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A GENERAL THEORY OF ABSTRACTION OPERATORS

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I present a general theory of abstraction operators which treats them as variable-binding term-forming operators, and provides a reasonably uniform treatment for definite descriptions, set abstracts, natural number abstraction, and real number abstraction. This minimizing, extensional and relational theory reveals a striking similarity between definite descriptions and set abstracts, and provides a clear rationale for the claim that there is a logic of sets (which is ontologically non-committal). The theory also treats both natural and real numbers as answering to a two-fold process of abstraction. The first step, of conceptual abstraction, yields the object occupying a particular position within an ordering of a certain kind. The second step, of objectual abstraction, yields the number sui generis, as the position itself within any ordering of the kind in question.

I. INTRODUCTION

Philosophers often advance claims about logical form in order to read off ontological consequences. For example, the asymmetric symbolization $\phi(t)$ for a primitive predication has allowed some to maintain that whereas the singular t might denote an object, nevertheless the predicate term ϕ does not do so. Parsimonious ontologists can have their particulars without being lumbered with any universals. Then there was Russell's attempt to show that definite descriptive phrases of the form 'the ψ' are not really singular terms. Any sentence apparently of the form 'ϕ(the ψ)' is to be reanalysed, according to Russell's theory of descriptions, as having the logical form $\exists x(\forall y(x = y \leftrightarrow \psi(y)) \wedge \phi(x))$. Building on this theory, Quine proposed to analyse all names as definite descriptions involving a predicate created from the name; whence 'Pegasus flies' would be rendered as $\exists x(\forall y(x = y \leftrightarrow \text{Pegasizes}(y)) \wedge \text{Flies}(x))$, and would be false, for want of an individual that Pegasizes. With the burden of existential commitment thus shifted from singular terms to quantifiers, Quine further proposed that one could work out what a theory says there is by looking at just those claims of the form $\exists x\phi(x)$ that follow from the theory.

In the light of these well known moves, there has been an interesting recent development. Crispin Wright and other neo-logicists (most notably Bob Hale) have proposed that we should be able to diagnose commitment to abstract objects from the uses we make of certain contextually introduced *singular terms*, rather than quantifiers – and singular terms, indeed, that are created by variable-binding, in exactly the same way as definite descriptions.

These new existentially committal singular terms are introduced into our discourse by certain abstraction principles. The recent resurgence of interest in neo-logicism about numbers has focused on that form of abstraction principle of which the best known example is Hume's principle:

$$\#xF(x) = \#xG(x) \leftrightarrow \exists R(\text{R maps the Fs 1–1 onto the Gs}).$$

The right-hand side states that the Fs are *equinumerous* with the Gs. It is a notion which, despite the Latin etymology, has nothing to do with numbers as such. As the well known example of salad plates and forks shows, one can verify the right-hand side without counting.

According to the Wrightian neo-logicist, this principle essentially provides (on its left-hand side) a new way of 'carving up the content' expressed on its right-hand side. The equinumerosity of any two concepts F and G must be allowed to be reanalysed as the identity of the numbers $\#xF(x)$ and $\#xG(x)$ respectively numbering those concepts. The concepts F and G need not themselves apply to any numbers. The Fs and the Gs could be ordinary physical objects acceptable to the nominalist. Yet the suggestion is that the very availability of the method of numerical abstraction furnished by Hume's principle reveals that numbers, too, are objects, but *abstract* objects, to which our newly extended discourse commits us.

Our discourse is extended so as to contain the variable-binding term-forming abstraction operator #; and Hume's principle is seen as an implicit definition of that operator. The condition of equinumerosity ensures that the abstractive terms must refer to numbers. Moreover, the abstractness of the newly recognized numbers is not due to their being referred to by the newly introduced 'abstractive' terms. Rather, the abstractness of these numbers stems from the fact that there must be infinitely many of them, even if (as is entirely possible) there are only finitely many physical objects. Since there are infinitely many numbers, and we have strong reasons to take them as being all of the same kind, and since it is necessary that it be possible for all but finitely many of them to be abstract, it follows that they are all abstract.

Hume's principle exhibits the general form taken as canonical by neo-Fregeans following Wright's lead. It involves, on the right-hand side of the biconditional, an equivalence relation between concepts; and on the left-hand side there are *two* abstraction terms flanking the identity sign.

II. THE LEADING IDEA

This paper reverses the reading-off of abstract existents from the behaviour of singular terms in abstraction principles. Moreover, the abstraction principles proposed here are of a quite different form. They do not involve equivalence relations on the right-hand side of a biconditional, with two abstraction terms in an identity on the left. In fact they are not biconditionals at all; rather they take the form of introduction and elimination rules, respectively for conclusions and major premises of the form

$$t = \alpha_{R} x \Phi(x)$$

where t is a place-holder for *any* singular term, R is a binary relation presumed given, Φ is the concept on which we are abstracting with respect to R, and α is the variable-binding abstraction operator used for that purpose. It is important to stress at the outset the very different general logical form of these abstraction principles, compared with those that have become widely entrenched in recent neo-Fregean discussions. But the approach proposed here can claim strong textual inspiration in Frege's own work – of which more anon.

I began this section by mentioning a reversal of readings. I suggest that ontological questions and questions of logical form are to be answered from within a general reflective equilibrium that can be struck by paying particular attention to considerations of *inferential uniformity* in our systematization of the behaviour of variable-binding term-forming operators (so-called *vbtos*).

Indeed, the position to be argued for is not exactly a reversal of the direction of thought described above, for, as already indicated, the very abstraction principles themselves are going to be significantly reformulated, in order to bring them all into a certain canonical form.[1] The leading idea is that one can begin with certain ontological commitments expressed clearly and up front, as it were, and use these to fashion rules of inference that handle abstractive *vbtos* in an illuminating and reasonably uniform way. Any departure from complete uniformity will be seen to derive from the complications *der Sache selbst*.

One might call the position to be developed here *abstractionist realism*. My aim is to give the broad idea of a treatment of abstraction operators that begins with a realist view about the objects involved, and seeks only to

[1] See the discussion of the desiderata for an inferential theory of meaning for logical operators in my *The Taming of the True* (Oxford UP, 1997), ch. 10.

clarify the logical forms of, and canonical inferences using, sentences containing terms that refer to them. The interest of the treatment lies in its unification of hitherto disparate abstraction phenomena.

Compared with Fregean logicism, the present treatment can be thought of as lying at the other extreme of a Euthyphronic contrast in abstract ontology. The Fregean seeks to show how abstract objects arise in response to our abstractions. These abstractions can be understood as the formation of the singular terms whose role it is to denote the abstract objects involved. Because a certain propositional content is analysed in a certain way, as involving these singular terms, abstract objects are generated, or brought into existence, as the bearers of those terms. They exist, one might say, only because we are prepared to think and speak about them in certain ways.

Against this kind of linguistic idealism one can oppose the present view. On this view, the abstract objects are not brought into existence by us. They do not spring up in response to our probings; rather, our probings reach out to them, seeking to represent them clearly. The objects themselves are independent of our conceptualizations of them, even if facts concerning them cannot outstrip our means of coming to know that they obtain. We need to distinguish clearly realism in ontology from realism about truth-value, or semantic realism. One can be an ontological realist and at the same time a semantic anti-realist.[2]

These introductory remarks, by way of foreshadowing, will be kept fairly brief, so that I can revisit an alternative inference-based neo-logicist approach which in my *Anti-Realism and Logic* (hereafter *ARL*) I called *constructive logicism*.[3] It is this latter approach that provides the real point of departure for the general ideas presented here. I shall also revisit Frege's *Grundgesetze* for the idea underlying what I shall call the relational, extensional, minimizing theory of logico-mathematical abstraction operators.[4]

III. NAÏVE COMPREHENSION AND UNFREE LOGIC

Curiously, the stress which Frege places in *Grundlagen* on the importance of Hume's principle (that two concepts have the same number if and only if they are equinumerous) is dissipated in *Grundgesetze*, where the two halves of the biconditional appear widely separated: in §53 Frege proves that if two concepts correspond 1–1, then their numbers are identical, and in §69 he proves the converse. But nowhere in *Grundgesetze* does he reassemble the

[2] See *The Taming of the True*, pp. 19–20.

[3] N. Tennant, *Anti-Realism and Logic: Truth as Eternal* (Oxford: Clarendon Library of Logic and Philosophy, 1987).

[4] Frege, *Grundgesetze der Arithmetik*, Band 1 (1893) (Hildesheim: Georg Olms, 1962).

biconditional or accord it prime philosophical importance. Had he done so, he would almost certainly have become the first neo-Fregean in response to Russell's paradox, rather than despairingly mourning, as he did in his reply to Russell, that

> ... with the loss of my rule V, not only the foundations of arithmetic, but also the sole possible foundations of arithmetic, seem to vanish.[5]

Frege's diagnostic lead has been followed ever since: Russell's paradox is said to arise from Frege's basic law V. This is an abstraction principle of the same form as Hume's principle (indeed, Hume's principle was fashioned after it), but with a simpler kind of equivalence between F and G on the right-hand side:

Basic law V. $\grave{\varepsilon}F(\varepsilon) = \grave{\varepsilon}G(\varepsilon) \leftrightarrow \forall x(F(x) \leftrightarrow G(x))$.

