von Neumann's definition of the ordinals, and on the definition of arbitrary functions as sets of ordered pairs. These very serviceable definitions fulfilled all the modelling demands of mathematics, from a structuralist point of view. So no one gave much thought to the question whether certain notions that had been furnished with serviceable set-theoretical surrogates might not be better characterizable directly, as *sui generis*, and as obeying a genuine *logic* of their own—a logic best laid out as a system of Gentzen-style introduction and elimination rules for the notions concerned. Sections (3)–(10) aim to do that for the orderly pairing operation and its associated coordinate-projection operations. Sections (11)–(14) will then put orderly pairing to work in our search for conceptually illuminating primitive rules governing addition and multiplication of cardinal numbers. These rules serve to define those two notions, and afford natural proofs of the Dedekind–Peano recursion axioms. (The task of laying out those proofs, however, will have to be deferred to another occasion.)

3 Notation

At this point let us give up the usual notation

$$\langle t, u\rangle$$

for an ordered pair, and adopt instead the notation

$$\pi(t, u).$$

The new notation stresses that the formation of an ordered pair is a matter of applying a binary function (of 'orderly *pairing*') to two arguments in a given order. The value of the function will, in general, depend both on the arguments and on the order in which they are given. The value will be an *individual* in the domain of discourse. (Remember that we are dealing here with a proposed extension of first-order logic, by focusing on certain functions.) One might be tempted to regard this individual $\pi(t,u)$ as somehow *composed out of* the arguments t and u. But that would be a mistake, in so far as we are concerned here with how to understand the first-order *logic* of the orderly pairing function π. All that one needs to grasp is that the function π maps t and u to their ordered pair. Exactly what kind of entity this ordered pair might be is a question that need not be pressed for the time being. It is enough to know that the ordered pair is an individual in the domain, falling within the range of the quantifiers.

A universally free logic will be used in all that follows, as in [16]. This logic is based on the Russellian construal of the truth-conditions of atomic predications, including identities: for an atomic predication $A(t_1,\ldots,t_n)$ to be true, all the terms $t_1,\ldots,t_n$ must denote (and of course the relation A must obtain among their denotations). One can express the fact that the term t denotes by writing $\exists x(x=t)$. One can abbreviate this as $\exists!t$. So we have the so-called

Rule of Denotation for Atomic Predications

$$\frac{A(t_1,\ldots,t_n)}{\exists!t_i} \quad (1 \leq i \leq n)$$

There is also a rule designed to capture the idea that in order for a functional term to denote, its argument terms must denote.

Rule of Denotation for Functional Terms:

$$\frac{\exists!f(t_1,\ldots,t_n)}{\exists!t_i} \quad (1 \leq i \leq n)$$

Note that since we must allow in general for *partial* functions, i.e. functions that are not everywhere defined, our free logic does *not* have the rule

$$\frac{\exists!t_1 \quad \ldots \quad \exists!t_n}{\exists!f(t_1,\ldots,t_n)}$$

But the orderly pairing function π is special: it *is* total. This is expressed by the so-called Rule of Totality (for π) below. The Rule of Totality tells one that from *any* two individuals, in *any* order, one can form the corresponding ordered pair. Of course, the expression 'one can form' is merely metaphorical. We do not compose ordered pairs out of their coordinates. Rather, ordered pairs exist as abstract objects, given only that their coordinates exist. That is to say, any two things can be paired, in any of their possible orders.

In addition to the pairing function π, one needs to treat the *coordinate projection functions* λ (for the first, or *left* coordinate) and ρ (for the second, or *right* coordinate). These are monadic functions. They are defined only on ordered pairs. So they are not necessarily total.

4 The formal rules for orderly pairing

For the purposes of providing introduction and elimination rules, we focus on the projection functions λ (projecting the left coordinate) and ρ (projecting the right coordinate). The introduction rule for λ tells us the canonical conditions under which one can infer an identity of the form $t = \lambda(u)$; likewise (*mutatis mutandis*) with the introduction rule for ρ.

(λ-I) $$\frac{u = \pi(t, v)}{t = \lambda(u)}$$

(ρ-I) $$\frac{u = \pi(v, t)}{t = \rho(u)}$$

Corresponding to these introduction rules for the projection operations λ and ρ are their elimination rules.[4]

[4]The reader unfamiliar with proof-theoretic notation is advised that the parenthetically enclosed numeral '(i)' has an occurrence labelling the step at which the indicated assumption-occurrences higher up at 'leaf nodes' of the sub-proof(s) are *discharged* by applying the rule in question. A discharged assumption no longer counts among the assumptions on which the conclusion of the newly created proof depends. Also, we say that a is *parametric* within a sub-proof just in case among the undischarged assumptions, and conclusion, of the sub-proof in question, a occurs only within sentences of the indicated form.

$$(\lambda\text{-E})\qquad \dfrac{t=\lambda(u) \qquad \begin{array}{c}\overline{u=\pi(t,a)}^{\,(i)}\\ \vdots\ a \text{ parametric}\\ \psi\end{array}}{\psi}\,(i)$$

$$(\rho\text{-E})\qquad \dfrac{t=\rho(u) \qquad \begin{array}{c}\overline{u=\pi(a,t)}^{\,(i)}\\ \vdots\ a \text{ parametric}\\ \psi\end{array}}{\psi}\,(i)$$

These elimination rules match their respective introduction rules, in that they carry existential import. Consider, for example, the rules for λ. The introduction rule for λ tells us that the only way to prove $t = \lambda(u)$ is to prove that u is an ordered pair whose left coordinate is t. But that requires that there be some right coordinate (called v in the introduction rule) that, with the left coordinate t, makes up the ordered pair u. The elimination rule for λ then exploits this existential assumption concerning the right coordinate. (*Mutatis mutandis* for ρ and the left coordinate.) The harmony between the introduction and elimination rules for λ is brought out by the following very obvious reduction procedure. It gets rid of any sentence occurrence that stands as the conclusion of a step of λ-introduction and as the major premise of λ-elimination:

$$\dfrac{\dfrac{\begin{array}{c}\Sigma\\ u=\pi(t,v)\end{array}}{t=\lambda(u)} \qquad \begin{array}{c}\overline{u=\pi(t,a)}^{\,(i)}\\ \Theta\\ \psi\end{array}}{\psi}\,(i) \qquad\longmapsto\qquad \begin{array}{c}\Sigma\\ (u=\pi(t,v))\\ \Theta[a/v]\\ \psi\end{array}$$

The operation of orderly pairing can be applied to *any* two things that exist. That is to say, the function π is everywhere defined (or total):

Rule of Totality $\qquad \dfrac{\exists!t \qquad \exists!u}{\exists!\pi(t,u)}$

5 Derivable results

The formal rules of the previous section suffice for proofs of the following central results. Proofs are given in the Appendix.

