Anti-realist Aporias

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Using a quantified propositional logic involving the operators it is known that and it is possible to know that, we formalize various interesting philosophical claims involved in the realism debate. We set out inferential rules for the epistemic modalities, ranging from ones that are obviously analytic, to ones that are epistemologically more substantive or even controversial. Then we investigate various aporias for the realism debate. These are constructively inconsistent triads of claims from our list: a claim expressing some sort of common ground in the debate; a characteristically anti-realist thesis about truth and knowability; and a characteristically realist thesis about determinacy of truth value. Various patterns of reductio proof for these inconsistent triads are generated, so as to display their variety. The reductio proofs use only the inferential rules set out earlier. The philosophical utility of each aporia for the anti-realist is then assessed. This involves consideration of the acceptability of the premiss expressing common ground; the strength and plausibility of the anti-realist premiss; the strength of the realist premiss that is the ultimate target of the reductio; and the analytic status of the inferential rules applied within the proof.

1. Introduction and background

In this paper we play variations on a theme begun in Crispin Wright’s Truth and Objectivity, and developed further in some of his more recent work. We examine aporias involving fundamental principles concerning truth and knowability. Each aporia is a deducibility statement of the form


The statement of common ground concerns our epistemic predicament. It is a statement about limitations on us as knowers. It is acceptable to both the realist and the anti-realist. The specifically anti-realist thesis expresses some form of epistemic constraint on truth, and is, of course, anathema to the realist. The specifically realist thesis expresses some form of determinacy of truth-value, and is, of course, anathema to the

1 I would like to thank Crispin Wright and Joe Salerno for stimulating discussions, an anonymous referee for helpful comments, and Mark Sainsbury for his editorial suggestions.

2 Wright, 1992.

3 Wright, unpublished.
anti-realist. There is a variety of exact logical forms for the three premisses, which it is our task to survey here.

The premisses are to be entertained with implicit background reference to some discourse or other. In order, therefore, to establish the consistency of any combination of claims from our list, it would be enough to “model” them in some discourse which sustained their truth. Such a discourse might or might not be effectively undecidable; and the knowers’ knowledge within it might vary from nothing more than logic, through to appreciation of the truth of the axioms for very powerful, undecidable and essentially incomplete theories.

One should also appreciate at the outset that the deducibility relation $\vdash$ is that of an intuitionistic, rather than a classical, quantified propositional logic. This is a precaution against begging any questions in favour of the realist. For, by means of classical rules of inference, the realist might be able to prove an inconsistency between the anti-realist’s thesis and the statement of common ground. In this way, the realist could claim that it was not his specifically realist thesis that was engendering the inconsistency. From the intuitionist’s point of view, however, the weaker rules of intuitionistic logic might not render his anti-realist thesis inconsistent with the statement of common ground; and the extra power of the classical rules in allegedly revealing such an “inconsistency” betrays only the inconsistency of the underlying realist thesis that is supposed to justify those very rules. And that, precisely, is what the anti-realist’s aporia, using only intuitionistic logic, and targeting the realist’s thesis as a third premiss, is designed to reveal. Note that we are dealing here with quantified propositional logic. Thus we do not have the metatheorem to the effect that any set of sentences that can be proved inconsistent using classical logic can be proved inconsistent using intuitionistic logic. This metatheorem holds for ordinary propositional logic, but not for quantified propositional logic.

In *The Taming of the True*, a distinction was drawn between two kinds of anti-realist argument for the rejection of the realist’s Principle of Bivalence. One kind is the so-called “single-sentence” argument. A very different (and more compelling) kind is called the “whole-discourse” argument. It was argued that one should prefer the latter kind of argument over the former. Appraising or justifying that preference is beyond the scope of this paper.

The only “single sentence” argument that was considered, however, was that to be found in the writings of Michael Dummett. It appealed to a “common ground” premiss in the form of an existence claim. The claim was that there exists an undecidable sentence. The argument would pro-

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4 Tennant, 1997, chs. 6 and 7.
ceed by saying “Let \( \phi \) be an undecidable sentence ...”. (Hence the adjec-
tival label “single-sentence” to characterize the type of argument that then
ensued.) The trouble with Dummett’s attempted reductio argument, how-
ever, was (as one will see again below) that it failed to target the charac-
teristically realist thesis! This was because the other two premisses of the
would-be reductio were already jointly inconsistent.

This need not be the case, however, with every single-sentence argu-
ment worthy of the name.

Our point of departure is an attempt by Joseph Salerno\(^5\) to formalize
clearly the theses that are plausibly involved in Wright’s argument (in
Wright 1992) that the anti-realist’s epistemic constraints on truth are in
tension with the realist’s principle of bivalence, \textit{modulo} certain facts
(acknowledged as “common ground”) about our epistemic predicament.
From Wright’s discursive treatment it is difficult to make out the exact
logical forms of the theses involved. The virtue of Salerno’s treatment is
that it ventured certain regimentations, and furnished a formal proof of the
aporia thus clarified.

The most pressing task was to make precise logical sense of Wright’s
expression of the common ground regarding our epistemic predicament.
Wright wrote (Wright 1992, p. 42) of

the a priori unwarrantability of the claim that the scales of in prin-
ciple available evidence must tilt, sooner or later, one way or the
other, between each statement and its negation.

Salerno offered more than one way of regimenting the thought behind this
phrase (which we shall call “Wright’s dictum”). The most plausible
among these was, he proposed, his so-called principle of Epistemic Mod-
esty, which will be introduced below. As we shall see, however, Salerno’s
treatment can be improved upon—in choice of theses, in choice of infer-
rential rules, and in overall philosophical impact.

Before embarking on our formal analysis of details, it is worth saying
why we favour the method of complete formalization employed below.
All too often philosophical discussions of the conclusions of certain infor-
mal and prosaic arguments are sidetracked by disagreements concerning
the sheer validity of the arguments in question. Critics might allege that
hidden and unacceptable premisses are being employed; or that certain
informal steps, taken as primitive in the prosaic reasoning, are, on closer
inspection, fallacious—involving, say, a quantifier switch or an equivoca-
tion on some important term. Completely formalizing the arguments
either makes it absolutely clear that such misgivings would be misplaced,
or enables the critic to pinpoint exactly which premiss or rule of inference
is to be disputed. Our aim below is to provide formally correct arguments

\(^5\) Salerno, forthcoming.
that proceed according to explicitly stated rules, with all their premisses made fully explicit and unambiguous. That will enable critics to set aside any misgivings about formal correctness according to the stated rules, and concentrate on the more important matters: assessing the validity of the rules of inference used, and determining a philosophical response to what the arguments reveal.

Our method will accordingly be to provide formal definitions of the central notions in terms of suitably chosen primitives; to formulate relevant theses as well-formed formulae in the appropriate formal language; to frame various rules of inference that might be used in the construction of formal proofs, so that questions about their epistemic or analytic status can profitably be raised; and to explore the various formal reductio proofs behind the anti-realist aporias of the informal kind presented by Wright. In short, we aim to provide an uncontentious battery of formal detail from which further philosophical discussion can safely proceed.

2. The central concepts

Using disjunction and negation, we can construct the propositional concept “ϕ is bivalent”:
\[ \Box \phi = df \phi \lor \neg \phi. \]
The claim “in the state of information w it is known that ϕ” will be represented by the formal sentence \( K_w \phi \). It will always be understood that only one, representative, knower is under consideration. That way we do not have to bother with the relational locution “x knows in w that ϕ” (\( xK_w \phi \)), and can deal simply with \( K_w \phi \).

A special state of information is a, the current state. We shall abbreviate \( K_a \phi \) to \( K\phi \).