Here the abstractive terms stand for the extensions, or *Werthverläufe*, of their embedded concepts. (Nowadays one would write $\{x|F(x)\}$ in place of $\grave{\varepsilon}F(\varepsilon)$. The *spiritus lenis* – the apostrophe above the initial occurrence of ε – is the actual *vbto*. The variable being bound is ε.)

It is simplistic, however, to attribute all the fault to basic law V. Frege was, after all, committed to having a denotation for every well-formed name in his formal language. Closer analysis of the source of Russell's paradox reveals that Frege's underlying logical assumptions are just as much to blame as basic law V.

For whatever the formula G, the well-formed abstractive term $\grave{\varepsilon}G(\varepsilon)$ was supposed to denote. That is to say, it was supposed to have a *Bedeutung*. Frege himself imposed the further requirement that in order to bear this out, one had to be in a position to determine, of any identity of the form '$\xi = \grave{\varepsilon}G(\varepsilon)$', whether it was true, provided only that the place marked by ξ was occupied by a well-formed name (see the famous discussion at §31 of Vol. 1 of *Grundgesetze*). This led immediately to the need to specify truth-conditions for identities of the form '$\grave{\varepsilon}F(\varepsilon) = \grave{\varepsilon}G(\varepsilon)$' – a need which Frege (mistakenly, as it happened) thought could be satisfied by basic law V.

Frege could not, however, have fixed the problem simply by restating basic law V with a more exigent right-hand side. Or at least, in order to do so in this way, he would *also* have had to abandon the naïve underlying assumption, to which he was committed, to the effect that every well-formed name would have a denotation. Suppose, for argument's sake, that Frege had not imposed the further requirement mentioned above, on the

[5] J. van Heijenoort, *From Frege to Gödel: a Source Book in Mathematical Logic, 1879–1931* (Harvard UP, 1967), pp. 127–8. For further discussion, see N. Tennant, review essay on Bob Hale and Crispin Wright's *The Reason's Proper Study*, *Philosophia Mathematica*, 11 (2003), pp. 226–41.

assignment of *Bedeutungen* to well-formed names. It would then have been simply a matter of his background logic (or of the referential semantics for his formal language) that, whatever formula $F(x)$ one might take,

$$\exists y(y = \dot{\varepsilon}F(\varepsilon)).$$

By taking $F(x)$ to be $x \notin x$, one would obtain Russell's paradox from the resulting instance

$$(\rho) \quad \exists y(y = \dot{\varepsilon}(\varepsilon \notin \varepsilon))$$

provided only that one could make the inferential steps

$$\frac{t \in \dot{\varepsilon}F(\varepsilon)}{F(t)} \quad \text{and} \quad \frac{F(t)}{t \in \dot{\varepsilon}F(\varepsilon)}.$$

In *this* derivation of Russell's paradox no use would have been made of basic law V. The latter principle, however, could always be said to be lurking problematically in the logical shadows, since one can derive the fated (ρ) from basic law V *within free logic itself*. (See the proof in free logic given below.) In free logic, one makes one's commitments explicit by using existential claims of the form

$$\exists!t =_{df} \exists x(x = t)$$

along with the appropriate modifications of the quantifier rules. The free logic that reflects the most 'robust sense of reality' is based on a Russellian conception of truth-conditions of atomic statements. On this conception, each term in an atomic statement must denote in order for the statement to be true. The free logic in question therefore contains the so-called *rule of atomic denotation*, for atomic statements $A(t)$, and the *rule of functional denotation*[6]

$$\frac{A(t)}{\exists!t} \quad \text{and} \quad \frac{\exists!f(t)}{\exists!t}.$$

A self-identity of the form $t = t$ is a special case of an atomic statement. Hence the step

$$\frac{t = t}{\exists!t}$$

is an application of the rule of atomic denotation. Such a step occurs as the penultimate step of the following proof, in Russellian free logic, of (ρ) from basic law V:

[6] See my *Natural Logic* (Edinburgh UP, 1978), ch. 7, for a detailed development of the free logic in question.

$$\dfrac{\dfrac{\dfrac{\dfrac{\grave{\varepsilon}F(\varepsilon) = \grave{\varepsilon}G(\varepsilon) \leftrightarrow \forall x(F(x) \leftrightarrow G(x))}{\grave{\varepsilon}F(\varepsilon) = \grave{\varepsilon}F(\varepsilon) \leftrightarrow \forall x(F(x) \leftrightarrow F(x))} \qquad \begin{array}{c}\text{Logic:}\\ \forall x(F(x) \leftrightarrow F(x))\end{array}}{\grave{\varepsilon}F(\varepsilon) = \grave{\varepsilon}F(\varepsilon)}}{\exists y(y = \grave{\varepsilon}F(\varepsilon))}}{\exists y(y = \grave{\varepsilon}(\varepsilon \notin \varepsilon))}\, F(x)/x \notin x$$

The moral of the story, then, is that no response to Russell's paradox can be based on a mere modification of basic law V (typically, by strengthening its right-hand side) without at the same time fundamentally overhauling Frege's logical preconceptions about well-formed singular terms having denotations. One needs both to avoid naïve comprehension and to adopt a free logic – at least, for any language in which $x \in x$ is to count as well-formed. (I am indebted here to Philip Ebert. A Russellian type-theory need not use free logic; but $x \in x$ is not in the language of such a theory.) An unfree logic visits naïve comprehension upon one, even in the absence of basic law V, if the abstractive terms for extensions are syntactically primitive. And basic law V, if left unmodified, leads to inconsistency even within (Russellian) free logic.

It is therefore surprising to find that Wright, who advocated Hume's principle as a Fregean abstraction principle *par excellence*, and who did not stratify his language in type-theoretic fashion, did not also adopt a free logic, even when giving up basic law V altogether.[7] Hence the 'bad company' objection, raised in *ARL* (p. 236), focusing on Wright's 'universal number' $\#x(x = x)$.[8]

IV. CONSTRUCTIVE LOGICISM

I proposed an alternative neo-Fregean approach to the foundations of arithmetic, called constructive logicism, in *ARL*. There my aim was to provide 'meaning-constituting' introduction rules, and harmoniously balancing 'meaning-explicating' rules, for the primitive expressions 0, #, *s* and *N*. This was in the spirit of Dummettian anti-realism, with its stress on rules of these kinds as the only source of meanings on whose basis one's logic could be justified as analytic. The innovation in *ARL* was to suggest that introduction and elimination rules could be provided for (singular) *term*-forming operators, such as $\#x\Phi(x)$, by taking as the canonical form, both of a conclusion

[7] See I. Rumfitt, 'Singular Terms and Arithmetical Logicism', *Philosophical Books*, 44 (2003), pp. 193–219, for a more extensive development of this criticism.

[8] See also G. Boolos, 'Is Hume's Principle Analytic?', in R. Heck (ed.), *Language, Thought, and Logic: Essays in Honour of Michael Dummett* (Oxford: Clarendon Press, 1997), pp. 245–61.

of an introduction rule and of the major premise of the corresponding elimination rule, a generalized identity of the form

$$t = \alpha x F(x)$$

where α was the *vbto* in question. This in itself is very much in a Fregean spirit, given Frege's concerns in §31 of Vol. 1 of *Grundgesetze* ('... so ist also die Frage, ob "$\xi = [\alpha x \Phi(x)]$" ein bedeutungsvoller Name einer Funktion erster Stufe mit einem Argumente sei ...').

It was important, for the development of constructive logicism in *ARL*, that one carried out one's formal derivations within a *free* logic, uncommitted to the existence of denotations for well-formed terms unless such commitment was explicitly incurred. This allowed a much more constructive 'bottom-up' development, incurring commitment first to the number 0, and thereafter, successively, to each non-zero natural number.

The treatment, moreover, was not only ontologically constructive, but also logically constructive. That is to say, it avoided any use of strictly classical rules of inference. In the proofs given of the Peano–Dedekind axioms for successor arithmetic, no use was made of the law of excluded middle, or of any of its equivalents, such as the rule of double-negation elimination, the rule of classical *reductio ad absurdum* or the rule of constructive dilemma.

The constructive logicist seeks not to define 0 and s, but to capture their meanings by direct stipulation. By means of 0 and s one can build up all the numerals – *terms* of the form $s \ldots s0$. The numeral $\underline{n}$ for the number n has n occurrences of s. These are the only number-terms in the language of Peano–Dedekind (successor) arithmetic. But number-talk, in application to other subject-matters, calls for the numbering of concepts. We are interested in saying how many Fs there are. To this end, abstractive terms of the form $\#xF(x)$ are introduced. The problem then is to link the correct intended use of the abstractive terms in *applied* arithmetic with the use of numerals in *pure* arithmetic. This was what I sought to do in *ARL*. Given the purpose just described, it was clear that an adequacy condition could be framed for the resulting theory, as follows. Whatever the sortal predicate F, and whatever the natural number n, the theory should prove the equivalence of the two sentences $\#xF(x) = \underline{n}$ and $\exists_n xF(x)$. The latter sentence, $\exists_n xF(x)$, to the effect that there are exactly n Fs, can be expressed in the well known way using only first-order logical resources. It involves no reference to or quantification over numbers (unless of course F itself is a numerical predicate). One has to realize that the subscript n is a meta-notational convenience. In the case where $n = 2$, for example, the sentence $\exists_n xF(x)$ is the sentence

$$\exists x \exists y (x \neq y \wedge Fx \wedge Fy \wedge \forall z (Fz \rightarrow (z = x \vee z = y))).$$

The numerical references are all confined to the sentence $\#x\mathrm{F}(x) = \underline{n}$; the numerical term on its left-hand side is abstractive, whereas the numerical term on its right-hand side is a numeral in the language of pure arithmetic.

