Forces Acting on an Aircraft:

\[ L = \frac{1}{2} \rho V^2 S C_L \]
\[ D = \frac{1}{2} \rho V^2 S C_D \]

*In Steady Level Flight: \( L = W \), \( D = T \)*

Wing Loading:

\[ \omega L = \frac{W}{s} \]

Mach Number:

\[ m = \frac{V_\infty}{a_\infty}, \quad a_\infty = \sqrt{\gamma RT} \]

Flight Regimes:

<table>
<thead>
<tr>
<th>( M )</th>
<th>Incompressible</th>
<th>Assumes constant density. Air behaves as a perfect gas.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M &lt; 0.3 )</td>
<td>Compressible Subsonic</td>
<td>Density changes are now substantial and should be accounted for. Assumes air behaves as a perfect gas.</td>
</tr>
<tr>
<td>( 0.3 &lt; M &lt; 0.8 )</td>
<td>Transonic</td>
<td>Mixed flow parts. Including subsonic and supersonic in certain regions.</td>
</tr>
<tr>
<td>( 0.8 &lt; M &lt; 1.2 )</td>
<td>Supersonic</td>
<td>Shock wave formation. Large temperature increases.</td>
</tr>
<tr>
<td>( 1.2 &lt; M &lt; 1.5 )</td>
<td>Hypersonic</td>
<td>Air is no longer behaving as a perfect gas. Shock</td>
</tr>
</tbody>
</table>
waves are present. Very high temperatures.

Standard Atmosphere: Agreed upon standard for what pressure, temperature, and density will likely be at a given altitude.

Pressure: \( P = \frac{F}{A} \), Normal force per unit area. Pressure decreases as you increase in altitude.

Density: \( \frac{m}{V} \), Mass per unit volume. Density decreases as you increase in altitude.

Temperature: Average kinetic energy of a collection of gas molecules.

Temperature Variation with Altitude:

- You cannot find altitude based only on temperature because a value of temperature can correspond to more than one altitude.
- The pressure and density variations with altitude are obtained from this empirical temperature variation by using the laws of physics.
- Temperature gradient regions:

\[
\alpha \equiv \frac{dT}{dn}
\]

\[
\frac{P}{P_i} = \left( \frac{T}{T_i} \right)^{-\frac{\gamma_0}{\alpha R}}
\]

\[
\frac{\rho}{\rho_i} = \left( \frac{T}{T_i} \right)^{-\frac{\gamma_0}{(\alpha R)}}
\]

\[
T = T_i + \alpha(n - h_i)
\]
- Isothermal Regions:

\[ \frac{\rho}{\rho_1} = \exp\left(-\left[\frac{g_0}{(RT)}\right](h-h_1)\right) \]

\[ \frac{\phi}{\phi_1} = \exp\left(-\left[-\frac{g_0}{(RT)}\right](h-h_1)\right) \]

Equation of State:

*assumes a perfect gas

\[ P = \rho RT \]

Manometers: Useful for measuring pressure

\[ \Delta P = -\rho g \Delta h \]

\[ P_b < P_a \]

\[ P_b - P_a = -\rho g (h_b - h_a) \]

\[ P_b = -\rho g h_b + P_a \]

\[ P_b = -\rho g h_b + P_a \]
**Continuity equation:**
Based on conservation of mass: Mass can be neither created nor destroyed.

**Mass flow rate:**
\[ \dot{m} = \rho v A \]

At two points:
\[ \dot{m}_1 = \dot{m}_2 \]
\[ \rho_1 A_1 v_1 = \rho_2 A_2 v_2 \]
If incompressible:
\[ v_1 A_1 = v_2 A_2 \]

**Bernoulli:**
\[ P_2 + \frac{1}{2} \rho v_2^2 = P_1 + \frac{1}{2} \rho v_1^2 \]

Derived from the conservation of momentum

Assumptions:
1. Inviscid
2. Neglect gravity (no body forces)
3. Steady
4. Incompressible \[ M < 0.3 \to \text{incompressible flow} \]
5. Same streamline
For any process:
- Energy Equation: relates temperature and velocity.

\[ C_p T_1 + \frac{1}{2} V_1^2 = C_p T_2 + \frac{1}{2} V_2^2 \]

- Isentropic Flow:
  Assumptions:
  1. Adiabatic: No heat transfer
  2. Reversible: No friction, no viscous forces
  Isentropic Relations:

\[
\frac{P_2}{P_1} = \left( \frac{\rho_2}{\rho_1} \right)^\gamma, \quad \frac{P_2}{P_1} = \left( \frac{T_2}{T_1} \right)^{\gamma/\gamma-1}
\]

\[
\left( \frac{\rho_2}{\rho_1} \right)^\gamma = \left( \frac{T_2}{T_1} \right)^{\gamma/\gamma-1}
\]

Cannot assume incompressible; density must be allowed to change. Cannot use Bernoulli.