1. $\dfrac{u = \lambda(t) \qquad v = \rho(t)}{t = \pi(u, v)}$

2. $\dfrac{u = \lambda(t)}{t = \pi(u, \rho(t))} \qquad \dfrac{v = \rho(t)}{t = \pi(\lambda(t), v)}$

3. $\dfrac{\exists!\pi(t, u)}{\exists!t} \qquad \dfrac{\exists!\pi(t, u)}{\exists!u}$

4. $\dfrac{\pi(t, u) = \pi(v, w)}{t = v} \qquad \dfrac{\pi(t, u) = \pi(v, w)}{u = w}$

5. $\dfrac{t = v \qquad u = w}{\pi(t, u) = \pi(v, w)}$

6. $\dfrac{\exists x\ \pi(x, t) = u}{t = \rho(u)} \qquad \dfrac{\exists x\ \pi(t, x) = u}{t = \lambda(u)}$

7. $\dfrac{t = \rho(u)}{\exists x\ \pi(x, t) = u} \qquad \dfrac{t = \lambda(u)}{\exists x\ \pi(t, x) = u}$

8. $\dfrac{\exists!\lambda(t)}{\exists!\rho(t)} \qquad \dfrac{\exists!\rho(t)}{\exists!\lambda(t)}$

Result (8) is called the Principle of Parity.

Results (4) and (5) can be combined into an abstraction principle that Kanamori [11] calls *Peano Pairing*:

$$\pi(t, u) = \pi(v, w) \leftrightarrow (t = v \wedge u = w).$$

6 Remarks on the rules

6.1 Rules of natural deduction in the style of Gentzen and Prawitz

The only functions for which we have provided explicit introduction and elimination rules (in the style of Gentzen [8] and Prawitz [14])[5] are the two

[5]Gentzen and Prawitz did not, themselves, consider the matter of natural deduction rules governing *term*-forming operators. They dealt only with *sentence*-forming logical

unary projection functions λ and ρ. Note, however, that the binary orderly-pairing function π does indeed have its own introduction and elimination rules. The introduction rule for π is

$$(\pi\text{-I}) \qquad \frac{u = \lambda(t) \qquad v = \rho(t)}{t = \pi(u, v)} \ ,$$

which is our Result (1) above. The elimination rules for π are

$$(\pi\text{-E}) \qquad \frac{t = \pi(u, v)}{v = \rho(t)} \qquad\qquad \frac{t = \pi(u, v)}{u = \lambda(t)}$$

which are none other than the rules (ρ-I) and (λ-I) respectively. The reduction procedure for these π-rules is obvious.

It is not possible to state adequate rules for the projection operations λ and ρ that do not involve mention of the operation π of orderly pairing. Nor is it possible to state adequate rules for π that do not mention λ and ρ. These three operations are conceptually *coeval*. It is therefore impossible to force one's introduction and elimination rules into a form where the operator concerned is the *sole* operator occurring in the statement of the rule. This, however, should not count against the rules that we have given as capturing the *logical* content of these operations.

The introduction and elimination rules for λ and ρ cannot be derived from the introduction and elimination rules for π alone. They can, however, be derived from those for π *together with the Principle of Parity*. Here, for example, is the derivation of (λ-E) using Parity in the direction from $\exists!\lambda(u)$ to $\exists!\rho(u)$:

operators such as connectives and quantifiers. The current treatment of introduction and elimination rules for term-forming operators is in the style introduced in [16], and further developed in [17], [18] and [19]. The leading idea is to frame an introduction rule for a term-forming operator Ω by specifying the conditions under which one would be justified in asserting an identity of the general form '$t = \Omega \ldots$', where we focus on one dominant occurrence of Ω (here, on the right-hand side). Likewise, one frames the corresponding elimination rule by using such an identity as the major premise, and specifying what one would be warranted in inferring from its assertion (presumed justified). A full statement of the rules for constructive logicism is to be found in §14 below.

$$\dfrac{\dfrac{\dfrac{t=\lambda(u) \qquad \dfrac{\dfrac{\dfrac{t=\lambda(u)}{\exists!\lambda(u)}}{\exists!\rho(u)}}{\rho(u)=\rho(u)}}{u=\pi(t,\rho(u))} \qquad \dfrac{\dfrac{t=\lambda(u)}{\exists!\lambda(u)}}{\exists!\rho(u)}}{\exists x\; u=\pi(t,x)} \qquad \begin{matrix}\overline{\;u=\pi(t,a)\;}^{(i)}\\ \vdots \\ \psi\end{matrix}}{\psi}\,{\scriptstyle (i)}$$

In similar fashion, one can derive (ρ-E) using Parity in the direction from $\exists!\rho(u)$ to $\exists!\lambda(u)$.

As we have also seen, the introduction and elimination rules for π, and the Principle of Parity, can be derived from the introduction and elimination rules for λ and ρ. The latter introduction and elimination rules, therefore, are a simpler primitive set to adopt. But we *could*, if we wished, adopt as primitive instead the introduction and elimination rules for π, along with the Principle of Parity.

Regardless of which of these two alternatives we adopted, we would still need to postulate the Principle of Totality, which is used for the proof of Result (5), the right-to-left direction of Peano Pairing.

6.2 Fregean abstraction

The logic of orderly pairing could be cast in a more traditional Fregean form, rather than in the form of introduction and elimination rules in the style of Gentzen and Prawitz. Fregean abstraction (for a term-forming operator Ω) involves laying down necessary and sufficient conditions, not mentioning Ω, for the truth of an identity statement involving *two* Ω-terms rather than one. In the case at hand, where we take π for Ω, such a statement of necessary and sufficient conditions is Peano Pairing itself:

$$\text{(PP)} \qquad \pi(t,u)=\pi(s,v) \quad \leftrightarrow \quad (t=s \wedge u=v).$$

The right-hand side of this biconditional involves no mention of π.