An improved statement of the rules proposed in *ARL* is as follows:

0-introduction
$$\dfrac{\begin{array}{c}\underbrace{{}^{(i)}\overline{\mathrm{F}(a)} \quad , \quad \overline{\exists! a}^{(i)}} \\ \vdots \\ \bot\end{array}}{0 = \#x\mathrm{F}(x)}\,(i)$$

0-elimination
$$\dfrac{0 = \#x\mathrm{F}(x) \qquad \exists! t \qquad \mathrm{F}(t)}{\bot}$$

#-introduction
$$\dfrac{\#x\mathrm{F}x = t \qquad \mathrm{R}xy[\mathrm{F}x \text{ 1–1 } \mathrm{G}y]}{\#x\mathrm{G}x = t}$$

(Here the condition $\mathrm{R}xy[\mathrm{F}x$ 1–1 $\mathrm{G}y]$ is that R effects a 1–1 correspondence of the Fs with the Gs. Purely logical rules were provided in *ARL*, pp. 276–81, for inferring to and from claims of this form.)

#-elimination
$$\dfrac{\#x\mathrm{G}x = t \qquad \begin{array}{c}\underbrace{{}^{(i)}\overline{\#x\mathrm{F}x = t} \quad , \quad \overline{\mathrm{R}xy[\mathrm{F}x \text{ 1–1 } \mathrm{G}y]}^{(i)}} \\ \vdots \\ B\end{array}}{B}\,(i) \qquad \text{F, R parametric}$$

s-introduction
$$\dfrac{\#x\mathrm{F}x = t \qquad \mathrm{R}xy[\mathrm{F}x \text{ 1–1 } \mathrm{G}y, r]}{\#x\mathrm{G}x = st}$$

(Here the condition $\mathrm{R}xy[\mathrm{F}x$ 1–1 $\mathrm{G}y, r]$ is that R effects a 1–1 correspondence of the Fs with all the Gs except r. Purely logical rules were provided in *ARL*, pp. 276–81, for inferring to and from claims of this form.)

s-elimination (first half)
$$\dfrac{\#x\mathrm{G}x = st \qquad \begin{array}{c}\underbrace{{}^{(i)}\overline{\#x\mathrm{F}x = t} \quad , \quad \overline{\mathrm{R}xy[\mathrm{F}x \text{ 1–1 } \mathrm{G}y, a]}^{(i)}} \\ \vdots \\ B\end{array}}{B}\,(i) \qquad a\text{, F, G, R parametric}$$

s-elimination (second half)
$$\dfrac{u = st \qquad \begin{array}{c}\overline{u = \#xHx}^{(i)} \\ \vdots \\ B\end{array}}{B}\,(i) \qquad \text{H parametric}$$

The second half of the rule of s-elimination in effect says that terms with s dominant can only denote objects within the range of denotations of

#-terms. In other words, such terms can be used for *counting*. The treatment in *ARL* did not involve this last rule. But *ARL* contained another rule, called 'the ratchet principle', as well as another form of 'elimination rule' for s which I claimed could be justified by invoking the ratchet principle (see *ARL*, pp. 291–2). The ratchet principle, however, is redundant.[9] It can be derived from the introduction rule for s. And the justification of the other form of 'elimination rule' for s just mentioned was limited to those uses of the rule where its major premise, of the general form $u = st$, were in fact of the more restricted form $\#x\mathrm{K}x = st$. This restriction turns out to be no restriction at all, provided only that we lay down the second half of s-elimination as I have done above. For this guarantees that any term with s dominant will in fact be co-referential with some term of the form $\#x\mathrm{K}x$.

Although the constructive logicist takes 0 and s as linguistically primitive, they are not assumed to be governed by the Peano–Dedekind axioms for pure arithmetic. That 0 and s do indeed satisfy the Peano–Dedekind axioms is what has to be *established* (constructively) by the constructive logicist, to which end I have laid down the rules above. They afford constructive proofs of the Peano–Dedekind axioms, despite the fact that these axioms contain no occurrences of #, precisely because of the way they yoke the meanings of 0 and s to that of #.

V. THE GENERAL PATTERN OF RULES FOR ABSTRACTION OPERATORS

I shall now lay out an alternative account of abstraction operators, which will include the operator # as a special case, but which will treat # in a manner somewhat different from the constructive logicist approach outlined above. A great attraction of the new treatment, however, is that it is uniform across the natural and the real numbers. Moreover, the rules governing # in these cases bear a striking structural similarity to those governing definite descriptions and set abstractions.

V.1. *The germ of the idea in Frege*

The germ of the proof-theoretic idea to be developed here is to be found in Vol. 1 of Frege's *Grundgesetze*, at p. 3:

> Begriff und Beziehung sind die Grundsteine, auf denen ich mein Bau aufführe. ['Concept and relation are the basic foundation stones on which I erect my structure': my translation.]

[9] See Rumfitt, 'Frege's Logicism', *Proceedings of the Aristotelian Society*, 73 (1999), pp. 151–80.

Clearly Frege means here by '*Grundsteine*' something like 'basic conceptual ingredients or building blocks' rather than 'basic or axiomatic truths'.

I quoted earlier also from §31:

> ... so ist also die Frage, ob '$\xi = \grave{\varepsilon}\mathbf{\Phi}(\varepsilon)$' ein bedeutungsvoller Name einer Funktion erster Stufe mit einem Argumente sei.... ['... thus the question is whether "$\xi = \grave{\varepsilon}\mathbf{\Phi}(\varepsilon)$" is a meaningful name of a function of the first level with one argument.']

The most general form of the kind of identity mentioned in the last-quoted passage would be '$t = \alpha x\mathbf{\Phi}(x)$', where t is any singular term (presumed to be understood), α is an abstraction operator, and $\mathbf{\Phi}$ is a concept or *Begriff* (one of Frege's two '*Grundsteine*'). For the anti-realist proof-theorist, the challenge can be construed as that of finding an introduction rule for '$t = \alpha x\mathbf{\Phi}(x)$' by exploiting as given some relation R between t and other objects. The relation [*Beziehung*] R would be the second of Frege's two '*Grundsteine*'. The dependency on R could well be explicitly acknowledged by having R as a subscript on every occurrence of the abstraction operator α, thus:

$$\alpha_{\mathrm{R}}x\mathbf{\Phi}(x).$$

Indeed, the subscript reminding us of the dependency of α on a relation R would be more informative if it told us what was essential about R for it to be able to feature in this way. Thus in general the subscript could take the form $\Theta(\mathrm{R})$, where Θ is a (possibly higher-order) specification of conditions on R:

$$\alpha_{\Theta(\mathrm{R})}x\mathbf{\Phi}(x).$$

I shall, however, take the liberty of suppressing such subscripts, though they are always implicitly there. My general abstractive term is not Kit Fine's '$t\mathrm{Abst}_{\mathrm{R}}C$', since his R is second-order.[10] My method differs also from Wright's use of Hume's principle, both because the general form of the identity claim on which I am focusing involves only one salient occurrence of the abstraction operator (rather than two, as is the case with Hume's principle), and because I am dealing with abstraction operators in general, and not just with the numerical abstraction operator #.

We must not lose sight of the fact, however, that the *Beziehung* R is every bit as important as the *Begriff* $\mathbf{\Phi}$ when it comes to abstracting by means of α. We should think always of abstracting *on* the concept $\mathbf{\Phi}$ *with respect to* the relation R. That is why I call the resulting theory a *relational* theory of abstraction. (I shall show presently why it is also to be called an *extensional* theory, as well as a *minimizing* one.)

[10] See K. Fine, 'The Limits of Abstraction', in M. Schirn (ed.), *Philosophy of Mathematics Today* (Oxford: Clarendon Press, 1998), pp. 503–629.

V.2. *Introduction and elimination rules*

Without further ado, I present the general rules for the introduction and elimination of an abstraction operator α.