**Subsonic Wind Tunnels:**
Most of the times can assume incompressible because we are dealing with low speeds.
- Velocity increases are the area decreases through the convergent nozzle, and the opposite occurs for the divergent part.

**Different Types of Pressure**
- Static Pressure: The pressure we would feel is we were moving along with the flow (Standard atmospheric table)
- Total Pressure (Stagnation Pressure): The pressure obtained at a point where the flow velocity has been decreased to zero.
- It is a property of flow
- Constant throughout: meaning we can use it to find the pressure at other points.
- Measured by a pitot tube.
- If we assume incompressible the Bernoulli’s equation can be used, which relates dynamic pressure, total pressure, and static pressure.

\[
\text{BERNOULLI: } P_0 + \frac{1}{2} \rho V^2 = P_s + \frac{1}{2} \rho V^2
\]

\[
P_0 - P_s = \frac{1}{2} \rho V^2 \quad \text{dynamic pressure}
\]

-Solving for true airspeed which deals with the actual density.

\[
\left[\frac{(P_0 - P_s) \rho}{\rho} \right]^{\frac{1}{2}} = V_{\text{true}}
\]

Equivalent airspeed: Airspeed measured by an airspeed indicator relates to sea level density. If assuming incompressible flow then,

\[
V_{\text{eq}} = \left[\frac{(P_0 - P_s) \rho}{\rho_{\text{s,l.}}} \right]^{\frac{1}{2}}
\]

An equation that relates true airspeed and equivalent airspeed:

\[
V_{\text{true}} = V_{\text{eq}} \sqrt{\frac{\rho}{\rho_{\text{s,l.}}}}
\]

Can find Mach number using isentropic Mach relations and tables.

**Supersonic Wind Tunnels:**

- For the velocity to increase the area muse increase.

Supersonic Wind tunnel:
Rocket Nozzle:

Reservoir: $P_0$, $T_0$, $\rho_0$  (flowing going into the wind tunnel)

Test Section: $P_{\text{exit}}$, $T_{\text{exit}}$, $\rho_{\text{exit}}$  (flowing moving out of the wind tunnel)

Area Mach Relations:

\[
\left( \frac{A}{A^*} \right)^2 = \frac{l}{m^2} \left[ \frac{2}{\gamma + 1} \left( 1 + \frac{\gamma - 1}{2} \frac{m^2}{l} \right) \right]^{\gamma + 1 / \gamma - 1}
\]

We are given a Mach number or an area ratio, we can get either of these values using the tables.

- Be careful using the table, your result will have two solutions and will depend on if the flow is subsonic or supersonic.
- Throat is the point where the smallest area of a wind tunnel or rocket nozzle can be found, but having a throat does not necessarily mean that you have a choke point where $M = 1$.
- $A^*$ is the area where the $M = 1$, this can be though of as a property of the flow like $P_0$, $T_0$, $\rho_0$. Even if there is not actual a physical point that this occurs, you can still solve for it and use this value to solve for other parameters.
- You will see indications that wind tunnel has a physical shock point. (that is if you are told that the flow goes from subsonic to supersonic)
- Even if there is a throat, but the Mach is not equal to one at the throat, then the flow will stay either subsonic or supersonic.
More types of Drag:

**Pressure Drag:** Mainly affects bluff bodies. It can cause high pressure upstream and low pressures downstream due to flow separation. This pressure acts perpendicular to the surface.

Flow Separation: Where the streamlines can no longer follow the curvature of the body.

Skin Friction Drag: Is produced by the friction of the air molecules with the surface of an object and creates a shear stress at the surface. This acts in the tangential direction.

Viscosity for the air at standard sea level temperature.

**SI:** $\mu = 1.7894 \times 10^{-5} \text{ kg/(m)(s)}$

**English:** $\mu = 3.7373 \times 10^{-7} \text{ slug/(ft)(s)}$

\[
\mu = 1.458 \left( \frac{T^{3/2}}{T + 11.04} \right) \times 10^{-6} \text{ kg/m.s}
\]

\[
\mu = 2.27 \left( \frac{T^{3/2}}{T + 199} \right) \times 10^{-8} \text{ slug/ft.s}
\]
**Reynolds Number:** A non-dimensional parameter that describes the behavior of viscosity. High Reynolds numbers indicates low viscosity and low Reynolds numbers indicate high viscosity.

\[ \text{Re} = \frac{\rho V x_c}{\mu} \]

Types of Flows:

**Laminar flow:** Streamlines are smooth and regular, and a fluid element moves smoothly along a streamline.

Boundary layer thickness

\[ \delta = \frac{5.2 x}{\sqrt{\text{Re}_x}} \]