This quantifier-free statement of Peano Pairing as a Fregean abstraction principle is implicitly universally quantified:

$$\forall x \forall y \forall z \forall w \;[\; \pi(x,z)=\pi(y,w) \quad \leftrightarrow \quad (x=y \wedge z=w)\;].$$

Taking the direction from right to left, we obtain an easy proof of the Principle of Totality:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\exists!a}{a=a}\quad\dfrac{\exists!b}{b=b}}{a=a\wedge b=b}\qquad\dfrac{\exists!b\quad\dfrac{\exists!b\quad\dfrac{\exists!a\quad\dfrac{\exists!a\quad\forall x\forall y\forall z\forall w[\pi(x,z)=\pi(y,w)\leftrightarrow(x=y\wedge z=w)]}{\forall y\forall z\forall w[\pi(a,z)=\pi(y,w)\leftrightarrow(a=y\wedge z=w)]}}{\forall z\forall w[\pi(a,z)=\pi(a,w)\leftrightarrow(a=a\wedge z=w)]}}{\forall w[\pi(a,b)=\pi(a,w)\leftrightarrow(a=a\wedge b=w)]}}{[\pi(a,b)=\pi(a,b)\leftrightarrow(a=a\wedge b=b)]}}{\pi(a,b)=\pi(a,b)}}{\exists!\pi(a,b)}
$$

We can also *define* the two projection functions λ and ρ in terms of the pairing function π, by using the definite description operator ι:

$$\lambda(t) \quad =_{df} \; \iota x\exists y\; t=\pi(x,y);$$

$$\rho(t) \quad =_{df} \; \iota y\exists x\; t=\pi(x,y).$$

Using these definitions, we can derive the introduction and elimination rules for λ and ρ. Note that if t is not an ordered pair, then both $\lambda(t)$ and $\rho(t)$ will be non-denoting terms. And that is how it should be.

It is interesting that orderly pairing is one of the cases in which the method of Fregean abstraction actually works, without incurring paradox (or well-grounded worries about potential paradox). This is because of intuition (i) below. Fregean abstraction applied to the notion of *set*, as opposed to orderly pairing, incurs paradox. Here the Fregean abstraction principle would be

$$\{x \mid Fx\} = \{x \mid Gx\} \;\leftrightarrow\; \forall x(Fx \leftrightarrow Gx).$$

This would have the special case

$$\{x \mid Fx\} = \{x \mid Fx\} \;\leftrightarrow\; \forall x(Fx \leftrightarrow Fx).$$

Since the right-hand side is a theorem, we can detach and conclude

$$\{x \mid Fx\} = \{x \mid Fx\},$$

hence

$$\exists!\{x \mid Fx\},$$

regardless of what the predicate Fx might be. Taking $\neg(x\in x)$ for Fx, we obtain Russell's Paradox.

7 Analytic intuitions

The *logic* of orderly pairing (and projection of coordinates) is captured by any sufficiently strong subcollection of the foregoing rules. That is, the rules in question capture what is essential to the *concept* of orderly pairing. The following five ingredients are essential, and, collectively, they exhaust the concept in question:

(i) any two things can be paired, and in either order;
(ii) not necessarily everything is an ordered pair;
(iii) two ordered pairs are identical just in case their respective coordinates are;
(iv) an ordered pair exists only if both its coordinates do; and
(v) a thing can have the one kind of coordinate only if it has the other kind as well (that is, the projection functions are defined only on ordered pairs, and indeed on exactly the same ones, namely all of them).

These intuitions (i)–(v) are captured by any of the three rule-collections that we have considered above.

But there is another source of intuitions about orderly pairing that needs to be addressed. These are intuitions about the global properties of the operation of orderly pairing. They are addressed in the next section.

The situation at which we have thus far arrived in formulating a logic for orderly pairing corresponds to the situation one is in with what I have called the 'logic of sets', after formulating introduction and elimination rules for the set-abstraction operator. (See Tennant [16], ch. 7, and [19].) Those rules capture the analytic connections among set-abstraction, membership and possession of defining properties (of sets). In particular, they reveal the axiom of extensionality as a *consequence* of a deeper analysis of the notion of set, as captured by the introduction and elimination rules. The rules are mute, however, on the question of what sets actually exist, either outright or conditionally. This is because the rules make up a *free* logic. The introduction and elimination rules for the set-abstraction operator provide a conceptual analysis of the notion of set, but not an ontological theory about what sets actually exist. The ontological questions are the ones properly left to set *theory*, as opposed to the logic of sets (or what Quine called 'virtual set theory'). Indeed, one might venture a little further than the logic of sets, by laying down rules (or axioms) for the *hereditarily finite sets*—of which, of course, there are infinitely many. One could take the view that the existence of the empty set is an analytic matter, as would be also the conditional existence of pair sets, unions, and power sets. But it would not be analytic that any particular infinite set exists.

This extension of the 'analytic logic of sets' would be a closer analogue to what we have been doing with the logic of orderly pairing. For, in the latter, we have commited ourselves to the existence of the ordered pair of any two existents. Provided, then, that there are at least two existents, it follows right away that there will be infinitely many. (This is so even if we leave the so-called *problem of conflation* unresolved—for which, see below.)

There is an interesting difference, however, between the logic of orderly pairing and the logic of sets, when it comes to considering the use that we make of them, respectively, for mathematics. As far as the notion of ordered pair is concerned, all that is important is already captured in the *logic* of orderly pairing. We do not need to settle the further 'ontological' questions addressed in the next section before having a serviceable notion of ordered pair.

8 Ontological intuitions

According to Kanamori ([11], p. 289),

(κ) Peano Pairing is 'the instrumental property which is all that is required of the ordered pair.'

The word 'instrumental' here adverts to the needs of mathematics. Mathematics needs ordered pairs that behave in a certain minimally constrained way, a way that will allow for a definition of functions as sets of ordered pairs, etc. Once that minimally constrained behavior is guaranteed (by combination, say, of set-theoretical axioms and a definition of ordered pairs as Kuratowskian pair-sets), the mathematician's needs are satisfied. It is then superfluous, as far as mathematics is concerned, to inquire further into the nature of ordered pairs themselves, *were they to be taken as sui generis.* Now, as it happens, set theory contains the Axiom of Foundation, which ensures that the set-membership relation $\in$ is well-founded. Set-theoretically defined ordered pairs therefore acquire a well-founded pedigree of ordered-pair formation from the well-foundedness of $\in$. But this does not provide a principled answer to the question whether pedigrees of ordered-pair formation should be well-founded when the operation of orderly pairing is taken as *sui generis*, and not as set-theoretically defined.