α-introduction

$$\dfrac{\begin{array}{c} \underbrace{{}^{(i)}\overline{\Phi(a)} \quad , \quad \overline{\exists! a}^{(i)}} \\ \vdots \\ \mathrm{R}at \end{array} \qquad \exists! t \qquad \begin{array}{c} \overline{\mathrm{R}at}^{(i)} \\ \vdots \\ \Phi(a) \end{array}}{t = \alpha x \Phi(x)}{\scriptstyle (i)}$$

α-elimination 1

$$\dfrac{t = \alpha x \Phi(x) \qquad \Phi(u) \qquad \exists! u}{\mathrm{R}ut}$$

α-elimination 2

$$\dfrac{t = \alpha x \Phi(x)}{\exists! t}$$

α-elimination 3

$$\dfrac{t = \alpha x \Phi(x) \qquad \mathrm{R}ut}{\Phi(u)}$$

Clearly, the rule α-elimination 2 is a special case of the rule of atomic denotation, where the atomic sentence $\mathrm{A}(t)$ is the identity $t = \alpha x \Phi(x)$. I state the rule α-elimination 2 separately, however, as a reminder that elimination rules do, after all, simply unpack the import of the corresponding subordinate subproofs called for in the introduction rule. The rule α-elimination 2 corresponds to the second subordinate subproof in the introduction rule, which is the requirement to establish (among other things) $\exists! t$ before using the introduction rule to infer $t = \alpha x \Phi(x)$. One cannot, as it were, get existence out, unless one has had to put existence in.

The rule α-elimination 1 (resp., 3) corresponds to the first (resp., third) subordinate subproof of the introduction rule, and tells us that we can infer its conclusion from its premises, since that deductive connection must have been established in order to infer $t = \alpha x \Phi(x)$ by means of the introduction rule.

V.3. *Three important results proved by means of the rules*

Abstraction theorem 1

$$\dfrac{{}^{(1)}\overline{\mathrm{R}at} \qquad \exists! t \qquad \overline{\mathrm{R}at}^{(1)}}{t = \alpha x \mathrm{R}xt}{\scriptstyle (1)}$$

Provided it exists, 't is the α of all things bearing R to it'.

Abstraction theorem 2

$$
\begin{prooftree}
\AxiomC{$\overline{\mathrm{R}ad}^{\,(1)}$}
\UnaryInfC{$\exists!a$}
\AxiomC{$\overline{\forall z(\mathrm{R}zc \leftrightarrow \mathrm{R}zd)}^{\,(2)}$}
\BinaryInfC{$\mathrm{R}ac \leftrightarrow \mathrm{R}ad$}
\AxiomC{$\overline{\mathrm{R}ad}^{\,(1)}$}
\BinaryInfC{$\mathrm{R}ac$}
\AxiomC{$\overline{\exists!c}^{\,(4)}$}
\AxiomC{$\overline{\mathrm{R}ac}^{\,(1)}$}
\UnaryInfC{$\exists!a$}
\AxiomC{$\overline{\forall z(\mathrm{R}zc \leftrightarrow \mathrm{R}zd)}^{\,(2)}$}
\BinaryInfC{$\mathrm{R}ac \leftrightarrow \mathrm{R}ad$}
\AxiomC{$\overline{\mathrm{R}ac}^{\,(1)}$}
\BinaryInfC{$\mathrm{R}ad$}
\RightLabel{(1)}
\TrinaryInfC{$c = \alpha x(\mathrm{R}xd)$}
\AxiomC{$\overline{\exists!d}^{\,(3)}$}
\RightLabel{(*Th.* 1)}
\UnaryInfC{$d = \alpha x(\mathrm{R}xd)$}
\BinaryInfC{$c = d$}
\RightLabel{(2)}
\UnaryInfC{$\forall z(\mathrm{R}zc \leftrightarrow \mathrm{R}zd) \rightarrow c = d$}
\RightLabel{(3)}
\UnaryInfC{$\forall y(\forall z(\mathrm{R}zc \leftrightarrow \mathrm{R}zy) \rightarrow c = y)$}
\RightLabel{(4)}
\UnaryInfC{$\forall x\forall y(\forall z(\mathrm{R}zx \leftrightarrow \mathrm{R}zy) \rightarrow x = y)$}
\end{prooftree}
$$

Those things are identical that are borne R by exactly the same things. This is a consequence, according to the rules, of R's being eligible to sustain an abstraction (via α). Hence it is a necessary condition on R's being so eligible that R must be extensional. (The now outmoded terminology 'internal relation' was introduced by Andrzej Mostowski, for any relation R such that $\forall x\forall y(\forall z(\mathrm{R}zx \leftrightarrow \mathrm{R}zy) \rightarrow x = y$.[11] Logicians nowadays use 'extensional' rather than 'internal'.) This is why the theory is an *extensional* theory of abstraction.

Abstraction theorem 3

$$
\begin{prooftree}
\AxiomC{$t = \alpha x\Phi x$}
\AxiomC{$\overline{\Phi a}^{\,(1)}$}
\AxiomC{$\overline{\exists!a}^{\,(2)}$}
\TrinaryInfC{$\mathrm{R}at$}
\RightLabel{(1)}
\UnaryInfC{$\Phi a \rightarrow \mathrm{R}at$}
\RightLabel{(2)}
\UnaryInfC{$\forall y(\Phi y \rightarrow \mathrm{R}yt)$}
\AxiomC{$t = \alpha x\Phi x$}
\AxiomC{$\overline{\mathrm{R}ct}^{\,(1)}$}
\BinaryInfC{Φc}
\AxiomC{$\overline{\forall z(\Phi z \rightarrow \mathrm{R}zb)}^{\,(3)}$}
\AxiomC{$\overline{\exists!c}^{\,(2)}$}
\BinaryInfC{$\Phi c \rightarrow \mathrm{R}cb$}
\BinaryInfC{$\mathrm{R}cb$}
\RightLabel{(1)}
\UnaryInfC{$\mathrm{R}ct \rightarrow \mathrm{R}cb$}
\RightLabel{(2)}
\UnaryInfC{$\forall u(\mathrm{R}ut \rightarrow \mathrm{R}ub)$}
\RightLabel{(3)}
\UnaryInfC{$\forall z(\Phi z \rightarrow \mathrm{R}zb) \rightarrow \forall u(\mathrm{R}ut \rightarrow \mathrm{R}ub)$}
\UnaryInfC{$\forall v(\forall z(\Phi z \rightarrow \mathrm{R}zv) \rightarrow \forall u(\mathrm{R}ut \rightarrow \mathrm{R}uv))$}
\BinaryInfC{$\forall y(\Phi y \rightarrow \mathrm{R}yt) \wedge \forall v(\forall z(\Phi z \rightarrow \mathrm{R}zv) \rightarrow \forall u(\mathrm{R}ut \rightarrow \mathrm{R}uv))$}
\end{prooftree}
$$

If $t = \alpha x\Phi x$, then every Φ bears R to t, and anything borne R by every Φ is borne R by anything bearing R to t. To put it a little more succinctly, t is 'R-minimal' in being borne R by every Φ. It is for this reason that I call the theory a *minimizing* theory of abstraction. I shall call $\forall u(\mathrm{R}ut \rightarrow \mathrm{R}uv)$ the *minimizing condition on t with respect to v*, where v, *ex hypothesi*, is borne R by every Φ.

[11] See A. Mostowski, 'An Undecidable Arithmetical Statement', *Fundamenta Mathematica*, 36 (1949), pp. 143–64, at p. 146.

VI. SPECIALIZING THE GENERAL FORM: DEFINITE DESCRIPTION AND SET ABSTRACTION

I examine now two very important instances of the general pattern. The first, ironically, shows that abstractive terms need not always denote abstract objects. The abstraction operator in question is obtained by taking for R the identity relation =. That is to say, $\alpha_=$ is the definite-descriptive term-forming operator ι. Because the identity relation is so familiar, it can serve as a subscript on its own, without our having to specify any condition Θ of the general kind mentioned above. But it is worth noting that one would obtain definite-descriptive abstraction by taking any relation R that is both reflexive and a congruence relation. In such a case $\Theta(\mathrm{R})$ would be

$$\forall x \mathrm{R}xx \wedge \forall \Psi \forall x \forall y(\mathrm{R}xy \rightarrow (\Psi x \rightarrow \Psi y)).$$

VI.1. *Definite description*

Here now is how the α-rules above specialize to ι-rules upon taking the identity relation = for the general relation R:

$$\dfrac{\begin{array}{c}\underbrace{{}^{(i)}\overline{\boldsymbol{\Phi}(a)}\quad , \quad \overline{\exists! a}^{(i)}}\\ \vdots \\ a = t\end{array}\qquad \exists! t \qquad \begin{array}{c}\overline{a = t}^{(i)}\\ \vdots \\ \boldsymbol{\Phi}(a)\end{array}}{t = \iota x \boldsymbol{\Phi}(x)}{\scriptstyle (i)}$$

$$\dfrac{t = \iota x \boldsymbol{\Phi}(x) \qquad \boldsymbol{\Phi}(u) \qquad \exists! u}{u = t} \qquad\qquad \dfrac{t = \iota x \boldsymbol{\Phi}(x)}{\exists! t} \qquad\qquad \dfrac{t = \iota x \boldsymbol{\Phi}(x) \qquad u = t}{\boldsymbol{\Phi}(u)}$$

Abstraction theorem 1, in this setting, is

$$\exists! t \;\rightarrow\; t = \iota x(x = t).$$

Abstraction theorem 2 is

$$\forall x \forall y(\forall z(z = x \leftrightarrow z = y) \rightarrow x = y).$$