Total Skin friction coefficient

\[ C_{\text{f, local}} = \frac{0.664}{\sqrt{\text{Re}_x}} \quad C_{\text{f, total}} = \frac{1.328}{\sqrt{\text{Re}_t}} \]

**Turbulent flow:** The streamlines break up and a fluid element moves in a random, irregular fashion.

Boundary layer thickness.

\[ \delta = \frac{0.37 x}{\text{Re}_x^{0.2}} \]

Total Skin friction coefficient

\[ C_{\text{f, local}} = \frac{0.0592}{(\text{Re}_x)^{0.2}} \quad C_{\text{f, total}} = \frac{0.074}{\text{Re}_t^{0.2}} \]

Laminar shear stress is less than the turbulent shear stress. Therefore, the skin friction is higher for turbulent flow. Turbulent boundary layer is thicker and grows faster.
The flow always starts out from the leading edge as laminar, and then at the transition point the boundary layer becomes completely turbulent where the boundary layer grows at a faster rate.

**Critical Reynolds Number:** This point where transition occurs is called the critical point, which corresponds to a critical Reynolds number.

\[ \text{Re}_{cr} = \frac{\rho U x_{cr}}{\mu} \]

**Airfoils**

For a 2D infinite airfoil (Cl,Cd,Cm) and a 3D finite wing (CL,CD,CM), the lift and drag coefficient are different.

This is because for an airfoil section, the end effects are removed when testing in a wing tunnel.

For a 3D wings these end effects produce a downward component called downwash. This causes an induced drag, which increases the total drag and reduces the lift.

Downwash causes the relative wind in the proximity of the airfoil section to be inclined slightly downward through a small angle called the induced angle of attack. This in turn reduces the angle of attack felt by the local airfoil section to a value smaller than the geometric angle of attack. This smaller angle of attack is called the effective angle of attack. The effective angle of attack for a 3D wing is equivalent to the geometric angle of attack for a 2D airfoil.

2 Main things going from 2D to 3D

1. Induced drag must be added to the finite wings:
2. The slope of the life curve for a finite wing is less than that for an infinite wing: \( \alpha < \alpha_0 \)
**Midterm 4:**

**Kepler’s Laws:**

Kepler’s 1st Law: All planets move about the sun in elliptical orbits, having the sun in one focus and the other focus is empty.

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

Kepler’s 2nd Law: A radius vector joining any planet to the Sun sweeps out equal areas in equal lengths of time.

Kepler’s 3rd Law: The square of the orbital period \( T \) of a planet is proportional to the cube of the semi-major axis \( R \) of its orbit: \( T^2 = kR^3 \)

- When moving in a central gravitational field (around a star or planet), one of four different trajectories will result:
  - circular orbit (Kepler’s laws apply)
  - elliptic orbit (Kepler’s laws apply)
  - parabola orbit
  - hyperbola orbit

**Orbital Parameters:**

- Most satellites as well as planetary orbits are elliptic. To uniquely define an elliptic orbit six parameters are needed:
  1. Semi-major axis (describes size and shape of orbit): \( a \)
2. Eccentricity (describes size and shape of orbit): \( e \)

\[
e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}
\]

3. Inclination (describes orientation of orbit): \( i \)

4. Right ascension of the ascending node (describes orientation of orbit): \( \Omega \)
5. Argument of perigee (describes the orientation of the semi-major axis in the orbital plane): $\omega$

![Diagram of celestial body and orbit parameters](image)

6. True anomaly (describes where the satellite is on the orbit): $\upsilon$

![Diagram of celestial body and orbit parameters](image)

**Vis Viva Equation:**

The orbital-energy-invariance law.

Specific total energy is constant throughout the orbit. If $a$ denotes Apoapsis and $p$ denotes Periapsis then,

\[
\mathcal{E} = \frac{v_a^2}{2} - \frac{GM}{r_a} = k^2 \left( \frac{2a - r_a}{2a r_a} - \frac{k^2}{r_a} \right) = -\frac{k^2}{2a}
\]

\[
\frac{v^2}{2} - \frac{k^2}{r} = -\frac{k^2}{2a}
\]

**vis-viva equation:** Velocity function of $r$: $\nu = f(r)$
The vis-viva equation provides a very useful relationship between the velocity and radius. Given radius $r$, we can $v$ and vice versa.
Example 1:

Compressed air flows through a convergent-divergent nozzle and exits supersonically. Given the throat and exit areas, as well as the stagnation pressure and temperature, determine the pressure, temperature, Mach number, the flow velocity at both the throat and exit of the nozzle. Assume the flow is isentropic throughout the entire nozzle.