Using the (unsubscripted) epistemic operator K (which, remember, really means \( K_a \)) we can construct the concept “ϕ is resolved”:
\[ \Box \phi = df K\phi \lor K\neg \phi. \]

We also need to express the claim (for a given ϕ that does actually hold) that it is possible, from state of information v, to come to know, in the actual world, that ϕ. In formal notation, we express this as follows: \( \exists w (v \leq w \land K_w \phi) \). Here we are taking states of information (in actual or potential developments within the actual world) to form the field of an accessibility relation \( \leq \). Thus \( v \leq w \) means that the state of information v could develop into the state of information w (as a result of either natural events over which knowers have no control, or the intentional actions of knowers, or both). When ϕ is a sentence, we shall use the symbolic abbreviation \( \Diamond v K\phi \) to cap-
ture the same thought. Thus any use of $\Diamond_\alpha K\phi$ carries with it the presupposition that in its unabbreviated form $\exists w (v \leq w \land K_w \phi)$, the embedded $\phi$ does not contain any free occurrence of $w$. The formal sentence just given might also be read as “It is feasibly knowable in $v$ that $\phi$”. (Once again, we can agree to suppress the subscript when it is $a$—in which case $\Diamond K\phi$ can be read as “It is feasibly knowable that $\phi$”.) The idea is that the possibility alluded to is that of our attaining the knowledge that $\phi$, where $\phi$ already holds. We are therefore stressing that it is a possibility for us, as knowers situated in the current state of information—or at least a possibility for some finite extension of ourselves. We are not being invited to think of $\Diamond K\phi$ as true when $\phi$ is a contingent falsehood. On the contrary: if $\phi$ is any contingent falsehood (such as, say, “Grass is purple”), then it is not feasibly knowable that $\phi$, in the sense of feasibility with which we are here concerned. Feasibility is not at all like the alethic modality of possibility.

Another way of putting this last moral is to say that not only the epistemic operator $K$, but also its modalization $\Diamond K$, is “factive”. That is, we have the valid inference “$\Diamond K\phi$, therefore $\phi$”, just as we have the inference “$K\phi$, therefore $\phi$”. (We stress again that in the abbreviatory context $\Diamond K\phi$, the embedded $\phi$ is a sentence, with no variables free.) Note that the current state of information $a$ is realized in the actual world; and all its accessible extensions are states that might transpire after $a$. That is, such states are realizable within possible future histories of the initial temporal segment of the actual world that takes us up to $a$. They do not have to be thought of as obtaining, in some cases, only in alethically alternative possible but non-actual worlds with different initial segments up to $a$.

But by the same token, not necessarily all the so-called “states of information” potentially available “in” the actual world, as possible extensions of the current state of information $a$, will actually be realized. The actual course of human epistemic history will cut a confined canyon through that partially ordered plain of possibilities. The “extension” relation $v \leq w$ among these “possible” states of information is to be understood as follows: if one happens to be in the state of information $v$, then one could, if called upon to do so, take certain steps so as to attain the state of information $w$. Now in saying this, we respect the following possibility: if, in $v$, the asserter were (for whatever reason) to choose not to take such steps as might be required (and would suffice) for him to attain the state of information $w$, then, as time passed, he might well have forfeited forever after, on behalf of the whole epistemic community the possibility of being so able to attain to (certain crucial pieces of information in) $w$. Potential relevant evidence not seized and committed to communal memory might be forever lost. The community might then, as a matter of “transcendent fact”, as it were, never be able to lay claim to the knowledge that $\phi$. But they
would be unapprised of that transcendent fact. They would also be unable to establish the impossibility involved; since, for all they knew, the relevant evidential traces might still survive in some still-accessible state of information. Thus they would not be justified in denying that φ. Ex hypotthesi, then, they would be unable to assert φ; but they would be unable to deny it either.

For any factive operator Θ, we shall call Θφ the Θ-strengthening of φ.

Using the epistemic modality ◇ K, we can construct the concept “φ is decidable”:

\[ \square \phi =_{df} \square K \phi \lor \square K \neg \phi. \]

We can then proceed further to construct the concept “φ is undecidable”:

\[ \Box \neg \phi =_{df} \neg (\square K \phi \lor \square K \neg \phi). \]

The concept “φ is an unsolved problem” we define thus: 6

\[ \Box K \phi =_{df} \neg (\Box K \phi \lor \Box K \neg \phi); \]

and the even more elusive “φ is a conundrum”7 thus:

\[ \Box K \neg \phi =_{df} \neg \Box (\Box K \phi \lor \Box K \neg \phi). \]

The concept “φ is a quandary” (due to Wright) can be defined as follows:

\[ \Box K \phi =_{df} \neg (\Box K \phi \lor \Box K \neg \phi); \]

and the even more exigent concept “φ is an enigma” can be defined as follows:

\[ \Box K \neg \phi =_{df} \neg \Box (\Box K \phi \lor \Box K \neg \phi). \]

Finally we construct the concept “φ is non-transcendent” (or, equivalently, “φ is in principle knowable, if true”):

\[ \Box \phi =_{df} \phi \rightarrow \Box K \phi. \]

These are all the complex concepts that we shall need for our discussion.

6 Strictly speaking, the formula here should be read as: “φ is not known to be resolved”.

7 Epithets such as “conundrum”, “quandary” and “enigma” are being invested here with a technical significance given strictly by the definitions provided, in terms of K and ◇. We lay no claim to having thereby explicated the standard meanings of these epithets in ordinary English. On the contrary; we have the definitions in mind, for future philosophical use, and are simply casting about for convenient abbreviatory labels from the English lexicon.
3. Some important philosophical claims involving the basic concepts

The concepts defined above can be embedded in various important philosophical claims. The following list provides, first, a mnemonic label; second, where practicable, a conveniently abbreviated formula; third, the formula in full detail; and fourth, an informative English name for the philosophical claim in question, whose initial letters are used for the mnemonic label.

<table>
<thead>
<tr>
<th>Realist Theses</th>
<th>Anti-realist Theses</th>
<th>Common Ground</th>
</tr>
</thead>
<tbody>
<tr>
<td>LEM (\forall \phi \supset \exists \phi) (\forall \phi(\phi \lor \neg \phi))</td>
<td>(\forall \phi \supset \exists \phi) (\forall \phi(\phi \rightarrow \neg \phi))</td>
<td>(\exists \psi \supset \neq \psi) (\exists \psi \neq K(\psi \lor \neg \psi))</td>
</tr>
<tr>
<td>GO (\forall \phi \supset \exists \phi) (\forall \phi(\neg \exists \phi \lor \neg \neg \phi))</td>
<td>(\forall \phi \supset \exists \phi) (\forall \phi(\phi \rightarrow \neg \phi))</td>
<td>(\exists \psi \neq \exists \psi) (\exists \psi \neq K(\psi \lor \neg \psi))</td>
</tr>
<tr>
<td>[CK] (\forall \phi \supset \exists \phi) (\forall \phi(\neg \exists \phi \lor \neg \neg \phi))</td>
<td>(\forall \phi(\neg \exists \phi \lor \neg \neg \phi))</td>
<td>(\exists \psi \neq \exists \psi) (\exists \psi \neq K(\psi \lor \neg \psi))</td>
</tr>
<tr>
<td>BR (\forall \phi(\exists \phi \lor \neg \exists \phi)) (\forall \phi((\exists \phi \lor \neg \exists \phi) \lor (\neg \exists \phi \lor \neg \neg \phi)))</td>
<td>(\forall \phi(\neg \exists \phi \lor \neg \neg \phi))</td>
<td>(\exists \psi \neq \exists \psi) (\exists \psi \neq K(\psi \lor \neg \psi))</td>
</tr>
<tr>
<td>BD (\forall \phi(\exists \phi \lor \neg \exists \phi))</td>
<td>(\forall \phi((\exists \phi \lor \neg \exists \phi) \lor (\neg \exists \phi \lor \neg \neg \phi)))</td>
<td>(\exists \psi \neq \exists \psi) (\exists \psi \neq K(\psi \lor \neg \psi))</td>
</tr>
</tbody>
</table>

The claim of Omniscience is bracketed above because, even though its logical form puts it roughly in the neighbourhood of the orthodox anti-realist Knowability Principle, it would nevertheless be an absurdly strong claim to visit upon the anti-realist. Indeed, a classic attempt to undermine
anti-realism, due to Fitch, involved trying to show that $KP$ actually implies $O$, with the absurdity of the latter taken as given.\footnote{A detailed anti-realist defence against the Fitch argument is presented in Tennant 1997 ch. 8. Exploring this aspect of the debate would take one too far afield here.} We shall point out only that the unabbreviated form of $KP$ is $\forall \phi(\phi \rightarrow \exists w(a \leq w \land K_w \phi))$. This does not say that every truth is actually known. Nor does it say that every truth will at some time be known. It says only that every truth could be known, in the sense that we could, if we so chose, eventually attain a state of information that warranted it. Note also the order of the quantifiers: $\forall \exists$. $KP$ is not claiming that there will or could be a state of information in which all truths would be known. To think otherwise would be to commit the quantifier-switch fallacy.