Abstraction theorem 3 is

$$t = \iota x \boldsymbol{\Phi} x \rightarrow (\forall y(\boldsymbol{\Phi} y \rightarrow y = t) \wedge \forall v(\forall z(\boldsymbol{\Phi} z \rightarrow z = v) \rightarrow \forall u(u = t \rightarrow u = v))).$$

VI.2. *Set abstraction*

My next example of the general pattern is provided by set abstraction. I use the familiar notation $\in$ (for the set-membership relation) but without assuming anything more about this relation than the extensionality required of it by abstraction theorem 2. The rules for abstraction with respect to such a *barely extensional* relation $\in$ ensure that the corresponding abstraction operator is precisely the set-term-forming operator $\{x|\mathbf{\Phi}x\}$. The rules, in effect, lay down the analytic connections that obtain among the notions of set, membership, and satisfaction of set-defining conditions. Moreover, they are ontologically non-committal. So even the most traditional analyticity theorist (one who maintains that no analytic truth can carry existential commitment) is able to view the following rules as providing the *logic of sets*. By taking $\in$ for the general relation R, subject to no condition Θ, the operation $\alpha x\mathbf{\Phi}(x)$ becomes that of *set abstraction* $\{x|\mathbf{\Phi}(x)\}$:

$$\dfrac{\begin{array}{c}\underbrace{{}^{(i)}\overline{\mathbf{\Phi}(a)}\quad , \quad \overline{\exists!a}^{(i)}}\\ \vdots \\ a \in t\end{array} \qquad \exists!t \qquad \begin{array}{c}\overline{a \in t}^{(i)}\\ \vdots \\ \mathbf{\Phi}(a)\end{array}}{t = \{x|\mathbf{\Phi}(x)\}}\,(i)$$

$$\dfrac{t = \{x|\mathbf{\Phi}(x)\} \quad \mathbf{\Phi}(u) \quad \exists!u}{u \in t} \qquad \dfrac{t = \{x|\mathbf{\Phi}(x)\}}{\exists!t} \qquad \dfrac{t = \{x|\mathbf{\Phi}(x)\} \qquad u \in t}{\mathbf{\Phi}(u)}$$

Abstraction theorem 1, in this setting, is

$$\exists!t \;\rightarrow\; t = \{x|x \in t\}$$

that is to say, everything is the set of its own members. Thus if we wish to have *Urelemente*, each of them will have to be self-membered. Otherwise, we shall be dealing only with pure sets.

Abstraction theorem 2 is

$$\forall x\forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

This is the axiom of extensionality of Zermelo's set theory. On my analysis of set abstraction, extensionality is a derived result. One might therefore claim that the analysis is deep. Alternatively, one could remind oneself that abstraction in this general mould *presupposes* the extensionality of the underlying relation, so the analysis afforded by the rules is not that deep after all. Against this harsher assessment, it can be claimed in mitigation that one could in any event simply lay down the rules as introducing the notions of set abstraction and membership simultaneously. The novice pondering the rules will then *learn*, by deduction, that membership is an extensional relation, and that the axiom of extensionality of modern set theory is true.

Abstraction theorem 3 is

$$t = \{x|\mathbf{\Phi}x\} \rightarrow (\forall y(\mathbf{\Phi}y \rightarrow y \in t) \wedge \forall v(\forall z(\mathbf{\Phi}z \rightarrow z \in v) \rightarrow \forall u(u \in t \rightarrow u \in v)))$$

i.e.,

$$t = \{x|\mathbf{\Phi}x\} \rightarrow (\forall y(\mathbf{\Phi}y \rightarrow y \in t) \wedge \forall v(\forall z(\mathbf{\Phi}z \rightarrow z \in v) \rightarrow t \subseteq v)).$$

If t is the set of all $\mathbf{\Phi}$s, then any set containing every $\mathbf{\Phi}$ has t as a subset; that is, t is *minimal* in having all the $\mathbf{\Phi}$s as members. Here the minimizing condition on t with respect to v is simply that t is a subset of v.

VII. RELATIONAL, EXTENSIONAL, MINIMIZING ABSTRACTION

I have been dealing with a binary relation R in general, and I have examined the special cases where R is = or $\in$. I am about to consider linear orderings as well, for which the usual symbol is <. Because we also have the familiar expression $x \leqslant y$ to abbreviate $(x < y \;\vee\; x = y)$, I shall formulate the following considerations in terms of < rather than R.

If we assume that the ordering relation < is irreflexive ($\forall x \;\neg\; x < x$) and connected (or trichotomous: $\forall x \forall y(x < y \;\vee\; y \leqslant x)$), then we shall have that

$$\forall y \forall z(\forall x(x < y \;\rightarrow\; x < z)) \vdash y \leqslant z.$$

Proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{
\dfrac{\dfrac{\forall x\forall y(x<y\vee y\leqslant x)\quad \overline{\exists!c}^{(3)}}{\forall y(c<y\vee y\leqslant c)}\quad \overline{\exists!b}^{(4)}}{c<b\vee b\leqslant c}
\qquad
\dfrac{\dfrac{{}^{(1)}\overline{c<b}\quad \dfrac{{}^{(2)}\overline{\forall x(x<b\to x<c)}\quad \dfrac{{}^{(1)}\overline{c<b}}{\exists!c}}{c<b\to c<c}}{c<c}\quad \dfrac{\forall x\neg x<x\quad \dfrac{{}^{(1)}\overline{c<b}}{\exists!c}}{\neg c<c}}{\bot}
\qquad \overline{b\leqslant c}^{(1)}
}{b\leqslant c}{}^{(1)}}{\forall x(x<b\to x<c)\to b\leqslant c}{}^{(2)}}{\forall z(\forall x(x<b\to x<z)\to b\leqslant z)}{}^{(3)}}{\forall y\forall z(\forall x(x<y\to x<z)\to y\leqslant z)}{}^{(4)}
$$

So in the presence of irreflexivity and connectedness (or trichotomy) for <, the minimizing condition on t with respect to v in abstraction theorem 3, namely,

$$t = \alpha x_{<}\mathbf{\Phi}x \rightarrow (\forall y(\mathbf{\Phi}y \rightarrow y < t) \wedge \forall v(\forall z(\mathbf{\Phi}z \rightarrow z < v) \rightarrow \forall u(u < t \rightarrow u < v)))$$

is simply that $t \leqslant v$.

I have shown thus far that both definite descriptions and set abstracts can be subsumed under a general pattern of abstraction with respect to a binary relation. For definite description, the binary relation in question is identity; for set abstraction, it is to be thought of as membership.

I say 'is to be thought of as', because when one looks at what is going on, it turns out that no condition is being imposed on the relation $\in$ other than its extensionality. The abstraction rules, within a free logic, provide the *logic of sets*, that is, a canon of reasoning about things that are essentially extensional (with respect to the binary relation $\in$ in question) *and no more.*

VIII. NATURAL NUMBERS: ABSTRACTING ON PROGRESSIONS

My professed intention at the outset was to provide an account of relation-based abstraction that would also accommodate numbers. So what about the natural numbers? How can this inferential treatment be extended so as to deal with them? The abstractions permitted by the rules can be parameterized by a non-trivial condition on the binary relation R. (By 'non-trivial' here, I mean a condition strictly stronger than the mere extensionality of R.) I now set about formulating such a condition $\Gamma(<)$ on a binary relation $<$. The abstraction operation $\alpha_{\Gamma(<)}x\Phi(x)$ will then be interpretable as numerical abstraction. Or almost....

Definition. $x[R]y \equiv_{df} Rxy \wedge \forall z(Rxz \rightarrow (y = z \vee Ryz))$.

x[R]y means that y is an *immediate* R-successor of x.

Definition. $S^*xy \equiv_{df} \forall F(\forall z \forall w(Fz \rightarrow (Szw \rightarrow Fw)) \rightarrow (Fx \rightarrow Fy))$.

The *definiens* states that any property that transmits under the relation S will transmit from x to y. That is to say, y is an S-ancestral of x.

Definition. $\exists_1 xFx \equiv_{df} \exists x \forall y(x = y \leftrightarrow Fy)$.

$\exists_1$ is the uniqueness quantifier. (Since Kleene,[12] the uniqueness quantifier has sometimes been written as '∃!'. The latter, however, is a notation that I reserve here for use as a predicate.)

Definition. $\Gamma(<)$ is the conjunction of the following:

$\exists x \forall y(x = y \vee x < y)$	Existence of an initial element
$\forall x \neg x < x$	Irreflexivity
$\forall x \forall y(x < y \vee (y = x \vee y < x))$	Connectedness (trichotomy)
$\forall x \forall y(x < y \rightarrow \forall z(y < z \rightarrow x < z))$	Transitivity
$\forall x \exists_1 y\, x[<]y$	Unique right-immediacy
$\forall x \forall y(x < y \rightarrow \exists_1 z\, z[<]y)$	Unique left-immediacy
$\forall x \forall y(x < y \rightarrow x[<]^* y)$	Finite connectivity

Definition. Any domain of elements ordered by a relation $<$ meeting condition Γ is called a *progression.*

12 S.C. Kleene, *Introduction to Metamathematics* (Princeton: Van Nostrand, 1952), p. 199.

Connectedness ensures that there is at most one initial element; whence, given the existence of at least one initial element, there is a *unique* initial element. In the statement of finite connectivity, the relation $[<]^*$ is the ancestral of *immediate* <-succession. The conditions of connectedness, irreflexivity, transitivity, and unique left- and right-immediacy ensure that all R-successors of the initial element form a single discrete linear order, which, by finite connectivity, will have order-type ω. (If we omitted finite connectivity, the remaining requirements would be satisfied by an ordering with order-type $\omega + (\omega^* + \omega)$.)