This problem is especially acute when we consider that the Rule of Totality seems so natural. Any two things can be paired in orderly fashion. The resulting ordered pair will be distinct from each of them; and will in turn be eligible to be a coordinate of yet other ordered pairs. That there is such an operation, in thought at least, seems undeniable. Note, however, that it would be impossible to render this ambitious thought within the confines of set theory. For then one would have to construe the operation of orderly pairing as a function, mapping objects a and b (in that order) to the ordered pair $\langle a, b\rangle$. The function would be construed as a set of ordered pairs whose first coordinate represents the input, and whose second coordinate represents the output. The input to any binary function is actually an ordered pair; and it would be the first coordinate of an ordered pair ('in' the graph of the function) whose second member would be the output of the function, i.e. the ordered pair itself. Hence the orderly pairing function would end up being the set of all ordered pairs of the form $\langle\langle a, b\rangle, \langle a, b\rangle\rangle$, where a and b are arbitrary individuals in the domain (of set theory). But no such set can exist, at least not according to a theory such as ZFC. Rather, it would have to be construed as a proper class. Even the theory NBG, however, would not throw much light on the operation of orderly pairing, construed now as the *proper class* of all ordered pairs of the form $\langle\langle a, b\rangle, \langle a, b\rangle\rangle$, where a and b are arbitrary individuals in the domain. For, the Rule of Totality, as it governs the operation of orderly pairing *sui generis*, would apply to any two *proper classes*. If the latter are *things* in the domain (which, according to NBG, they are), then, it seems, we can pair them in whichever order we please. It would seem, then, that this operation of orderly pairing cannot even be captured as a *proper class* of ordered pairs.

What we are seeing here is a special case—involving the operation of orderly pairing—of a more general problem that afflicts the set-theoretic aspiration to provide a set-theoretic surrogate (i.e., a *set*) for any mathematical entity, including operations that are everywhere defined. Within set theory itself, for example, we have the operation of power-set formation, which is everywhere defined. (This is so even when we allow for *Urelemente*. The power set of any *Urelement* is the singleton of the empty set.) The power-set operation $\wp$ cannot *be* a set (of ordered pairs), since it would be too large. Yet we can use the functional expression $\wp(x)$ in set theory without risk of contradiction or paradox. This is because every occurrence of a term of the form $\wp(t)$ can be replaced by a set-abstract of the form

$$\{x \mid \forall y(y \in x \rightarrow y \in t)\}.$$

Because power-set formation is thus explicitly definable in set-theoretic terms, we are never at any theoretical 'loss for words' when talking about power sets, even though, on pain of contradiction, we do suffer an *ontological* deficit in not being able to identify any particular *set* within the set-theoretic universe as 'being' the operation of forming power sets. We can reason set-theoretically 'as if' $\wp$ were a primitive operation, *sui generis* and everywhere defined, because we know that anything we might wish to say in terms of $\wp$ can be translated, via the foregoing definition, into consistent set-theoretical terms. Having the lexical primitive $\wp$ affords a mere *conservative extension* of ordinary set theory, i.e. of first-order set theory formulated only in terms of set (abstraction and) membership. What we have just observed concerning the operation of power-set formation applies also to the operations of union and of pair-set formation. These operation are term-wise definable within set theory. So having primitive functional symbols to express them would result in only a conservative extension of ordinary set theory.

Let us return now to ordered pairing. Kanamori's claim (κ) may well be allowed by one who insists only on respecting the minimally demanding structural intuitions about ordered pairs in mathematical contexts where ordered pairs are simply devices for constructing set-theoretical surrogates of mathematical objects of various more complicated kinds. But the claim may be questioned by one who wishes to know more about the essential natures of ordered pairs, when orderly pairing is taken as *sui generis.*

Two questions can be asked in this regard:

(i) If one decomposes an ordered pair by projecting its coordinates, and repeats this operation on coordinates that are themselves ordered pairs, could one continue *ad infinitum*?—or is the 'pedigree' of any ordered pair finite?

(ii) Can such a 'pedigree', even if finite, nevertheless contain loops? That is, could a finite sequence of applications of λ and/or ρ to any ordered pair t produce t as its result?

The answer to (i) should be that the 'pedigree' of any ordered pair should indeed be finite. This is an *ontological* intuition about 'grounding'. The thought is that the operation of orderly pairing cannot be indefinitely iterated; it must have *begun* with things that are *not* ordered pairs. (Let us call such things *monads*, with no echo of Leibniz. Monads might be *Urelemente*, but they could also be sets. The set ω, for example, is a monad, since it is not an ordered pair.) Hence also the answer to (ii) is that there cannot be any loops in the pedigree of an ordered pair. That pedigree must be a finite *tree.*

These ontological intuitions go strictly beyond what is required of the notion of ordered pair in order for it to serve as it does in the set-theoretical reconstruction of mathematics. Given the Axiom of Foundation in set theory, and the Kuratowski definition of ordered pair, it is clear that the 'orderly pairing' pedigrees of Kuratowskian ordered pairs must be finite trees. But note that this is only because the membership relation among *sets* is in general well-founded. If one is dealing with orderly pairing *sui generis*, outside the context of set theory, then one has to do more in order to ensure that the pedigrees of one's ordered pairs are indeed finite trees. This can be seen from the examples given in the next section.

9 Deviant interpretations

Surprisingly, the foregoing rules for the orderly pairing operation π, and its associated projection functions λ and ρ, underdetermine the notion of ordered pair to the extent that it is possible to identify certain ordered pairs with the individuals from which they are formed. Suppose given two distinct individuals a and b. Our rules force the four ordered pairs $\langle a, a\rangle$, $\langle a, b\rangle$, $\langle b, a\rangle$, and $\langle b, b\rangle$, to be distinct from each other. Those rules do not, however, force all four of these ordered pairs to be distinct from a and to be distinct from b. The following arrangement, for example, is compatible with our rules:

$$\begin{array}{cccc} \langle a, a\rangle & \langle a, b\rangle & \langle b, a\rangle & \langle b, b\rangle \\ \bullet & \bullet & \bullet & \bullet \\ a & b & & \end{array}$$

The reader can easily check that the individual a could be held identical to any one of the four ordered pairs that can be formed from a and b, and the individual b could be held identical to any one of the remaining three. The resulting denotation diagram is easily extendible to a model for our rules. (Extension of course is required in order to cope with the indefinitely many further possible iterations of orderly pairing.) Let us call this *the problem of conflation.*

Observe also that the one-element model satisfies our rules. In this model, consisting of a single individual a, say, all ordered pairs are identical to a. These include $\langle a, a\rangle$, $\langle a, \langle a, a\rangle\rangle$, $\langle\langle a, a\rangle, a\rangle$, $\langle\langle a, a\rangle, \langle a, a\rangle\rangle$, etc. This is a very degenerate and extreme case of the problem of conflation.

Even if the problem of conflation were to be solved by further stipulation of (axioms or) rules, another problem would remain: there could be infinitely descending paths of successive projections of coordinates. Let us call this the *problem of regress*.