We have also bracketed Complete Knowledge, since it would be an absurdly strong claim (for non-trivial discourses) to visit upon the realist. Both $O$ and $CK$ are extreme substitution instances of schematic logical forms that we shall be employing below for the anti-realist and realist theses respectively. Indeed, they offer some kind of reassurance that all the more likely substitution instances will have come within our purview through the application of the schematic method about to be pursued.

In addition to the claims above, we shall have occasion to consider some of their $K$-strengthenings or $\Box K$-strengthenings, such as

\[
\begin{align*}
\text{K-LEM} & \quad K \forall \phi \supset \Box \phi & K \forall \phi(\phi \rightarrow \Box \phi) & K \text{-Law of Excluded Middle} \\
\text{\Box K-P} & \quad \Box K \forall \phi \Box \phi & \Box K \forall \phi(\phi \rightarrow \Box K \phi) & \Box K \text{-Knowability Principle}
\end{align*}
\]

It should be clear, from the definitions already given, what formulas in primitive notation answer to various abbreviations. For example:

\[
\begin{align*}
\neg K \rightarrow KP & \text{ abbreviates } \neg K \rightarrow \forall \phi(\phi \rightarrow \Box K \phi) \\
K \Box(KP) & \text{ abbreviates } K(\forall \phi(\phi \rightarrow \Box K \phi) \rightarrow \Box K \forall \phi(\phi \rightarrow \Box K \phi))
\end{align*}
\]

We turn now to discuss further the logical properties of the ingredient notions $K$ and $\Box K$.

4. Some inference rules governing the known and the knowable

In this section we set out certain formal inference rules governing both the simple epistemic operator $K$ and the composite epistemic modality $\Box K$.
Note that these are implicitly subscripted by $a$. We continue to suppress subscripts in the interests of simplicity. Note, however, that in the statement of the rules to follow, one could, if one wished, supply subscripts as called for at the appropriate places. The reader can rest assured that all the formal reasoning remains both correct and valid upon restoration of explicit parameters in this way. Indeed, the analytic status of all the analytic rules that involve the contextually defined operator $\Diamond K$ is guaranteed by, among other things, the logical behaviour of the existential quantification in the sentence $\exists w(a \leq w \land K_w \phi)$ that is abbreviated to the form $\Diamond K \phi$ that appears in the rule of inference.

Some of the inference rules to be given below are analytically valid. Others are more substantive; and yet others are even quite controversial, not just among meaning theorists and philosophical logicians, but also among epistemologists. We need to set out all the rules, however, so that we can better assess the cogency and philosophical value of proofs employing them.

4.1 Analytic rules of inference

The elimination rule expressing the factive character of a propositional operator $\Theta$ is as follows:

$$
\frac{\phi}{\phi}^{(i)}
$$

$$
\vdots
$$

$$(\Theta E) \frac{\Theta \phi \psi}{\psi}^{(i)}$$

Both $K$ and $\Diamond K$ are factive operators. This is part of their very meaning. Trivial applications of the rules $(KE)$ and $(\Diamond KE)$ allow one to derive the simple elimination inferences mentioned earlier:

$$
\frac{K\phi}{\phi} \quad \frac{\Diamond K \phi}{\phi}
$$

Contraposing, we have $\neg \phi \vdash \neg K \phi$ and $\neg \phi \vdash \neg \Diamond K \phi$. Thus any unresolved proposition is both an unsolved problem and a conundrum; and any undecidable sentence is both a quandary and an enigma.
The following rule \((\Omega \vdash)\) expresses the fact that the propositional operator \(\Omega\) transmits under deducibility:

\[
\begin{array}{c}
\theta_1, \ldots, \theta_n \\
\vdots \\
\Omega \chi
\end{array}
\]

\((\Omega \vdash)\) \(\frac{\Omega \theta_1, \ldots, \Omega \theta_n}{\Omega \chi}\) \((i)\) where \(0 \leq n\) and \(\chi\) depends on exactly \(\theta_1, \ldots, \theta_n\)

There is no disputing that facts about provability yield facts about knowability. Proof is an exemplary means of coming to know. So the following rule \((\Diamond K \vdash)\) is analytically valid:

\[
\begin{array}{c}
\theta \\
\vdots \\
\Diamond K \chi
\end{array}
\]

\((\Diamond K \vdash)\) \(\frac{\Diamond K \theta}{\Diamond K \chi}\) \((i)\) where \(\chi\) depends on exactly \(\theta\)

That is, whatever is deducible from a knowable proposition (or from no assumptions at all) is itself knowable. Note that with \((\Diamond K \vdash)\) we do not allow the case where \(n\) exceeds 1; whereas with \((K \vdash)\) (see below) we do. This is for the simple reason that two separately knowable propositions might not be knowable conjointly. (Quantum mechanics provides familiar examples; and others can be thought up.)

So far we have not yet provided any rules characterizing the independent logical contribution of \(\Diamond\) within \(\Diamond K\). This is no surprise; for there really is no independent logical contribution being made by \(\Diamond\) on its own. It is only ever used in association with \(K\). There is, however, one (derivable) rule that is worth stating as though it were primitive. We shall call it \(\Diamond\)-introduction. We have to bear in mind, however, that this is not a rule for introducing the modal operator of possibility in all contexts (that is, as a prefix to any proposition whatsoever). Rather, since the possibility concerned is that of coming to know that a proposition \(\phi\) holds, the premiss for the rule has to be of the form \(K \phi\):

\[
\begin{array}{c}
K \phi \\
\Diamond K \phi
\end{array}
\]

\((\Diamond I)\) \(\frac{K \phi}{\Diamond K \phi}\)
This rule is clearly analytic; it simply says that what is actually known is feasibly knowable. It is easy to derive this rule once given the axiom \( a \leq a \):

\[
\frac{a \leq a}{\exists w (a \leq w \wedge K_w \phi)}
\]

4.2 More substantive rules of inference

Epistemologists often dispute the validity of the rule \( (\Omega \vdash) \) when \( \Omega \) is replaced by \( K \):

\[
\frac{(\vdash I) \theta_1 \ldots \theta_n} {(\vdash K \chi) \theta_1 \ldots \theta_n \chi \theta_n}{\chi (i)} \quad \text{where } 0 \leq n \text{ and } \chi \text{ depends on exactly } \theta_1 \ldots \theta_n
\]

At best, the objector would complain, the operator \( K \) would then be interpretable, not as “It is known that ...”, but rather “It is deducible from what is known that ...”. Note that in the special case where \( n = 0 \), \( (K^+) \) tells us that we can infer \( K \chi \) when we have shown that \( \chi \) is a logical theorem. A single step of \( (\vdash K^\chi) \) would then secure \( \vdash K \chi \); or, alternatively, one could simply apply the rule \( (\vdash K \chi) \). When the epistemic operator is used in conformity with the rule \( (K^+) \), epistemic logicians talk of “virtual knowledge”. This is an idealized notion, to be construed as knowledge on the part of an ideally rational agent. On this construal, \( K \) enjoys closure under logical deducibility; so the sense of \( K \) is not at all that of “occurrently knows”, which of course is not closed under logical deducibility.

One way to mitigate the apparent substantiveness of \( (K^+) \) would be to insist, as Salerno does, that the subordinate proof be known to the knower whose knowledge is concerned. At present the rule is to be read as though this requirement is not to be imposed. Within the context of particular proofs constructed by means of \( (K^+) \), however, one might be able to argue that the proof-giver and the proof-receiver will thereby be apprised of the existence of the subordinate proof for each application of that rule, so that the distinction between the two ways of taking the rule cannot really be sustained.