Numerical abstraction (here, abstraction of natural numbers) is effected in two stages. The first stage involves abstracting in accordance with the inferential rules, with respect to a relation $<$ satisfying the condition Γ just defined. Let the term abstract in question be

$$\gamma_< x\mathbf{\Phi}(x).$$

This term denotes a position within the <-progression. Its denotation will be the actual element in that progression which is the 'first' (in the sense of $<$) to come after all the $\mathbf{\Phi}$s – which, given the conditions for $\gamma_<$-introduction, must form a finite initial segment of the progression in question. Since the denotation of $\gamma_< x\mathbf{\Phi}(x)$ is thus an actual element in the progression, it is not, in general, a 'genuine' natural number. Instead, it is the <-least non-$\mathbf{\Phi}$. It could, for all we know, be a concrete object. Whether it is or not depends on the progression in question. *If* the progression happened to be that of the natural numbers themselves, then each natural number n would be the least non-predecessor of n, i.e., the least number not among 0, ..., $n - 1$.

If we have two distinct progressions, involving the relations $<_1$ and $<_2$ respectively, we shall have two distinct kinds of γ-abstraction. The objects $\gamma_{<_1} x\mathbf{\Phi}(x)$ and $\gamma_{<_2} x\mathbf{\Phi}(x)$ will sit within their respective progressions, and will in general be distinct. The abstractive terms, therefore, give us what might be called *intra-progressional* positions (indeed, the actual occupants of those positions). What is further needed here, in order to abstract severely enough to attain the natural numbers themselves, is a way of correlating such intra-progressional positions with one another, so as to obtain the *inter*-progressional positions themselves, independently of any particular progression. This is like trying to get at directions of lines, independently of any lines. So what better way to do this than to employ the form of *objectual* (rather than conceptual) abstraction that Frege himself employed in the case of line-directions?

I therefore introduce an abstraction *function* (not: variable-binding operator), which I represent as #[ ], and which is subject to the following abstraction principle:

$(\nu)\quad \#[\gamma_{<_1} x\Phi(x)] = \#[\gamma_{<_2} x\Psi(x)] \;\leftrightarrow\; \exists \mathrm{R}\ \mathrm{R}xy[\Phi x, \Psi y].$

The left-hand side displays *objectual* abstraction, even though the right-hand side places a condition on the embedded *concepts* Φ and Ψ. Those concepts have already been abstracted upon (via $\gamma_{<_1}$ and $\gamma_{<_2}$ respectively) to produce the *inputs* to the step of objectual abstraction effected by #[ ]. The right-hand side says that there is a one–one correspondence R between the Φs and the Ψs. (The bound second-order variable R here is not to be confused with the earlier placeholder for a binary relation. The latter has already been instantiated to $<_1$ and $<_2$ in the present discussion.)

(ν) is not a form of Hume's principle; for Hume's principle involves *conceptual* abstraction on the left-hand side. What we have on the left-hand side here is objectual abstraction. Moreover, this objectual abstraction cannot apply directly to the object itself, in the way the direction-producing abstraction-function D[ ] can apply in the case of line-directions:

$$\mathrm{D}[l_1] = \mathrm{D}[l_2] \;\leftrightarrow\; l_1 \parallel l_2.$$

Here the lines l_1 and l_2 can be directly named, by proper names, i.e., structureless singular terms; and the abstraction of directions via D[ ] can still be effected, since the condition on the right-hand side involves reference directly to l_1 and l_2. The analogous situation does not obtain, however, in the case of (ν). For the objectual abstraction to be effected, the inputs to the abstraction process must be denoted by complex singular terms, themselves in abstractive form, so that we can delve into them to extricate the predicates Φ and Ψ in terms of which to state the condition on the right-hand side.

This fact might make it look as though what we have here, after all, is a form of Hume's principle; for surely, the naïve thought might go, is not what we see on the left-hand side of (ν) an abstractive process on Φ and on Ψ respectively – a two-stage process in each case, to be sure, but one which is really only the composition of the two abstraction operations γ and #[ ]? Even though the second stage #[ ] is objectual, the first stage γ is conceptual, and so the overall two-stage operation will also be conceptual.

The answer to the naïve question just posed is negative. For the overall abstraction operation on the left of the identity, applied to Φ, is composite by virtue of the objectual function #[ ] being applied to the output of the conceptual abstraction $\gamma_{<_1}$; whereas the one on the right of the identity, applied to Ψ, is composite by virtue of #[ ] being applied to the output of a *different* conceptual abstraction, namely $\gamma_{<_2}$. That $\gamma_{<_1}$ and $\gamma_{<_2}$ are indeed different stems from the fact that $<_1$ and $<_2$ can be wholly different progressions. They might even have disjoint domains.

So the naïve perception of (v) as a form of Hume's principle is mistaken. The naïve objector might not give up just yet, however. He might persist by arguing as follows:

> Suppose one were to reconstrue $\alpha_{R}x\Phi(x)$, with its R-dependent abstraction operator, as $\alpha x[R, \Phi]$, so as to make the α-part independent of R itself. Would that not make the two-stage operations, on Φ and on Ψ respectively, the same?

Again the answer is negative. The proposal is to make α binary, applying to the (binary) relation R and to a (monadic) concept Φ. The resulting way of construing the composite abstraction operation (first applying α and then applying #[ ]) would still fail to make (v) into a form of Hume's principle. For there is no provision, in Hume's principle, for the relation R in addition to the two predicates Φ and Ψ. (The relation R in question is not to be confused with the *bound* second-order variable R on the right-hand side in the usual statement of Hume's principle.)

It appears, then, that my two-stage proposal for the objectual abstraction of genuine natural numbers from conceptually abstracted positions-within-particular-progressions is not simply Hume's principle in disguise. On the positive side, the proposal enables us also to understand the structuralist thesis that what matters is position-within-a-progression, rather than any particular progression. We may begin by thinking that indeed we cannot know what numbers are, but that we can still say *how to abstract them*. As we reflect further, however, we may realize that knowing how to abstract them is all there is to knowing what they are.

This account of numerical abstraction, to be sure, involves quite a heavy presuppositional burden: a presupposition to the effect that there are indeed progressions – that is, domains orderable by relations $<$ satisfying the condition Γ. Will not the complaint arise from the disappointed logicist that this is not at all what is meant by a logicist account of number? Surely, the complaint will go, we need to be shown how to 'obtain the numbers' by using only logical materials?

There are two strands to disentangle here: ontological and epistemological. To take the ontological one first, we cannot, as logicists, aspire to conjure something out of nothing. There have to be enough 'logical objects' for us to be able to find the numbers among them. Such was Frege's hope; and of course he overprovided, on the ontological side, to the point of inconsistency. And while the neo-logicist is going to avoid Frege's mistake, he still has to put forward principles of sufficient existential strength to vouchsafe the numbers. Indeed, he must show that the numbers themselves are *necessary* existents.

It is for this reason that I do not mind helping myself to so much (up to isomorphism) at the very outset. I am happy to premise my logicist thinking about number on the logically possible existence of at least one progression (a domain with an ordering $<$ satisfying condition Γ). In the second-order case, the consistency of a given theory does not, in general, guarantee the (logically possible) existence of a model for the theory. This holds even when the only second-order quantifications involved are monadic, as is the case in the statement of the condition Γ, in which the finite connectivity of $<$ is the only requirement whose statement involves second-order quantification.

There is also the following consideration, when it comes to assessing how much one is entitled to when attempting to reveal a logicist commitment to the existence of a given kind of number. Are the logicist materials that one is using of a much higher consistency-strength than is the original mathematical theory which one is trying to derive? The answer is in the affirmative, as far as second-order Frege arithmetic is concerned (i.e., second-order logic with Hume's principle). It is from Frege arithmetic that Wright seeks to derive Peano arithmetic. Boolos showed that Frege arithmetic is equi-interpretable with second-order arithmetic Z_2. This is an exceedingly powerful system to assume, when one's goal is the 'justificatory re-derivation' of the more modest theory of Peano arithmetic.

Although the matter will require more precise investigation, I claim that the second-order theory $\Gamma(<)$ given above will be no more powerful than Z_2, and may well be less powerful. So on the epistemological side we are no worse off with my proposal than the Wrightian neo-logicist is with (HP).

The conviction that it is logically possible for there to be an infinite progression – in the weaker sense that all its members could co-exist, rather than in the stronger sense that they could form a completed totality – can also be sustained by appeal to the constructive logicist account of number laid out above. It amounts to the conviction that one should always be able to tack on a new element at the rightmost end of any finite left-to-right discrete linear ordering. I do not believe that this conviction needs to be sustained by a Kantian appeal to the form of intuition of time. It strikes me as a logical and conceptual matter that one can always 'keep going on' in building up a progression.