Here is an example. Think of an infinite binary tree with root a. The nodes in this tree will be ordered pairs. Each node's daugher nodes will be its left and right coordinates. The operation of taking a left or a right coordinate can be iterated indefinitely. This seems strongly counterintuitive to anyone who thinks that ordered pairs must in some sense ultimately be 'built up out of' things that are not, themselves, ordered pairs. The claim 'Everything is an ordered pair' is consistent with our rules thus far, and would indeed force all interpretations to be deviant. But even the denial of that claim would not suffice to eliminate the problem of regress. For let b be something that is not an ordered pair. Add b to the domain consisting of the earlier infinite binary tree rooted on a. Then 'close' the domain with all the further ordered pairs that can be generated from the individuals already at hand. The result is a model of all our rules thus far, plus the axiom 'Not everything is an ordered pair.' And since this model contains the infinitely descending branches of the infinite binary tree rooted on a, the problem of regress has not gone away. Indeed, just one such branch would pose the problem.

So, if one wishes to respect the ontological intuitions expressed above, one will have to lay down more rules.

10 Supplementing the rules

The simplest supplemental rules will rule out the one-element model for the logic of orderly pairing:

$$\frac{t = \lambda(t)}{\perp} \qquad \frac{t = \rho(t)}{\perp}$$

So these rules would solve the extreme form of the problem of conflation. But they would not be enough to solve the problem of conflation in general. For, while it would ensure that a and $\langle a, a\rangle$ are distinct, one would still be able to construct models in which other undesirable conflations occur. To be sure, the distinct individuals a and b would now each have to be distinct from each of the four ordered pairs that can be formed from them. These six individuals would give rise to as many as thirty further ordered pairs. But nothing in the rules thus far can prevent a from being held identical to, say, $\langle\langle a, a\rangle, b\rangle$ and b from being held identical to, say, $\langle a, \langle b, b\rangle\rangle$, and letting the other twenty eight ordered pairs be new distinct individuals. In this (ontologically) deviant interpretation, the left (and right) coordinate of the left coordinate of a would be a itself; and the right (and left) coordinate of the right coordinate of b would be b itself. That is, there would be two loops (of length 2) in the orderly-pairing pedigree of a; and likewise for b.

These ontologically deviant interpretations could be ruled out by prohibitions more powerful than the ones just given, to the effect that pairs cannot be either of their coordinates. That prohibition merely rules out shortest possible loops, of length 1, in orderly-pairing pedigrees. What is needed now is a way of ruling out all possible (finite) loops.

This could be done by adopting the family of rules of the following form (for $n \geq 1$):

$$\frac{\gamma_1(t_1) = t_2 \quad \gamma_2(t_2) = t_3 \quad \ldots \quad \gamma_n(t_n) = t_1}{\perp}$$, where each γ_i is either λ or ρ, and $n \geq 1$.

An alternative, and equivalent, set of schemes would be

$$\frac{\gamma_n(\gamma_{n-1}(\ldots \gamma_1(t)\ldots)) = t}{\perp}$$

This would solve the problem of conflation. But the problem of regress still remains. Perhaps the best way to solve the problem of regress is to lay down a principle of induction. Recall our earlier notion of a *monad*, i.e. something that is not an ordered pair. The following are definitional rules of reflection for that notion:

$$(M\text{-I}) \quad \dfrac{\begin{array}{c} \overline{t = \pi(a,b)}^{(i)} \\ \vdots \ a,\, b \text{ parametric} \\ \perp \end{array}}{M(t)}{(i)} \qquad\qquad (M\text{-E}) \quad \frac{t = \pi(u,v) \quad M(t)}{\perp}$$

Equipped with the notion of a monad, we can now state a principle of induction for orderly pairing, in which a and b are parametric within the subordinate proofs in which they occur:

$$\dfrac{\begin{array}{c}\overline{M(a)}^{\,(i)}\\ \vdots\\ \Phi(a)\end{array}\qquad t=\pi(u,v)\qquad \begin{array}{c}\underbrace{\overline{\Phi(a)}^{\,(i)}\quad \overline{\Phi(b)}^{\,(i)}\quad \overline{\exists!a}^{\,(i)}\quad \overline{\exists!b}^{\,(i)}}\\ \vdots\\ \Phi(\pi(a,b))\end{array}}{\Phi(t)}{}^{(i)}$$

Here, $\Phi(x)$ is any first-order formula with x as its sole free variable. So the Principle of Induction is a first-order rule-schema.

11 Applications of the logic of orderly pairing in a constructive logicist account of arithmetic

The constructive logicist account of number in [17] was furnished by means of a formal language of first-order logic with the variable-binding term-forming operator $\#x\Phi(x)$ ('the number of Φs'), the name 0 and the unary function sign s ('successor'). Natural deduction rules were laid down for these expressions, in a way that arguably captures their intended meanings. (See Section 14 below for a statement of all those rules in their current preferred form.) Detailed formal derivations were given for the Peano–Dedekind postulates governing 0 and s, and these derivation were carried out in intuitionistic relevant logic. This constructive logicist account also delivers every instance of *Schema N*:

$$\#xF(x) = \underline{n} \leftrightarrow \text{there are exactly } n\ F\text{s},$$

thereby meeting what was proposed as a material adequacy condition on any theory of the natural numbers. In Schema N, $\underline{n}$ is the numeral for the number n, and the right-hand side contains no reference to numbers. An illustrative instance would be

$$\#xF(x) = ss0 \leftrightarrow \exists x \exists y(\neg x = y \wedge \forall z(Fz \leftrightarrow (z = x \vee z = y))).$$

The material adequacy condition ensures, in Fregean spirit, that the account captures the general *applicability* of natural numbers (for counting finite collections of things, concrete or abstract). The derivation of the Peano–Dedekind postulates ensures that the account captures the pure mathematics of the natural numbers (as abstract entities considered in their own right).

The account did not extend, however, to addition and multiplication of natural numbers. An appropriate extension is to be offered here. One can exploit the resources of the logic of orderly pairing in order to furnish adequate rules governing both addition and multiplication. The aim is to provide a rule-theoretic analysis that is conceptually lucid and convincing, and that affords straightforward deductions of the Peano–Dedekind postulates governing addition and multiplication. (Here it should be borne in mind that [17] has already secured the Peano–Dedekind postulates governing zero and successor, in a system based on similar rules.)