Another rule whose analytic status is dubious, but which will find application at least once below within an aporia whose line of argument
appears to require it, is the following rule \((K\neg)\). This rule says, in effect, that anything inconsistent with the knower’s knowing what he does is known by him not to hold. The boldface \(K\) indicates that all assumptions other than \(\phi\) on which \(\bot\) depends are of the form \(K\psi\):

\[
\frac{K\phi}{\vdots}
\]

\((K\neg)\)

\[
\frac{K\neg(K\neg)}{K\neg\phi}
\]

This rule comes very close to committing one to the controversial \(KK\)-thesis: if \(\phi\) is known, then it is known that \(\phi\) is known. For, by one step of \(\neg\)-elimination applied to \(K\phi\) and \(\neg K\phi\), and one step of \((K\neg)\), one can infer \(K\neg\neg K\phi\) from \(K\phi\):

\[
\frac{K\phi \quad \neg K\phi}{\bot (1)}
\]

\((K\neg)\)

\[
\frac{\bot (1)}{K\neg\neg K\phi}
\]

For the classicist, this is enough to obtain \(KK\phi\) from \(K\phi\). Simply apply \((KE)\) and double negation elimination:

\[
\frac{\bot \psi \quad DNE (1)}{\neg\neg \psi (1)}
\]

\((KE)\)

\[
\frac{\bot \psi (1)}{\psi (1)}
\]

\[
\frac{\neg\neg \psi (1)}{K\psi (1)}
\]

Substituting \(K\phi\) for \(\psi\), we now have \(K\neg\neg K\phi\vdash KK\phi\).

The step of \(DNE\) would be justified by intuitionistic lights if its conclusion were a decidable proposition. In the present instance, that conclusion is of the form \(KK\phi\). Now if the operator \(K\) sustained an “occurrence” reading—or at least one according to which the knower was not required to know all the deductive consequences of what he knew—then one might be able to put forward a plausible case for the claim that \(KK\)-sentences were indeed decidable. If, however, one were to adopt a rule as strong as \((K^+)\), then, in any language for which deducibility were an undecidable relation, whether one knew that one knew a given proposition to be true would itself be an undecidable matter. In that case \(DNE\) would not be justified from the intuitionist’s point of view. But even for the intuitionist who, in the circumstances just described, did not accept double negation elimination, the result \(K\phi \vdash K\neg\neg K\phi\) would appear bad enough. In the context of our overall dialectic, however, it is not for us to defend the rule.
(\(K\neg\)) (or the other controversial rule (\(\lambda\)) to be stated below) against philosophical objections. For, as we shall see, the most obvious attempts to produce the kind of “quandary-based” aporia of the kind sought by Wright involve appeal to these formal rules. To the extent, then, that the rules themselves are dubious, so would be these Wrightian aporias. We leave it as a formal challenge to the quandary theorist to produce a quandary-based aporia using only basic rules more clearly analytical than (\(K\neg\)) and (\(\lambda\)).

4.3 Controversial rules of inference

The idealization of knowledge implicit in the rule (\(K^+\)) does not deviate as much from the actual concept of knowledge as would the idealization imposed by the following, much stronger, \(S4\)-like rule of inference for \(K\):

\[
\begin{align*}
\text{K} & \\
\vdots & \\
\text{(K+) \quad \chi} & \Rightarrow \frac{\text{K}\chi}{\text{K}\phi}
\end{align*}
\]

where \(K\) consists of propositions of the form \(K\phi\)

The reason why (\(K^+\)) is unreasonably strong (and implausibly analytic) is that it definitely commits one (constructively) to the \(KK\)-thesis. For, one particular application of the rule (\(K^+\)) is the following:

\[
\begin{align*}
\frac{K\phi}{KK\phi}
\end{align*}
\]

It would behove us always to seek to apply the weaker (and analytic) rule (\(\Diamond K^+\)) whenever possible, rather than the idealizing rule (\(K^+\)), or the over-idealizing rule (\(K^+\)).

Our final rule governing \(\Diamond\) (within the context \(\Diamond K\)) may be somewhat more controversial than our earlier ones:

\[
\begin{align*}
&\text{K, K}\phi \\
\vdots & \\
\text{(\(\lambda\)) \quad \Diamond K\phi \Downarrow} & \Rightarrow \frac{\Downarrow}{\Downarrow}
\end{align*}
\]

We are not concerned to defend (\(\lambda\)) as analytic; for surely it is not. The rule says that if knowing \(\phi\) would be inconsistent with our knowing what we do, then so would be \(\phi\)'s knowability. It might be objected that the rule (\(\lambda\)) is infirmed by the Gödel phenomena in formal arithmetic, taking \(\phi\) as the consistency statement for a system such as Peano Arithmetic (\(PA\)).
and interpreting \( K \) as based on proof within \( PA \) itself. The objection would be that there are possible ways of knowing the consistency of \( PA \) that are not in any conflict with our knowledge of \( PA \) itself. In reply to such an objection, it should be pointed out that it follows from the second incompleteness theorem that the objector would have to require \( K \) (in the context \( \Diamond K \)) to be based on proof within some proper extension of \( PA \)—for how, otherwise, is one to come to know the consistency of \( PA \)? So the objection would be fundamentally misguided. If one’s reading of \( K \) (in all contexts) were to be limited to what is known on the basis of proof in \( PA \) alone, then indeed the second incompleteness theorem would tend to confirm the rule (\( \lambda \)), rather than infirm it. But this would not yet be enough to make (\( \lambda \)) analytic.\(^9\)

4.4 A derived rule

Note that \( KP \vdash \Diamond K \neg KP \). To prove this, simply instantiate \( KP \) with \( KP \) itself. The result is a conditional with \( KP \) as antecedent and \( \Diamond K \neg KP \) as consequent. Then detach.

Consider next the following rule:

\[
\frac{\psi_1 \ldots \psi_n \vdash \chi}{KP, \psi_1 \ldots \psi_n \vdash \Diamond K \chi}
\]

It can be derived as follows. In the following schema, simply substitute \( \Diamond K \) for \( \Theta \):

\[
\begin{array}{c}
\psi_1 \ldots \psi_n \\
\vdash \forall \phi (\phi \to \Theta \phi) \\
\chi \to \Theta \chi \\
\end{array}
\]

5. Some logical connections among the claims

We now look at the various existence claims that might be thought eligible to be expressions of common ground. They all try, in one way or

\(^9\) I owe to an anonymous referee the observation that the rule (\( \lambda \)) renders \( \Diamond K \phi \) inconsistent with \( K \neg K \phi \); and that this would be far too strong. It certainly would be, given a classical understanding of the negation sign; but the intuitionist might be less worried by this result. In any event, this observation underscores the fact that the rule (\( \lambda \)) is controversial. I am not advocating it; I am stating it only because it turns out to be used in certain aporias whose cogency needs to be assessed.
another, to assert the existence of a proposition that in some sense lies beyond the limits of our cognitive assessment.

Every undecidable proposition is an enigma. Thus $EU$ implies $EE$. Every enigma is a quandary. Thus $EE$ implies $EQ$. Every enigma is a conundrum. Thus $EE$ implies $EC$.

$EU$ implies $IKD$. For, if there is an undecidable sentence then of course it is impossible to know that every sentence is decidable. The converse, however, does not hold, even in classical logic. It could be impossible to know that all claims are decidable, without there being any particular claim that is undecidable. For this reason, $IKD$ is, prima facie, more plausible a statement of the common ground (the shared confession of epistemic shortcomings) than $EU$.

$EQ$ implies $EM$:

\[
\begin{align*}
\forall \phi \exists \phi & \quad (1) \\
K \forall \phi \exists \phi & \quad (1) \\
K \exists q & \quad (2) \\
\neg K \exists q & \quad (2) \\
\exists \psi \neg K \exists \psi & \quad \bot \quad (2) \\
\bot & \quad (2) \\
\neg K \forall \phi \exists \phi & \quad (3)
\end{align*}
\]

Note that the proof just given involves the passage from $K \forall \phi \exists \phi$ to $K \exists q$, for arbitrary propositions $q$. This is the smallest degree of closure of $K$ under derivability that can be exacted by means of the rule ($K^+$). As discussed above, this rule confers on $K$ a slightly idealized sense. But in the context of the application of ($K^+$) in the preceding proof (at the step marked (1)), the idealization involved can be kept to a minimum. Here $K$ need be interpreted no more “ideally” than “knows that, or is committed by one primitive logical step by what one knows, to the fact that”. It is worth noting that exactly the same degree of logical closure for $K$ is assumed in the proof of the Fitch paradox. In this proof, one infers from $K \forall \phi \exists \phi$ to $K \exists q$. If we regard universal quantification as a generalized conjunction of instances, we see that the step from $K \forall \phi \exists \phi$ to $K \exists q$ is of the same kind. Any misgivings about allowing $K$ to enjoy this minimal degree of logical closure are orthogonal, however, to the phenomenon of “quantificational opacity” of $K$ which we shall remark on below.