A modern argument with a logical flavour to this effect would be as follows. A weaker condition results from Γ by dropping the requirement of finite connectivity, and weakening unique right-immediacy to the claim that any element *that has a $<$-successor* has an immediate $<$-successor. This weaker condition, which may be called 'Γ^f', is wholly first-order. Moreover, it is clear that it is satisfied by any finite left-to-right discrete linear ordering. So

Γ^f has arbitrarily large finite models. Hence by the compactness theorem for first-order logic, Γ^f has an infinite model. Any infinite model of Γ^f has an initial segment that will satisfy not only Γ^f, but also finite connectivity and unique right-immediacy. That is, it will satisfy Γ.

It may be objected that this reasoning for the logical possibility of an infinite progression is a *petitio*. For the argument that invokes the compactness theorem in order to pass from the existence of arbitrarily large finite models to the existence of an infinite model in effect assumes the logical possibility of an infinite progression. The argument proceeds by considering the theory that results from adjoining to Γ^f infinitely many distinct non-identities of the form $\neg\, a_i = a_j$ $(i < j)$, involving names a_0, a_1, ... These names themselves constitute an infinite progression of the very kind whose logical possibility we are trying to establish.

But what about using infinitely many names that are given in no particular order, and adjoining to Γ^f the negations of the identity claims that can be formed by using two distinct names? Then the *petitio*, if it is still lurking, would have to be sought elsewhere. If we appeal to the following general result concerning sentences ϕ,

If ϕ has arbitrarily large finite models, then ϕ has an infinite model

then we are open to the objection that this general result reverses to WKL_0 over RCA_0.[13] If we appeal instead to the specific result

If Γ^f has arbitrarily large finite models, then Γ^f has an infinite model

then we find that its consequent (and hence the conditional itself) can be proved in RCA_0. Without any well developed theory of reversal, *modulo* some subtheory of RCA_0 that fails to prove the existence of an infinite progression, it is difficult to calibrate the existential strength of the specific result.

These considerations, inconclusive though they may be, nevertheless incline one to accept the hard fact that the logicist cannot aspire to get something for nothing, especially when that something is an infinite progression. We may just have to take it as an article of logicist faith that it is logically possible for there to be infinitely many things.

So much for the first, ontological, strand of logicism. What about the second, epistemological, strand? How does this proposed 'neo-logicist' treatment enable us *to come to know* the basic axioms of arithmetic? How does it enable us to derive them from a deeper and, one hopes, more secure 'logical' foundation?

[13] See, e.g., S.G. Simpson, *Subsystems of Second Order Arithmetic* (Berlin: Springer, 1999).

The answer to this question can, in principle, be provided, but this is not the occasion to spell it out. The aim would be to show that the basic arithmetical facts about the number 0 and the successor function *s*( ) on natural numbers drop out from the condition Γ on the progressions from which those numbers are abstracted. 0 will be defined as the image, under #[ ], of (what will turn out to be) *the* initial element of any ordering $<$ satisfying Γ. Likewise, $s(n)$ will be defined as the image, under #[ ], of the γ–abstractum, in any ordering $<$ satisfying Γ, of the pre-images, under #[ ], of the members of that ordering whose images, under #[ ], are 0, ..., *n*.

IX. REAL NUMBERS: ABSTRACTING ON CONTINUOUS ORDERINGS

I turn now to a logicist treatment of the real numbers that proceeds in much the same way as the foregoing treatment of the naturals. I shall be performing γ-abstractions to obtain objects in appropriate positions *within* a given ordering, and then performing an objectual #-abstraction to obtain the reals *sui generis*, as positions within an order-type regardless of the ordered domains which may be of that type. Indeed, one of the attractions of this approach is how similar are the methods for obtaining the naturals and the reals, respectively.

The method calls for an ordering $<$ of some domain, subject to some condition Θ. In the case of the reals, the domains to consider are those of (what one would like to call real-valued) *magnitudes*. These magnitudes are necessarily expressed in terms of some unit of the appropriate dimension (mass, time or distance, for example). I shall call the unit magnitude 1. There is also the trivial magnitude, namely 0, which will be the additive identity. We do not, however, strictly need the name 0 for the formulation of the condition Θ(<), since the unique existence of an element with the properties of 0 can be secured without naming it.

I shall nevertheless continue to use the name 0 in formulating Θ(<). It will also be convenient to add three further defined notions.

Definition. $<(\mathrm{F}, \mathrm{G}) \equiv_{df} \exists x\mathrm{F}x \wedge \exists x\mathrm{G}x \wedge \forall x(\mathrm{F}x \rightarrow \forall y(\mathrm{G}y \rightarrow x < y))$

i.e., the predicates F and G have instances, and every F is less than every G.

Definition. $\mathrm{F}z^{\dagger}\mathrm{G} \equiv_{df} \forall x(x \neq z \rightarrow ((\mathrm{F}x \rightarrow x < z) \wedge (\mathrm{G}x \rightarrow z < x)))$

i.e., z is greater than all Fs distinct from it, and less than all Gs distinct from it.

Definition. $x[+_y]w \equiv_{df} x + y = w$

$[+_y]$ is a binary relation, eligible for the formation of ancestrals.

Consider now the following well known properties of a densely and continuously ordered Abelian semigroup, with addition as the group operation, and with a unit 1 distinct from its additive identity 0.[14]

$0 \neq 1$	
$\forall y(0 = y \vee 0 < y)$	0 is initial
$\forall y\, y + 0 = y$	0 is an additive identity
$\forall y \forall z(y < z \rightarrow \exists w(0 < w \wedge y + w = z))$	Differences
$\forall y \forall z(y < z \rightarrow \forall w(w + y < w + z))$	Order-additivity
$\forall x \forall y \forall z \forall w(x + z < y + w \rightarrow (x < y \vee z < w))$	Order-decomposability
$\forall x \forall y \forall z\, x + (y + z) = (x + y) + z$	Associativity of +
$\forall x \forall y\, x + y = y + x$	Commutativity of +
$\forall x \neg x < x$	Irreflexivity of <
$\forall x \forall y(x < y \rightarrow \forall z(y < z \rightarrow x < z))$	Transitivity of <
$\forall x \forall y(x < y \vee (y = x \vee y < x))$	Connectedness of <
$\forall x \forall y(x < y \rightarrow \exists z(x < z \wedge z < y))$	Density of <
$\forall F \forall G(<(F, G) \rightarrow \exists z\, Fz^{\dagger}G)$	Continuity of <
$\forall x \forall y \forall z((x < z \wedge 0 < y) \rightarrow$ $\exists w(x[+_y]^* w \wedge \exists v(v < y \wedge z = w + v)))$	Archimedean principle

The first two axioms imply that 0 < 1. The last two axioms, the continuity and Archimedean principles, are second-order. The definition of the ancestral relation $x[R]^*y$ involves second-order quantification. The Archimedean principle is not stated by Tarski. It is, however, needed if we wish to rule out infinitesimals.

From the Archimedean principle it follows that every magnitude is the sum of (i) an integral multiple of the unit 1 (i.e., either 0 or something of the form 1 + ... + 1, with finitely many occurrences of 1), and (ii) some 'sub-unit' magnitude r ($0 \leqslant r < 1$). (I refrain from calling r a 'fractional' magnitude, since I do not wish to imply that r will be a rational number. This remainder r could be irrational.) As an easy consequence of the continuity principle we have

$$\forall x(0 < x \rightarrow \exists y(0 < y \wedge x = y + y)) \qquad \text{Halving}$$

It follows from halving that we have the quantities of half a unit, a quarter of a unit, an eighth of a unit ... and so on. I shall call these 'powers' of ½ *Cauchy quantities*. (They are sometimes called 'bicimals', which is an unappealing neologism.) I use scare quotes with 'powers' because neither multiplication nor exponentiation is primitive. Corresponding to each

[14] See, for example, A. Tarski, *Introduction to Logic and to the Methodology of Deductive Sciences* (Oxford UP, 1965), p. 214. My treatment here differs from Tarski's in so far as I am concerned only with the non-negative numbers, and wish to rule out infinitesimals.

Cauchy quantity, however, is a definite descriptive term of the language denoting it. It follows by repeated applications of connectedness that r will be determinately locatable with respect to any finite sum of Cauchy fractions. We have therefore 'rigidified' all the intervals between integral multiples of the unit magnitude, in terms of the unit itself. Thus we do not need the full-blown rationals, with multiplication and division, in order to effect meaningful comparisons between scalar magnitudes of different dimensions that are not integral multiples of their respective units. These considerations show that any structure satisfying the axioms above is isomorphic to the real line $[0, \infty)$.

It should now be evident that the rules for γ-abstraction with respect to any ordering $<$ satisfying these axioms will ensure that $\gamma_{<}x\Phi(x)$ is the least upper bound of the Φs; or, equivalently, that it is the Dedekind cut-number determined by taking the Φs as forming the left class and the non-Φs as forming the right class. The condition for $\gamma_{<}$-introduction ensure that the Φs are not closed on the right, i.e., every Φ is strictly less than the cut-number.