It is important to appreciate at the outset that the aim is not to furnish pair-theoretic surrogates for the natural numbers, in the way that set theory furnishes set-theoretic surrogates for them (namely, the finite von Neumann ordinals). The natural numbers will remain as objects of abstraction, *sui generis*. In the terminology of the present study, numbers will be monads, not ordered pairs. So we resist the temptation to exploit the operation of orderly pairing in order to furnish a surrogate for the successor function, such as

$$s(n) =_{df} \pi(0, n).$$

On such an approach, as soon as one had secured the existence of zero, the succeeding 'numbers' 1, 2, 3, ... would be generated as

$$\pi(0,0),\ \pi(0,\pi(0,0)),\ \pi(0,\pi(0,\pi(0,0))),\ \ldots$$

But such a definitional fix is to be eschewed. The numbers have, after all, already been obtained in their own right (cf. [17]), as the necessarily existing *Bedeutungen* of arithmetical terms (both pure and applied) that obey appropriate sense-constituting rules of inference. That much is achieved simply in securing the Peano–Dedekind postulates governing zero and successor. Ordered pairs will, however, feature in the current *extension* of what was accomplished in [17], an extension intended to furnish a logicist analysis of the operations of addition and multiplication. In providing the sought extension, we can avail ourselves of the *logic* of orderly pairing. The extended account will still be *logicist*, both in spirit and in letter. Details will emerge below.

11.1 A first (but inadequate) stab at rules for addition

Here is a first and obvious way to try to characterize addition, using only the resources of the constructive logicist account of [17].

If the number of Fs is t, and the number of Gs is u, and nothing is both F and G, then the number of things that are either F or G is the *sum* $(t+u)$. This simple-minded thought is captured by the following rule:

$$\dfrac{\#xFx = t \qquad \#xGx = u \qquad \begin{array}{c} \underbrace{{}^{(i)}\overline{Fa}\ ,\ \overline{Ga}^{(i)}} \\ \vdots \quad a \text{ parametric} \\ \bot \end{array}}{\#x(Fx \vee Gx) = (t+u)}\,{}_{(i)}$$

As a special case, we would be able to derive the result that if the number of Fs and the number of Gs exist, and nothing is both F and G, then the sum of those numbers is the number of things that are F *or* G:

$$\dfrac{\exists!\#xFx \qquad \exists!\#xGx \qquad \begin{array}{c} \underbrace{{}^{(i)}\overline{Fa}\ ,\ \overline{Ga}^{(i)}} \\ \vdots \quad a \text{ parametric} \\ \bot \end{array}}{\#x(Fx \vee Gx) = (\#xFx + \#xGx)}\,{}_{(i)}$$

This way of characterizing addition, however, suffers from the limitation that F and G must be disjoint concepts. Yet even for overlapping concepts F and G one should be able to make sense of the sum of $\#xFx$ and $\#xGx$. So the characterization of addition afforded by the foregoing rule seems insufficiently general.

It is insufficiently general on another score too: how is it to be used so as to capture $t + u$ when both t and u are (pure) numerals? The most obvious way to furnish a concept F with exactly t satisfiers is to take Fx to be 'x is a natural number preceding t'. (Likewise, take Gx to be 'x is a natural number preceding u'.) But that brings with it the problem that when t and u are non-zero, (these choices of) F and G must overlap in extension. The proposed rule would therefore be useless in fixing what the sum of t and u is.

This problem will be solved only by resorting instead to choices of Fx and of Gx that *are* disjoint, and for which t is the number of Fs and u is the number of Gs. In characterizing the natural numbers, even in the absence of addition and multiplication, we know that we cannot assume the existence of infinitely many concrete objects; so the numbers themselves need to be taken as objects in order to ensure the existence of indefinitely large, albeit finite, extensions of (at least, numerical) concepts. Since we have to resort to the abstract realm to ensure the existence of arbitrarily finitely many things, we may as well exploit the potential infinitude of the abstract realm in providing an account of the operations of addition and multiplication.

The current account does just that—but by exploiting the resources of orderly pairing. Regardless whether the extensions of F and of G contain any concrete objects, we can ensure respectively equinumerous *and disjoint* extensions F' and G' as follows:

$$F'(x) : \lambda(x)=0 \wedge F(\rho(x));$$
$$G'(x) : \lambda(x)=1 \wedge G(\rho(x)).$$

That is, the satisfiers of F' will be ordered pairs of the form $\pi(0, x)$ where x satisfies F; and the satisfiers of G' will be ordered pairs of the form $\pi(1, x)$ where x satisfies G. Since 0 is distinct from 1, F' is disjoint in extension from G', even if F and G themselves overlap. This enables one to make any individual that satisfies both F and G eligible to be reckoned 'twice over' in computing the sum of the number of Fs and the number of Gs.

In general, an arithmetical term is either of the form $\#xF(x)$, or of the form $s(t)$ for some arithmetical term t. (Note that 0 is defined to be $\#x\ \neg x=x$, so 0 qualifies as a term of the former kind.) The constructive logicist has already secured the general result that

$$\underline{n} = \#x(x < n).$$

11.2 A second (but still inadequate) stab at rules for addition

It may be thought that, in accounting for the conditions under which one can assert or infer statements of the form

$$t = u + v$$

where u and v are arithmetical terms in general, it should suffice to account for the conditions under which one can assert or infer statements of the form

$$t = \#xF(x) + \#xG(x).$$

Recalling that 1 is defined as $s(0)$, and that we have the theorem $\neg 0=1$, it might be thought that the introduction rule for addition could accordingly be framed as follows.

$$(+\text{-Intro}) \quad \frac{t = \#x((\lambda x = 0 \wedge F\rho x) \vee (\lambda x = 1 \wedge G\rho x))}{t = \#xF(x) + \#xG(x)}$$

The corresponding elimination rule would then simply be the converse:

$$(+\text{-Elim})\quad \frac{t = \#xF(x) + \#xG(x)}{t = \#x((\lambda x = 0 \wedge F\rho x) \vee (\lambda x = 1 \wedge G\rho x))}$$

To proceed in this fashion, however, would be to lose sight of the need, on which Frege insisted so vigorously, to provide an account of the truth-conditions of an assertion of the *general* form $t = u + v$, whatever *bedeutungsvolle Eigennamen* might be put in place of u and v. In the natural-deduction setting, the Fregean logicist is obliged to provide an account of the conditions under which one may *infer to* and *from* assertions of the form $t = u+v$, whatever terms might be put in place of u and v. Among the candidate terms are the parameters that the natural-deduction theorist uses for quantificational inferences such as ∀-Introduction and ∃-Elimination within proofs. The foregoing rules do not provide such an account.