If one accepts the foregoing proof (which I shall assume the reader is willing to do) then it follows by transitivity that each of $EE$ and $EU$ implies $EM$. $EM$ does not, however, (even classically) imply $EU$. For it could be a truth, unknown, that $\forall \phi \exists \phi$. In such a situation, because
∀φ∃ψφ were unknown, EM would be true (i.e., it would be true that ¬K∀φ∃ψφ); but, because ∀φ∃ψφ were true, EU would be false (i.e., it would be false that ∃ψ¬∃ψφ).

Indeed, EM does not (even classically) imply EQ. EM is of the form ¬K∀φ∃ψφ, while EQ is of the form ∃ψ¬K∃ψφ. Notice that this inference (whatever property ⊤ might be) fails because of the meaning of K. The following situation would make the premiss (EM) true, but the conclusion (EQ) false. This situation illustrates the phenomenon of “quantificational opacity” alluded to above.

Everything in the range of the quantifiers has, as it happens, been considered by the knower and he has come to know that it has property ⊤. So there is nothing in the range of the quantifiers of which it can be said that he does not know it to have property ⊤. Hence EQ is false. But EM is true, for the simple reason that the knower does not happen to know that he has indeed encountered every object in the range of the quantifiers!—whence he has no way of knowing that ∀φ∃ψφ. This is so even though he knows, of each object, that it has property ⊤.

EM is of the form ¬K-go, while IKD is of the form ¬◊K-go. So clearly IKD implies EM, but not conversely. Even modulo KP, EM does not imply IKD. So EM is weaker than IKD. It follows that by using IKD in place of EM as an expression of common ground, one should be able to weaken one or both of the other premisses of an attempted anti-realist aporia. We shall see below that this is indeed so.

To prove that EQ implies EP, we use the rule (K↑):

\[
\begin{align*}
\exists ψ & \rightarrow K(ψ \land K\neg ψ) & \exists ψ & \rightarrow K(ψ \land K\neg ψ) \\
\end{align*}
\]
EP does not imply EQ. For, one can know that a proposition is decidable without knowing that it has been decided. That EC implies EP is obvious; as is the failure of the converse (for non-omniscient knowers). Against the background of intuitionistic logic (which is all that one is entitled to assume here, without begging any important philosophical questions) it would appear that IKD and EQ are logically independent. First, there is nothing in IKD:

\[ \neg \exists \phi (\neg K \phi \lor \neg K \neg \phi) \]

that could warrant the existence claim of EQ:

\[ \exists \psi \neg K (\neg K \psi \lor \neg K \neg \psi) \]

So IKD does not imply EQ. Secondly, even if one had such a sentence \( \psi \) as claimed to exist by EQ—one that is not known to be decidable in principle—that, in and of itself, could surely not tell against the possibility of our eventually coming to know that all sentences (including \( \psi \) itself!) are decidable in principle. One who wished to deny this possibility (precisely by asserting IKD) would have to be prepared to deny it on other grounds than that some sentence could be exhibited that was not known (at the time it was adduced) to be decidable in principle. So EQ does not imply IKD.

The foregoing considerations can be summarized in the following diagram of logical implications. The absence of an arrow indicates a failure of implication:

\[ \begin{align*}
EE & \quad \supset \quad IKD \\
& \quad \supset \quad EQ \\
& \quad \supset \quad EC \\
& \quad \supset \quad EP \\
& \quad \supset \quad EM
\end{align*} \]

The striking aspect of this diagram is that EQ and IKD present themselves as the most useful strengthenings of EM for an anti-realist aporia. By thus strengthening the statement of common ground, one is thereby in a position to weaken the anti-realist thesis and the realist thesis involved in the
aporia, and also to avail oneself of only purely analytic rules in precipitating the inconsistency.

It would appear that \( IKD \) is a most interesting alternative to \( EM \) as an expression of the common ground that Wright sought to express by the dictum quoted above. \( IKD \) asserts that it is not possible to come to know the truth of Gödelian Optimism, the doctrine that all claims are in principle decidable, even if we do not at present know how to decide them.\(^\text{10}\) We are not suggesting that the anti-realist could establish \( IKD \) as the conclusion of an argument with more obvious premises. We note simply that \( IKD \) serves well as a ‘working epistemological hypothesis’ of anti-realism. As already pointed out, since \( IKD \) strengthens \( EM \), it allows the other premises of an anti-realist aporia to be weakened. On the other hand, since \( IKD \) weakens \( EU \), it is more plausible to the anti-realist, whose knowability thesis \( KP \) is anyway inconsistent with \( EU \), as we shall see in the next section.

There is another virtue that \( IKD \) has as an expression of common ground: it should definitely strike the realist as unquestionable. The realist, after all, believes in the possibility of verification-transcendent truth. For such a truth \( \phi \) it would be false that \( \Diamond K\phi \lor \Diamond K \neg \phi \). Hence the universal claim \( \forall \phi (\Diamond K\phi \lor \Diamond K \neg \phi) \) would be false, according to the realist. Now, if a proposition \( \theta \) is false then it is not possible to know \( \theta (\neg \Diamond K\theta) \); so \( IKD \) follows.

The principle \( IKD \) is therefore somewhat Janus-faced in the debate between the realist and the anti-realist. It serves each side well, from its own point of view, and for curiously different reasons. For the anti-realist, \( IKD \) is a more elaborate expression of the denial of Bivalence; while for the realist, it is a consequence of the possibility of verification-transcendent truth. It is, ironically, a perfect compromise between the two sides as an expression of common ground. It remains, then, to see what constructive reasoning will deliver from \( IKD \) in conjunction with principles that do not command assent from both sides.

\(^{10}\) Note that in this context the term “decidable” does not involve the existence of mechanical decision procedures. It adverts rather to there being a determinate and discoverable truth-value for the proposition in question. It may indeed take ingenuity to discover the truth-value in any given case; the point, however, is that there is an intellectually negotiable route to that truth-value, regardless of the paucity and inherent limitations of our effective decision procedures for determining the truth-values of sentences of any class to which the sentence in question belongs.
6. The search for an anti-realist aporia

Consider the following natural deduction in (intuitionistic relevant) quantified propositional logic:

\[
\begin{align*}
\exists \psi \neg (\Theta \psi \lor \neg \psi), & \quad \forall \phi (\phi \rightarrow \Theta \phi) \vdash \bot.
\end{align*}
\]

Now substitute \( \Diamond K \) for \( \Theta \). We thereby have

\[
\exists \psi \neg (\Diamond K \psi \lor \Diamond K \neg \psi), \quad \forall \phi (\phi \rightarrow \Diamond K \phi) \vdash \bot.
\]

That is, the existence of a sentence that is “undecidable in principle” (EU) is inconsistent with the anti-realist tenet that all truths are in principle knowable (KP). Note that for the proof of this inconsistency we do not need to break into the compositional structure of the epistemic modality \( \Diamond \). Nothing in the “interplay” between \( \Diamond \) and \( K \) contributes to the inconsistency uncovered.

The inconsistency we have established is between just EU and KP. It has nothing to do with the law of excluded middle or with any other principle of strictly classical reasoning. If, therefore, anyone were to present a reductio of the set \{EU, KP, LEM\}, in an attempt to discredit LEM (modulo EU and KP), they would be making a strategic error. The proponent of LEM could point to the prior inconsistency of the reduced set \{EU, KP\}, claiming (correctly) that the problem lies with the combination of these two premisses, rather than with LEM itself.
This point was made in *The Taming of The True*,¹¹ and the misguided reductio “of” LEM was the following:

\[
\begin{align*}
\forall \phi (\phi \rightarrow \Theta \phi) & \quad \forall \phi (\phi \rightarrow \Theta \phi) \\
q & \quad \neg q & \quad \neg q \rightarrow \Theta \neg q \\
\forall \phi (\phi \lor \neg \phi) & \quad \Theta q & \quad \Theta \neg q \\
q \lor \neg q & \quad \Theta q \lor \Theta \neg q & \quad \Theta q \lor \Theta \neg q \\
\exists \psi \neg (\Theta \psi \lor \Theta \neg \psi) & \quad \bot \tag{2}
\end{align*}
\]

Misguided though this exercise was, one can see what it was supposed to accomplish: a *reductio ad absurdum*, upon substituting $\Theta K$ for $\Theta$, of three principles taken jointly—$EU$, $KP$ and LEM. Let us consider these principles more closely, noting their characteristics for the sought aporia.