In keeping with the realist view discussed above, the cut-number, i.e., the denotation for $\gamma_{<}x\Phi(x)$, is 'already there' in the domain ordered by $<$. I am not following the usual method of 'extending' the domain of rational numbers so as to include irrational real numbers for the first time via cuts (or least upper bounds). Rather, I am assuming that all the reals are already in view, so to speak, and seeking only to display the logical behaviour of the linguistic means whereby one keeps them in view.

By analogy with the way in which I used the function #[ ] for the objectual abstraction of natural numbers from positions within progressions, I have an abstraction principle for obtaining real numbers *sui generis*, which will be stated presently. As before, I first employ conceptual abstraction to obtain (the objects in) certain positions within the various orderings that may be available. Then I employ objectual abstraction on those positions within the orderings, in order to obtain (dimensionless) real numbers *sui generis*. The orderings, for the case of real numbers, are not the progressions with which I dealt in the case of natural numbers; instead, they are orderings of scalar magnitudes, each one identified by a dimension such as time, distance or mass. These orderings are somewhat similar to what Hale calls complete q-domains.[15] A direct comparison is made difficult by the fact that Hale makes use of the notions of multiplication and of ratio in stating his conditions on domains; I am working here without those notions.

These scalar magnitudes form densely and continuously ordered Abelian semigroups with units specific to the kind of scalar magnitude in question

[15] See B. Hale, 'Reals by Abstraction', *Philosophia Mathematica*, 8 (2000), pp. 100–23, at p. 108.

(such as one second, one metre or one gramme, respectively, for the examples just mentioned). I shall call them *scalar orderings* for the sake of convenience.

Definition. If Rwy is a relation between members x of a scalar ordering $<_1$ (with initial element 0_1 and unit 1_1) and members y of a scalar ordering $<_2$ (with initial element 0_2 and unit 1_2), then we say that R *preserves order* if and only if, whenever Rx_1y_1 and Rx_2y_2, we have $x_1 <_1 y_1$ if and only if $x_2 <_2 y_2$; and we say that R *preserves addition* if and only if R(0_1, 0_2), R(1_1, 1_2) and, whenever Rx_1y_1 and Rx_2y_2, we have R($x_1 + x_2$, $y_1 + y_2$).

Preservation of addition is easily seen to guarantee R-correlation of Cauchy fractions: that magnitude which is half of 1_1 will be correlated by R with that magnitude which is half of 1_2; and likewise for quarters, eighths, and so on.

Definition. If Rxy is a relation between members x of an ordering $<_1$ and members y of an ordering $<_2$, then we say that R is an *isomorphism between* $<_1$ and $<_2$ if and only if R is 1–1 from the domain of $<_1$ onto the domain of $<_2$ and preserves both order and addition.

I can now state the abstraction principle for real numbers as follows.

$(\nu')\ \#[\gamma_{<_1}x\Phi(x)] = \#[\gamma_{<_2}x\Psi(x)] \leftrightarrow \exists R(Rxy[\Phi x, \Psi y] \wedge R \text{ preserves} < \text{and} +).$

The condition on the right-hand side, given the last definition, is simply that R is an isomorphism between the respective restrictions, to the **Φ**s and the **Ψ**s, of the two scalar orderings $<_1$ and $<_2$.

IX.1. *The similarity with the case of natural numbers*

This condition could already have been imposed, without loss of generality, on the relation R involved on the right-hand side of the earlier abstraction principle for natural numbers. The condition would have been unusual in such a context, however, since in the case of natural numbers the mere existence of a 1–1 correlation (i.e., the equinumerosity of the **Φ**s and the **Ψ**s) suffices. That the correlation in question can also, without loss of generality, be taken (in the case of the natural numbers) to be order-preserving sometimes escapes attention.

It is indeed an insufficiently appreciated point that there can be no concept of a countable infinity which will not intrinsically deliver the concept of a progression, with the underlying ordering deriving from the finite collections involved. The standard definition of a countably infinite set is that the set in question is equinumerous with the natural numbers; so that definition cannot be relied upon to make a non-trivial case for the conceptual point at issue here. We seek, then, a definition of what it is for a set X to be countably infinite, one which does not presuppose the natural numbers in their usual ordering. The only candidate definition of infinite set that we can use in this connection is due to Dedekind: X is infinite if and only if for some $x \in X$, X is equinumerous with $X - \{x\}$. We can then say that X is *countably infinite* if and only if X is infinite and every infinite subset of X is equinumerous with X itself. Given any countably infinite set X, we can create a progression out of equivalence classes E of its finite subsets F. The equivalence relation in question is simply equinumerosity. The underlying ordering of the progression is obtained as follows: $F < F'$ if and only if any member of F is equinumerous with some proper subset of any member of F'. (I realize that the formalization of this argument in ZF without infinity will draw on both the axiom of power set and the axiom scheme of replacement. These principles, however, can commend themselves to one innocent of any commitment to the existence of a completed infinity.)

Without any prospective alternatives to the Dedekindian definitions, it appears to be safe to say that one cannot have 'bare countable infinity' without there being also a simultaneously associable sense of a progression. That is why there is no loss of generality if, in principle (ν'), we require that the relation R on the right-hand side must establish not just the equinumerosity of the Φs and the Ψs, but their order-isomorphism as well. Precisely because there is no loss of generality, it is not necessary to require this order-isomorphism either.

IX.2. *Back to the reals*

With real numbers, however, the requirement that R must be an isomorphism is essential if the definition is to achieve its intended aim. We should nevertheless realize now that the abstraction principles for naturals and for reals are exactly analogous. And this, I submit, is the bonus that offsets the apparent loss of ontological innocence in not having my abstraction principles 'bring the abstract objects into existence' in a creative way that ensures that those principles themselves afford the only possible epistemic access to them that we might enjoy.

I have proceeded directly from a treatment of the naturals to a treatment of the reals. In mathematical developments of real analysis, it is usual to treat the rationals after the naturals and before the reals. This, however, is occasioned by the need (on the part of those at the other end of the Euthyphronic spectrum noted on p. 108 above) to 'generate' the non-natural rationals, and thereafter to 'generate' the irrational reals. The creation of ratios makes the erstwhile discrete ordering of the natural-number progression dense; whereupon the creation of cuts (or least upper bounds) makes the dense ordering of the rationals continuous. Nor should the reader be concerned at my omission of negative numbers. It is easy enough to get these into the picture after the construction of all positive reals.[16] If I had to provide a treatment of the rationals along lines consonant with my general approach to abstraction, I would provide introduction and elimination rules for statements of the form $t = m/n$, where m and n are naturals. It is possible to do this without having multiplication as an explicit operation; but the details must be left to another occasion.

Finally, I revisit the matter of consistency-strength, broached above in connection with the naturals, which were based on a theory Γ of progressions. In the case of the reals, I used the second-order theory which I called Θ, a theory of the ordered additive structure of the reals greater than or equal to 0. This theory included second-order Archimedean and continuity principles; but it did not involve multiplication. The omission of multiplication from Θ had a philosophical purpose. The idea would be that repeated addition of 1 to 0 corresponds to the 'laying down of units' in a scalar measurement process; whereafter, upon getting within a unit's reach of the sought scalar entity, repeated addition of 'negative powers of 2' (which I called Cauchy fractions) would continue the measurement process to any desired degree of accuracy. That should suffice, conceptually, for the concept of a real number – that is, of an entity lying in a continuum, an entity which is arbitrarily closely approximable, 'bicimally, from below'.

It is worth noting that there is an obvious two-sorted first-order reformulation of the theory $\Theta(0, 1, +, <)$ which I shall call T. T is still without multiplication, but now also without any form of the Archimedean principle, which has no first-order schematic analogue like that of the continuity principle. T is interpretable in the theory of real closed fields (which of course has multiplication primitive as well). By a classic quantifier-elimination result due to Tarski,[17] the latter theory in turn is equi-interpretable with the usual theory of the ordered field of reals that includes the first-order axiom scheme of continuity. There is an unpublished recent result of Friedman's, to the effect that EFA (exponential function arithmetic) proves the consistency of the theory of real closed fields. This would neatly lower the presuppositional power of (the first-order surrogate T for) Θ, as the neo-logicist's raw materials for deriving a theory of real numbers. In the two-sorted first-order version T of Θ, however, there is no way of defining the predicate 'x is a natural number'. Yet one very much wants to be able to pick out the naturals among the reals! In order to do so, one would simply have to add 'x is a natural number' as a primitive

[16] See, for example, E. Landau, *Foundations of Analysis* (New York: Chelsea, 1951).

[17] See Tarski, *A Decision Method for Elementary Algebra and Geometry*, 2nd edn (California UP, 1951), p. 42.

predicate, subject to the obvious axioms. The resulting first-order theory would then, as is to be expected, have a much higher consistency-strength, namely, that of Z_2.

Gödel's lesson seems unavoidable: one cannot get a mathematical something from a logical nothing. The approach commended here takes Gödel's lesson seriously. It gives up trying to pull mathematical rabbits out of a logical hat. It seeks, instead, the logical rules that govern the breeding of those rabbits – wherever they happen, of necessity, to come from.[18]

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