12 Rules for addition

Regardless of the syntactic shape of the terms t, u and v—be they abstractive terms, or terms of the form $s(t)$, *or parameters*—we must specify what it is for t to be the sum of u and v. Within the confines of standard second-order logic, this can be done somewhat laboriously as follows. Note that the expression $\Phi xy[\Psi x\text{1-1}\Theta y]$ means that the two-place relation Φ effects a 1-1 correspondence of the Ψs with the Θs. Purely logical rules were provided in [17], pp. 276–81, for inferring to and from claims of this form.

$$\frac{u = \#xFx \quad v = \#xGx \quad \begin{array}{c} \underbrace{\overline{F'a}^{\,(i)}\ ,\ \overline{G'a}^{\,(i)}} \\ \vdots \\ \bot \end{array} \quad Rxy[Fx\text{1-1}F'y] \quad Sxy[Gx\text{1-1}G'y] \quad t = \#x(F'x \vee G'x)}{t = u + v}{\scriptstyle (i)}$$

This rules takes into account the possibility that F and G, the concepts whose numbers u and v are to be added, might overlap in extension. When that happens, one finds *different* concepts F' and G' that are disjoint, and respectively equinumerous with F and with G, and one takes the number of the disjunctive concept $F' \vee G'$ as the required sum. This supplies the logical letter for the spirit of Frege's §33.

In all applications of the rule just formulated, one could always find the desired disjoint concepts F' and G' by exploiting the simple device of pairing each object in the extension of F with some object (say, 0) and pairing each object in the extension of G with some distinct object (say, 1). Thus one would have

$$F'x \equiv_{df} \lambda x = 0 \wedge F\rho x$$
$$G'x \equiv_{df} \lambda x = 1 \wedge G\rho x$$

Thus F' and G' will apply only to ordered pairs; and will be disjoint, since 0 is distinct from 1. Moreover, F' will be equinumerous with F, and G' with G. For the 1-1 mappings R and S one could take $x = \rho y$, for which one can easily prove

$x{=}\rho y[Fx$ 1-1 $(\lambda y = 0 \wedge F\rho y)]$ and
$x{=}\rho y[Gx$ 1-1 $(\lambda y = 1 \wedge G\rho y)]$

Thus the third, fourth and fifth subproofs for the foregoing rule would always be to hand. One might as well, therefore, cut down the postulational work and adopt the following more streamlined rule. Bear in mind that this is possible because we now have the logic of orderly pairing as part of the logical apparatus that the logicist is entitled to bring to bear on the definitional problem at hand, namely, how to set up rules that genuinely define addition of natural numbers in fully Fregean spirit. The more streamlined rule is as follows:

$$(+\text{-Intro}) \quad \frac{u{=}\#xFx \quad v{=}\#xGx \quad t{=}\#x((\lambda x{=}0 \wedge F\rho x) \vee (\lambda x{=}1 \wedge G\rho x))}{t{=}u+v}$$

Note that $F(x)$ and $G(x)$ are schematic, to be replaced, in applications of this rule, by any two formulae with exactly the variable x free. That means, in effect, that the assertability condition called for by the rule of (+-Intro) is second-order existential:

$$\frac{\exists F\exists G(u{=}\#xFx \wedge v{=}\#xGx \wedge t{=}\#x((\lambda x{=}0 \wedge F\rho x) \vee (\lambda x{=}1 \wedge G\rho x)))}{t{=}u+v}$$

Accordingly, the elimination rule takes the following form, where the predicate-parameters $\mathcal{F}$ and $\mathcal{G}$ in the subordinate proof of φ are parametric for existential elimination. That is to say, they do not occur in the premise $t = u + v$, nor in φ, nor in any assumptions, other than those indicated, on which the upper occurrence of φ depends.

$$(+\text{-Elim})\qquad \dfrac{t=u+v \qquad \begin{array}{c}\underbrace{\overline{u=\#x\mathcal{F}x}^{\,(i)}\ ,\ \overline{v=\#x\mathcal{G}x}^{\,(i)}\ ,\ \overline{t=\#x((\lambda x=0\wedge\mathcal{F}\rho x)\vee(\lambda x=1\wedge\mathcal{G}\rho x))}^{\,(i)}}\\ \vdots\\ \varphi\end{array}}{\varphi}{\scriptstyle (i)}$$

The foregoing rules clearly provide a precise sense for addition on *all* cardinal numbers, both finite and infinite.

13 Rules for multiplication

With a primitive notion of ordered pair, we can now formulate an introduction rule for multiplication as well.

$$(\times\text{-Intro})\qquad \dfrac{u=\#xFx \quad v=\#xGx \quad t=\#x\exists y\exists z(Fy\wedge Gz\wedge x=\pi(y,z))}{t=u\times v}$$

By considerations similar to those that led us to the right elimination rule for $+$, we have the following elimination rule for $\times$, where the restrictions on the predicate-parameters $\mathcal{F}$ and $\mathcal{G}$ are as they were for (+-Elim):

$$(\times\text{-Elim})\qquad \dfrac{t=u\times v \qquad \begin{array}{c}\underbrace{\overline{u=\#x\mathcal{F}x}^{\,(i)}\ ,\ \overline{v=\#x\mathcal{G}x}^{\,(i)}\ ,\ \overline{t=\#x\exists y\exists z(\mathcal{F}y\wedge\mathcal{G}z\wedge x=\pi(y,z))}^{\,(i)}}\\ \vdots\\ \varphi\end{array}}{\varphi}{\scriptstyle (i)}$$

The foregoing rules clearly provide a precise sense for multiplication on *all* cardinal numbers, both finite and infinite.

14 A full statement of the rules of constructive logicism

An improved statement of the rules for #, 0 and s proposed in [17] is as follows.[6] Together with the rules stated in §4 for orderly pairing, and those stated in §12 and §13 for addition and multiplication respectively, they form

[6] This formulation is to be found in [19], at p. 113. It is slightly more condensed than the set of rules in [17], owing to the elimination of a redundancy first remarked by Ian Rumfitt in [15].

a complete basis for a constructive logicist derivation of full Peano-Dedekind arithmetic.[7]

0-Introduction

$$\dfrac{\begin{array}{c}\underbrace{\overline{F(a)}^{\,(i)}\ ,\ \overline{\exists! a}^{\,(i)}}\\ \vdots\\ \bot\end{array}}{0 = \#xF(x)}\,(i)$$

0-Elimination

$$\dfrac{0 = \#xF(x) \quad \exists! t \quad F(t)}{\bot}$$

#-Introduction

$$\dfrac{\#xFx = t \quad Rxy[Fx\ 1\text{–}1\ Gy]}{\#xGx = t}$$

Recall that the condition $Rxy[Fx\ 1-1\ Gy]$ is that R effects a $1-1$ correspondence of the Fs with the Gs.

#-Elimination

$$\dfrac{\#xGx = t \qquad \begin{array}{c}\underbrace{\overline{\#xFx = t}^{\,(i)}\ ,\ \overline{Rxy[Fx\ 1\text{–}1\ Gy]}^{\,(i)}}\\ \vdots\ F,\ R \text{ parametric}\\ B\end{array}}{B}\,(i)$$

s-Introduction

$$\dfrac{\#xFx = t \quad Rxy[Fx\ 1\text{–}1\ Gy, r]}{\#xGx = st}$$

(Here, the condition $Rxy[Fx\ 1-1\ Gy, r]$ is that R effects a $1-1$ correspondence of the Fs with all the Gs except r. Purely logical rules were provided in [17], pp. 276–81, for inferring to and from claims of this form.)