First of all, $EU$ was an attempt to express some kind of principle of “epistemic modesty”, a principle which, moreover, could be taken as *common and uncontested ground*¹² in the debate between the realist and the anti-realist. (Other principles that are to be reckoned to the common ground are the analytic inferential principles governing the key notions in play, such as the elimination rules for $K$ and for $\Theta K$, the rule $(\Theta K^+)\Theta K$ and (perhaps) the rule $(\Theta K^+)\Theta K$.) Secondly, $KP$ is one interesting attempt to express the anti-realist’s characteristic thought about truth: namely, that truth must be *epistemically constrained*. Thirdly and finally, our straightforwardly quantified propositional statement LEM is one interesting attempt to express the realist’s characteristic thought about truth: namely, that truth is bivalent. Put another way: the realist thinks that any proposition’s truth-value is determined to be either the value True or the value False, independently of our knowing, and independently of our means of coming to know, which of those two values it is.

The strategic intention (not realized in the foregoing reductio) is that by carefully formulating three such theses—the “common ground”; the characteristically anti-realist thesis; and the characteristically realist thesis—one will then be able to establish, formally, their joint inconsistency, *without an inconsistency arising from any two of them* (even in the presence of the analytic inferential principles mentioned above). Thus, both the realist and the anti-realist will be able consistently to extend the common ground with their respective characteristic theses; and, without the common

¹¹ Tennant, 1997 p. 181.

¹² This point is due to Salerno, forthcoming.
ground explicitly expressed, their characteristic theses would not yet be in conflict. This requirement—that the inconsistency should arise from the substantive interaction of all three premises, and not from any pair of them—will be called the “bareness requirement”.

If the strategic intention is fulfilled, the anti-realist proceeds to deny the realist’s thesis, holding to the common ground and the characteristically anti-realist thesis. This, indeed, is why we choose to call the reductios in question anti-realist aporias. The search for them was motivated, in the first instance, by the anti-realist’s desire to show that the realist’s assumption of bivalence not only had no justification, but could be convicted of incoherence—modulo, of course, other assumptions that the anti-realist, for his part, would take to be unchallengeable (or, at the very least, defensible against any challenge).

For the realist, of course, the situation would be reversed. Faced with a genuine so-called “anti-realist aporia”, he would seek to turn the tables. He would proceed to deny the anti-realist’s thesis, holding to the common ground and the characteristically realist thesis. It is not our purpose in this paper to adjudicate between such reactions, but rather to examine the range of circumstances in which such adjudication would appropriately be called for. That is, we wish to find the genuine and interesting aporias of the general kind just described.

Note, however, that we have, thus far, only a description of the general kind of aporia that holds our interest. We have not yet produced any concrete examples! This is because the three premises $EU$, $KP$ and $LEM$ of the second reductio given above do not satisfy the bareness requirement. This was established by the first reductio, which involved only two of those premises, namely $EU$ and $KP$.

From the anti-realist’s point of view, $EU$ is an extraordinarily strong statement, involving, as it does, an outright existence claim for a very exigent property. From this we conclude that $EU$ is unsuitable as an expression of what ought to be the common ground between the realist and anti-realist. It behooves us to find a better alternative. Note also that since $EE$ implies $EU$, $EE$ is inconsistent with $KP$; whence any aporia involving $EE$ and $KP$ will fail the bareness requirement.

The search is still on for a philosophically effective aporia, involving all three premises of the kind described above, and satisfying the bareness requirement. It turns out that there is more than one such aporia, and the differences among them are interesting. Trying to reach some understanding of how these differences arise is an independently interesting philosophical exercise.
7. Salerno’s aporia

As already noted, one alternative expression of common ground, suggested by Salerno (Salerno, forthcoming), is the principle $EM$. By moving from $EU$ to $EM$ one is weakening the expression of common ground. Accordingly, in order to complete the sought aporia, the other two premisses $KP$ and $LEM$ have to come in suitably strengthened versions, in order to take up the logical slack. Salerno opts for their $K$-strengthenings $K-KP$ and $K-LEM$ respectively. So the three premisses for his aporia (let us call it $(\alpha)$) are as follows:

\[
\begin{align*}
(1) \quad EM & \quad \neg K \vee \psi \phi \phi \phi & \quad \neg K \vee \phi (\neg K \vee \phi \neg K \phi) & \quad \text{Common ground} \\
(2) \quad K-KP & \quad K \vee \phi \neg \phi \phi & \quad K \vee \phi (\phi \rightarrow K \phi) & \quad \text{Anti-realist thesis} \\
(3) \quad K-LEM & \quad K \vee \phi (\phi \neg \phi) & \quad K \vee \phi \neg \phi & \quad \text{Realist thesis}
\end{align*}
\]

Salerno’s own reductio proof for his aporia $(\alpha)$ can be cast into our natural-deduction format as follows.

\[
\begin{align*}
(1) \quad \forall \phi (\phi \rightarrow \Theta \phi) & \quad \neg p & \quad \neg p \rightarrow \Theta \neg p \\
(2) \quad p \rightarrow \Theta p & \quad \Theta p & \quad \Theta \neg p \\
(3) \quad p \neg p & \quad \Theta p \vee \Theta \neg p \\
(4) \quad p \neg p & \quad \Theta p \vee \Theta \neg p \ \\
(K) \quad K \vee \phi (\psi \phi \neg \phi) & \quad K \vee \phi (\phi \rightarrow \Theta \phi) & \quad \forall \phi (\Theta \phi \rightarrow \Theta \neg \phi) & \quad \neg K \vee \phi (\Theta \phi \neg \phi)
\end{align*}
\]

This establishes

\[
\neg K \vee \phi (\Theta \phi \neg \phi), \ K \vee \phi (\phi \rightarrow \Theta \phi), \ K \vee \phi (\phi \neg \phi) \vdash \bot
\]

Substituting $\Theta K$ for $\Theta$, we obtain

\[
\neg K \vee \phi (\hat{\Theta} K \phi \neg \phi), \ K \vee \phi (\phi \rightarrow \hat{\Theta} K \phi), \ K \vee \phi (\phi \neg \phi) \vdash \bot
\]

which is the Salerno aporia $(\alpha)$ promised above, involving, respectively, Epistemic Modesty, the $K$-strengthening of the Knowability Principle, and the $K$-strengthening of the Law of Excluded Middle. Note that once again the interplay between $\hat{\Theta}$ and $K$ plays no role in precipitating the absurdity.

In trying to make sense of Wright’s dictum via $EM$, Salerno has actually taken us out of the realm of single-sentence arguments. No existential
premiss is involved; accordingly, no parametric sentence $\psi$ is invoked, of supposedly problematic epistemic status, in pursuit of a contradiction.

8. An alternative to Salerno’s aporia

It turns out that one can do even better than the Salerno proof, which applies $(K^+)$. Note how in the last proof $LEM$ was applied in full generality, in so far as the step of $\forall$-elimination on $LEM$ was performed with respect to the arbitrary proposition $p$, which subsequently gave way to a step of $\forall$-introduction immediately before the step of $(K^+)$. One can use, however, a less robustly realist premiss en route to inconsistency than (the $K$-strengthening of) full $LEM$, once given the other two premisses $K$-$KP$ and $EM$. Note first that the bivalence of any sentence $\phi$ justifies classical reductio on $\phi$ (which we shall call $CR(\phi)$):

$$
\begin{array}{c}
\phi \lor \neg \phi \\
\phi \\
\neg \phi
\end{array}
$$

Now let $\phi$ be of the form $\mathcal{D}q$, for parametric proposition $q$. Clearly, the use of $CR[\mathcal{D}q]$ can be justified by the principle of Bivalence of Decidability ($BD$), which is of the form $\forall \phi (\mathcal{D}\phi \lor \neg \mathcal{D}\phi)$. Likewise, the use of $CR[\mathcal{R}q]$ can be justified by the principle of Bivalence of Resolution ($BR$), which is of the form $\forall \phi (\mathcal{R}\phi \lor \neg \mathcal{R}\phi)$.