[7] The formal derivations collectively justifying this claim are deferred to another paper.

s-Elimination (first half)

$$\dfrac{\#xGx = st \qquad \begin{array}{c} \underbrace{\overset{(i)}{\overline{\#xFx = t}}\ ,\ \overset{(i)}{\overline{Rxy[Fx \ 1\text{–}1 \ Gy, a]}}} \\ \vdots \ a,\ F,\ G,\ R \text{ parametric} \\ B \end{array}}{B}{\scriptstyle (i)}$$

s-Elimination (second half)

$$\dfrac{u = st \qquad \begin{array}{c} \overset{(i)}{\overline{u = \#xHx}} \\ \vdots \ H \text{ parametric} \\ B \end{array}}{B}{\scriptstyle (i)}$$

The second half of the rule of s-Elimination in effect says that terms with s dominant can only denote objects within the range of denotations of #-terms.

14.1 Commutativity and associativity of the two operations

Note that the new introduction and elimination rules for $+$ and $\times$ say nothing about zero and successor. The remaining Peano–Dedekind postulates, the well-known 'recursion equations' for $+$ and $\times$:

$$\forall x \; x + 0 = x$$

$$\forall x \forall y \; x + sy = s(x + y)$$

$$\forall x \; x \times 0 = 0$$

$$\forall x \forall y \; x \times sy = (x \times y) + x$$

will be derivable as the 'pure numerical' reflections of their respective introduction and elimination rules.

There is something intellectually satisfying in the way in which these rules for addition and for multiplication directly secure the general commutativity and associativity of the two operations.[8]

The rules of inference and the derivations of constructive logicism are purely conceptual; it would be a mistake to think that anything like Kantian

[8] Once the derivations are carried out in full detail, we realize that more is required in order to establish the commutativity and associativity of addition than just the commutativity and associativity of disjunction, contrary to the impression given by John Burgess in [1], at p. 26–7.

intuitions, however pure, still obtrude upon the route to the basic laws of arithmetic.

15 Appendix: Proofs of (1)-(8)

Proof of (1).

$$\dfrac{v=\rho(t) \qquad \dfrac{u=\lambda(t) \qquad \dfrac{\dfrac{}{t=\pi(b,v)}{\scriptstyle(2)} \qquad \dfrac{\lambda\text{-I}\ \dfrac{\dfrac{}{t=\pi(u,a)}{\scriptstyle(1)}}{u=\lambda(t)} \qquad \dfrac{\dfrac{}{t=\pi(b,v)}{\scriptstyle(2)}}{b=\lambda(t)}\ \lambda\text{-I}}{u=b}\ \text{Subst.=}}{t=\pi(u,v)}\ \text{Subst.=}}{t=\pi(u,v)}\ (1)\ \lambda\text{-E}}{t=\pi(u,v)}\ (2)\ \rho\text{-E}$$

Proof of (2).

$$\lambda\text{-E}\ \dfrac{u=\lambda(t) \qquad \dfrac{\dfrac{}{t=\pi(u,a)}{\scriptstyle(1)} \qquad \dfrac{\dfrac{}{t=\pi(u,a)}{\scriptstyle(1)}}{a=\rho(t)}\ \rho\text{-I}}{t=\pi(u,\rho(t))}\ \text{Subst.=}}{t=\pi(u,\rho(t))}{\scriptstyle(1)}$$

The other half of (2) is proved similarly.

Proof of (3). These are special cases of the rule of denotation for functional terms. They can also, however, be derived as follows.

$$\dfrac{\exists!\pi(u,v) \qquad \dfrac{\lambda\text{-I}\ \dfrac{\dfrac{}{a=\pi(u,v)}{\scriptstyle(1)}}{u=\lambda(a)}}{\exists!u}\ \text{Rule of Denotation for Atomic Predications}}{\exists!u}{\scriptstyle(1)}$$

The other half of (3) can be derived similarly.

Proof of (4).

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi(t,u)=\pi(v,w)}{\exists!\pi(v,w)}}{\pi(v,w)=\pi(v,w)}}{v=\lambda(\pi(v,w))} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\pi(t,u)=\pi(v,w)}{\exists!\pi(t,u)}}{\pi(t,u)=\pi(t,u)}}{t=\lambda(\pi(t,u))} \quad \pi(t,u)=\pi(v,w)}{t=\lambda(\pi(v,w))}}{v=t}$$

The other half of (4) is proved similarly.

Proof of (5).

$$\dfrac{\dfrac{\dfrac{t=v}{\exists! t} \quad \dfrac{u=w}{\exists! u}}{\exists!\pi(t,u)} \qquad \dfrac{{}^{(1)}\dfrac{}{a=\pi(t,u)} \quad \dfrac{\dfrac{{}^{(1)}\dfrac{}{a=\pi(t,u)} \quad t=v}{a=\pi(v,u)} \quad u=w}{a=\pi(v,w)}}{\pi(t,u)=\pi(v,w)}}{\pi(t,u)=\pi(v,w)}{}^{(1)}$$

Our rules therefore suffice for Peano Pairing.

Proof of (6).

$$\dfrac{\exists\ \pi(x,t)=u \qquad \dfrac{\dfrac{}{\pi(a,t)=u}{}^{(1)}}{t=\rho(u)}\rho\text{-I}}{t=\rho(u)}{}^{(1)}$$

The other half of (6) is proved similarly.

Proof of (7).

$$\rho\text{-E}\,\dfrac{t=\rho(u) \qquad \dfrac{\dfrac{}{u=\pi(a,t)}{}^{(1)}}{\exists x\ \pi(x,t)=u}}{\exists x\ \pi(x,t)=u}{}^{(1)}$$

The other half of (7) is proved similarly.

Proof of (8).

$$\dfrac{\begin{array}{r}\exists!\lambda t\\ \text{i.e. } \exists x\; x=\lambda(t)\end{array} \qquad \dfrac{\overline{b=\lambda(t)}^{\,(2)} \qquad \dfrac{\dfrac{\overline{t=\pi(b,a)}^{\,(1)}}{a=\rho(t)}\ \rho\text{-I}}{\exists!\rho(t)}}{\exists!\rho(t)}{\scriptstyle(1)}\ \lambda\text{-E}}{\exists!\rho(t)}{\scriptstyle(2)}$$

The other half of (8) is proved similarly.

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