At this point we offer a word of strategic explanation to the reader. We wish to continue setting out our proofs in full and rigorous detail. Some of them, however, would be too wide to display within a usual page-width. These longer proofs are therefore split up into manageable chunks. Earlier chunks will be able to be embedded into later ones, thereby reducing the sideways spread of each proof display. We also exploit the fact that certain proof-schemata can be provided at a grosser level of logical analysis than is revealed within our various theses. When this is the case, we use the schematic letters $\Theta$ and $\Omega$, for propositional operators governed by both the usual elimination rule and the rule expressing the fact that the operator transmits under deducibility. We are then free to substitute either $K$ or $\Diamond K$ (uniformly) for occurrences of $\Theta$ and of $\Omega$. In this way a single proof-template can spawn quadruplets of aporias. We proceed now to illustrate the method.
The following proof $\Xi$ establishes $\Theta q \lor \Theta \neg q$ from $\forall \phi (\phi \rightarrow \Theta \phi)$ and $CR$ on $(\Theta q \lor \Theta \neg q)$:

$$
\begin{array}{c}
(\Xi) \\
(1) \quad \forall \phi (\phi \rightarrow \Theta \phi) \\
q \rightarrow q \Theta q \Theta q \lor \Theta \neg q \neg (\Theta q \lor \Theta \neg q) \\
\Theta q \lor \Theta \neg q \neg q \quad \forall \phi (\phi \rightarrow \Theta \phi) \\
\neg q \rightarrow \Theta \neg q \\
\Theta \neg q \Theta q \lor \Theta \neg q \neg (\Theta q \lor \Theta \neg q) \\
\bot (CR) \\
\end{array}
$$

Now consider the proof-schema $\Pi$:

$$
\begin{array}{c}
\Pi \\
(1) \quad \forall \phi (\phi \rightarrow \Theta \phi), CR[ \Theta q] \\
\Theta q \lor \Theta \neg q \\
\Omega \forall \phi (\phi \rightarrow \Theta \phi) \forall \phi (\Theta \phi \lor \Theta \neg \phi) (1) \\
\Omega \forall \phi (\Theta \phi \lor \Theta \neg \phi) \neg \Omega \forall \phi (\Theta \phi \lor \Theta \neg \phi) \\
\bot
\end{array}
$$

which establishes the schematic result

$$
\neg \Omega \forall \phi (\Theta \phi \lor \Theta \neg \phi), \Omega \forall \phi (\phi \rightarrow \Theta \phi), \text{CR on } (\Theta q \lor \Theta \neg q) \vdash \bot.
$$

Recall

- $BR$ allows one to derive $CR$ on $(Kq \lor K \neg q)$;
- $BD$ allows one to derive $CR$ on $(\Diamond Kq \lor \Diamond K \neg q)$; and
- $KP \vdash \Diamond K \rightarrow KP$.

We therefore have the following results from the previous schematic deducibility statement, upon the indicated substitutions for $\Omega$ and $\Theta$:
9. Aporias exploiting the existence of quandaries

The proof $\Xi$ above can be embedded in yet another proof-schema as follows:

\[
\begin{align*}
\forall \phi (\phi \to \Theta \phi), CR[\Theta q] \\
\text{(1)}
\end{align*}
\]

\[
\begin{align*}
\forall \phi (\phi \to \Theta \phi) & \qquad \Theta q \land \neg q \\
\neg \Omega (\Theta q \land \neg q) & \qquad \text{(2)}
\end{align*}
\]

\[
\begin{align*}
\exists \psi & \neg \Omega (\Theta \psi \land \neg \psi) \\
\bot & \qquad \text{(2)}
\end{align*}
\]

This establishes the schematic deducibility statement

\[\exists \psi \neg \Omega (\Theta \psi \land \neg \psi), \quad \Omega \forall \phi (\phi \to \Theta \phi), \quad CR \text{ on } (\Theta q \land \neg q) \vdash \bot.\]

Substituting $K$ for $\Omega$ and $\diamond K$ for $\Theta$ we obtain

\[\exists \psi \neg K (\diamond K \psi \land \neg \psi), \quad K \forall \phi (\phi \to \diamond K \phi), \quad CR[\Theta q] \vdash \bot.\]

Since $CR[\Theta q]$ is derivable from $BD$, we have

\[EQ, \quad K\text{-}KP, \quad BD \vdash \bot.\]

This reductio, however, involves the less-than-perfectly-analytic rule $(K^+)$. Substituting $\diamond K$ for both $\Omega$ and $\Theta$ we obtain

\[\exists \psi \neg \diamond K (\diamond K \psi \land \neg \psi), \quad \diamond K \forall \phi (\phi \to \diamond K \phi), \quad CR[\Theta q] \vdash \bot.\]

Now recall that $KP$ implies $\diamond K\text{-}KP$. Hence we have

\[EE, \quad KP, \quad BD \vdash \bot,\]

and indeed by means of the perfectly analytic rule $(\diamond K^+)$. Unfortunately, however, it fails to meet the bareness requirement—since $EE$ is so strong that it is already inconsistent with $KP$ on its own.
We now label as $\Sigma_0$ a chunk of proof that we have encountered before:

\[
\begin{align*}
(\Sigma_0) \quad \frac{\forall \phi (\phi \rightarrow \Theta \phi)}{q} \quad \frac{\forall \phi (\phi \rightarrow \Theta \phi)}{q \rightarrow \Theta q} \quad \frac{\forall \phi (\phi \rightarrow \Theta \phi)}{\neg q \rightarrow \Theta \neg q} \\
\forall \phi (\phi \lor \neg \phi) \quad \Theta q \quad \Theta \neg q \\
\frac{q \lor \neg q}{\Theta q \lor \Theta \neg q}
\end{align*}
\]

We now form the chunk $\Sigma_1$ as follows. Note that the step of $\lozenge$-introduction need only be applied in the case where $\Omega$ is $\lozenge K$:

\[
\begin{align*}
(\Sigma_1) \quad \frac{\forall \phi (\phi \lor \neg \phi), \forall \phi (\phi \rightarrow \Theta \phi)}{\Sigma_0} \\
(\lnot K) \quad \frac{\forall \phi (\phi \lor \neg \phi)}{K \forall \phi (\phi \lor \neg \phi)} \quad \frac{\forall \phi (\phi \rightarrow \Theta \phi)}{K \forall \phi (\phi \rightarrow \Theta \phi)} \quad \Theta q \lor \Theta \neg q \\
\frac{K (\Theta q \lor \Theta \neg q)}{K (\Theta q \lor \Theta \neg q)} \quad (2) \\
\frac{[ \lozenge K (\Theta q \lor \Theta \neg q)] \rightarrow \neg \Omega (\Theta q \lor \Theta \neg q)}{\exists \psi \rightarrow \neg \Omega (\Theta \psi \lor \Theta \neg \psi)} \quad (3) \\
\frac{K \exists \psi \rightarrow \neg \Omega (\Theta \psi \lor \Theta \neg \psi)}{\lor} \quad (3)
\end{align*}
\]

Finally, we embed $\Sigma_1$ within a further schema as follows. Note that this schema contains both an application of the rule ($\lambda$) and an application of the rule ($K \rightarrow$). Also, the first indicated application of $\lozenge$-introduction can

\[
\frac{\lor}{\bot}
\]
be omitted in the case where \(\diamond K\), and the second one can be omitted in the case where \(\Omega\) is \(K\).

\[
\begin{align*}
K^3 \forall \neg\Omega(\Theta \forall \neg\Theta \forall \psi), K^3 \forall \phi(\phi \lor \neg \phi), K^3 \forall \phi(\phi \rightarrow \Theta \phi) &\quad (1) \\
& \vdash \quad ;
\end{align*}
\]

\[
\begin{align*}
\forall \phi(\phi \rightarrow \Theta \phi) &\quad (2) \\
\forall \phi(\phi \rightarrow \Theta \phi) \lor \forall \phi(\phi \rightarrow \Theta \phi) \rightarrow \Theta \forall \phi(\phi \rightarrow \Theta \phi) &\quad \Sigma_1 \\
\Theta \forall \phi(\phi \rightarrow \Theta \phi) &\quad ;
\end{align*}
\]

\[
\begin{align*}
(K^-) &\quad \bot \\
(K^-) \lor \forall \phi(\phi \rightarrow \Theta \phi) &\quad (2) \\
\lbrack\forall \forall \phi(\phi \rightarrow \Theta \phi)\rbrack &\quad \bot (1) \\
\lbrack\forall \forall \phi(\phi \rightarrow \Theta \phi)\rbrack &\quad \bot (1)
\end{align*}
\]

This gives us the schematic deducibility statement

\[
K^3 \forall \neg\Omega(\Theta \forall \psi \lor \neg \psi), \neg\Omega \lor \forall \forall \phi(\phi \rightarrow \Theta \phi), K^3 \forall \forall \phi(\phi \lor \neg \phi) \vdash \bot
\]

Substituting for \(\Omega\) and \(\Theta\) as indicated, we obtain the following instances:

<table>
<thead>
<tr>
<th>(\Omega)</th>
<th>(\Theta)</th>
<th>Rules Used</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K)</td>
<td>(K)</td>
<td>(K)-EP.</td>
<td>(K)-LEM (\vdash \bot)</td>
</tr>
<tr>
<td>(\diamond K)</td>
<td>(K)</td>
<td>(K)-EQ.</td>
<td>(K)-LEM (\vdash \bot)</td>
</tr>
<tr>
<td>(\diamond K)</td>
<td>(K)</td>
<td>(K)-EC.</td>
<td>(K)-LEM (\vdash \bot)</td>
</tr>
<tr>
<td>(\diamond K)</td>
<td>(K)</td>
<td>(K)-EE.</td>
<td>(K)-LEM (\vdash \bot)</td>
</tr>
</tbody>
</table>

10. Summary and Philosophical Assessment

We have established the following aporias, involving a statement of Common Ground (\(CG\)), an Anti-Realist Thesis (\(AR\)), and a target Realist Thesis (\(R\)) for reductio:

<table>
<thead>
<tr>
<th>CG</th>
<th>AR</th>
<th>R</th>
<th>Rules Used</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>(EM)</td>
<td>(K)-KP</td>
<td>(K)-LEM</td>
<td>(K)-</td>
<td>Salerno’s ((\alpha))</td>
</tr>
<tr>
<td>(EQ)</td>
<td>(K)-KP</td>
<td>(K)-LEM</td>
<td>(K)-</td>
<td>Wright; but EQ (\vdash) EM</td>
</tr>
</tbody>
</table>
The last four results represent the outcome of our best efforts to exploit Wright’s notion of a quandary (and closely related notions that he did not consider, such as the notions of an unsolved problem, an enigma, or a conundrum) in pursuit of a more telling anti-realist aporia than Salerno’s (α). What Wright wanted was a crucial weakening of the usual anti-realist premiss \( KP \), seeking to maintain in its stead the more modest \( \neg KP \), or even \( \neg \Box KP \). But it emerges that the price of such a weakening is, apparently, forced resort to the more substantive, and perhaps even epistemologically contentious, inferential rules \((K/H)\), \((K/\neg)\) and \((\lambda)\). Their net effect is to idealize the notion of knowledge so as to require it to be closed under deducibility, and also to take one perilously close to presupposing the controversial KK-thesis (that if one knows a proposition, then one knows that one knows it). We do not claim to have shown definitively that a quandary-based aporia must involve recourse to such controversial inferential principles; but we present it as a challenge to the champion of quandaries to show that this is not so.

The fourth aporia in the master list above marks a slight improvement on (α) in so far as the target realist thesis is the weaker \( BD \) rather than \( LEM \). This is the only improvement that we have found which involves \( EM \) as the expression of common ground.

It is rather interesting that \( BD \) expresses a kind of second-order realism, maintaining only that questions about decidability are themselves decidable. This does not entail that all propositions are decidable; that stronger claim is \( LEM \) itself.

Overall, the best two alternatives to the \( EM \)-involving aporias would appear to be those involving \( IKD \) in its stead:

\[
\begin{array}{cccc}
IKD & KP & LEM & \Box K^+ \\
EM & K-KP & BD & K^+ \\
IKD & KP & BD & \Box K^+ \\
\neg K-C\bar{K} & K-O & BR & K^+ \\
\neg \Box K-C\bar{K} & \Box K-O & BR & K^+ \\
E\bar{Q} & K-KP & BD & K^+ \\
E\bar{E} & KP & BD & \Box K^+ \\
\end{array}
\]

Better than (α) on \( AR, R, \) rules
Better than (α) on \( R \)
Best result of all
\( AR \) much too strong
\( AR \) still too strong
\( AR \) perhaps too strong
Fails the bareness requirement

\[
\begin{array}{cccc}
K-EP & \neg K-\neg O & K-LEM & K^+ , K-\neg, \lambda \\
K-E\bar{Q} & \neg K-\neg KP & K-LEM & K^+ , K-\neg, \lambda \\
K-EC & \neg \Box K-\neg O & K-LEM & K^+ , K-\neg, \lambda \\
K-EE & \neg \Box K-\neg KP & K-LEM & K^+ , K-\neg, \lambda \\
\end{array}
\]

\( AR \) too strong
Wright
\( AR \) too strong
CG too strong
Both these aporias avoid the use of any of the controversial inferential rules just mentioned. They use only the analytic rule ($\Diamond K^+$) (as far as transmissibility under deducibility is concerned). Secondly, they still succeed (via IKD) in explicating rather nicely Wright’s dictum about the a priori unwarrantability of what is, in effect, a statement of Gödelian Optimism. (The modal suffix would appear to call for our modal prefix $\Diamond$ in an accurate regimentation of the dictum.) Finally, $BD$ is but a special case of $LEM$; so the second of these aporias targets an even narrower realist claim than does the first.

The upshot of all these considerations is that the second of these two aporias ($IKD, KP, BD \vdash \bot$ [using ($\Diamond K^+$)]) would appear to be the best overall alternative to any $EM$-involving aporia for the anti-realist. Note, however, that the reductio involved for this aporia is not a single-sentence argument, as that notion was defined above. For $IKD$ (like $EM$) is not an existential premiss; nor, for the anti-realist, would it entail one. $IKD$, as remarked above, is slightly stronger than $EM$. And that is why one can get by with just $KP$ and $BD$ alongside $IKD$ for this aporia.

By opting for the weaker expression $EM$ of common ground, Salerno resorted, in (α), to the $K$-strengthenings of $KP$ and of $LEM$. We have managed, in the fourth aporia on the list above, to weaken $K-LEM$ down to $BD$. But that continues to impose upon the user of $EM$ the apparent need to use the more substantive inferential rule ($K^+$) rather than the perfectly analytic rule ($\Diamond K^+$). Once again, we offer no proof that resort to the stronger rule ($K^+$) is necessary; we are content to leave it as a challenge to the champion of $EM$ to show that this is not so.

The champion of $EM$ might be justified in complaining that we have been too hasty in saddling him with apparent commitment to the use of non-analytic rules of inference—specifically, the use of ($K^+$). Surely, the objection would go, it is analytic that a proposition that is known to follow, logically, from propositions that one knows, is itself known? Thus, in the proof of Salerno’s aporia, note that a sub-proof establishes that $GO$ follows logically from $KP$ and $LEM$. Thus, the contention would be, the premisses $K-KP$ and $K-LEM$ logically imply $K-GO$. And this last inference is perfectly analytic. This line of argument would appear most convincing if the notion of knowledge at issue were that of “the best of our pooled, communal, knowledge”. For in that case the sub-proof has been made available to the community, and those equipped to reflect on it would agree with the subsequent “analytic” inference to the overall result $K-KP, K-LEM \vdash K-GO$. But if the reading of $K$ were to be restricted
as envisaged at the outset of these investigations—so as to represent the knowledge of an individual knower—then it seems that it is a substantive further assumption, and one hard to disclose in formal terms, that the individual knower (about whom the philosophical logician is reasoning) is committed to knowing that the inference from $KP$ and $LEM$ to $GO$ is valid. At the very best, then, it would appear that the Salerno aporia ($\alpha$) is based on some “at least weakly substantive” rule governing $K$. And to that extent, any alternative aporia using only the perfectly analytic rule ($\Diamond K^+$) can claim a certain philosophical edge in the final assessment.

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REFERENCES
Salerno, J. Forthcoming: “Revising the Logic of Logical Revision”, Philosophical Studies, 99, 2.
Wright, C. Unpublished: “Intuitionism and Indeterminacy”.

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