\[ \rho^* = 0.25 \text{m}^2 \]

\[ P_0 = 500 \text{kPa} \]

\[ T_0 = 1000 \text{K} \]

\[ A_e = 0.5 \text{m}^2 \]

\[ P_e \]

\[ V_e \]

\[ T_e \]

\[ \rho^*, V^*, T^*, M^* \]

\[ M^* = 1 \text{ since Mach exits supersonically} \]

**Use Mach equations to solve at throat**

\[ \frac{P_0}{\rho^*} = \left[ 1 + \frac{\gamma-1}{2} M^*^2 \right]^{\frac{\gamma}{\gamma-1}} \]

\[ \rho^* = \frac{P_0}{\left[ 1 + \frac{\gamma-1}{2} M^*^2 \right]^{\frac{\gamma}{\gamma-1}}} = \frac{500}{\left[ 1 + \frac{1.4-1}{2} (1)^2 \right]^{1.4/1.4-1}} \]

\[ P^* = 264.15 \text{kPa} \]

\[ \frac{T_0}{T^*} = 1 + \frac{\gamma-1}{2} M^*^2 \]

\[ T^* = \frac{T_0}{1 + \frac{\gamma-1}{2} M^*^2} = \frac{1000}{1 + \frac{1.4-1}{2} (1)^2} \]

\[ T^* = 833.33 \text{K} \]

\[ a^* = \sqrt{\gamma R T^*} \]

\[ R = 287 \text{ J/kg K} \]
$$a_v = \sqrt{(1.4)(287)(533.32)} = 578.65 \text{ m/s}$$

$$M_v = \frac{V_v}{a_v} \quad \Rightarrow \quad 1 = \frac{V_v}{578.65 \text{ m/s}}$$

$$V_v = a_v = 578.65 \text{ m/s}$$

$$\left( \frac{Ae}{A_u} \right)^2 = \frac{1}{\mu_e^2} \left[ \frac{2}{\gamma+1} \left( 1 + \frac{\gamma-1}{2} \mu_e^2 \right) \right]^{\frac{\gamma+1}{\gamma-1}}$$

$$\frac{Ae}{A_u} = \frac{0.5}{0.25} = 2 \quad \Rightarrow \quad \text{USE ISENTROPIC MACH TABLES}$$

**LINEARLY INTERPOLATE**

$$\frac{2.2 - M_u}{2.2 - 1.15} = \frac{2.005 - 2}{2.005 - 1.919}$$

$$M_u = 2.197$$

$$\frac{P_o}{P_e} = \left[ 1 + \frac{\gamma-1}{2} \mu_e^2 \right]^{\frac{\gamma}{\gamma-1}} \quad \Rightarrow \quad 46.98 \text{ kPa}$$

$$\frac{T_o}{T_e} = 1 + \frac{\gamma-1}{2} \mu_e^2 \quad \Rightarrow \quad 508.51 \text{ K}$$

$$a_e = \sqrt{gRT_e} \quad \Rightarrow \quad 452.15 \text{ m/s}$$

$$V_e = M_e a_e \quad \Rightarrow \quad 993.37 \text{ m/s}$$
Example 2:

Consider a spacecraft in orbit around the Earth with a flat rectangular solar array that is always pointed such that the side covered with solar cells is facing the sun and the side without solar cells is facing deep space. If the solar array is 2 m wide by 2 m long, with $\alpha = 0.89$ and $\varepsilon = 0.81$, determine the steady-state temperature of the solar panel modeled as a black body.

\[ T = \frac{4 \alpha \frac{\text{Width} \times \text{Length}}{\text{Width} \times \text{Length}}} {\varepsilon \frac{I_{\text{sun}}}{\sigma}} \]

\[ \sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4 \]

\[ I_{\text{sun}} = 1394 \text{ W/m}^2 \]

To find $I_{\text{in}}$, we need to consider the surface area that is facing the sun (receiving incoming radiation):

\[ I_{\text{in}} = \text{Width} \times \text{Length} \]

\[ I_{\text{in}} = 2m \times 2m = 4m^2 \]

To find $I_{\text{out}}$, we need to consider the surface area that is radiating energy outward (both sides of the solar array are radiating energy):

\[ I_{\text{out}} = 2[\text{Width} \times \text{Length}] \]

\[ I_{\text{out}} = 2[4m^2] = 8m^2 \]
Now, we can substitute all of the values we have to find the steady-state temp:

\[ T = \sqrt[4]{\frac{a \sin \frac{I_{\text{sun}}}{\sigma}}{2 \frac{S_{\text{out}}}{\sigma}}} \]

\[ T = \sqrt[4]{\frac{0.89 \frac{4}{0.81} \frac{1394}{5.67 \times 10^{-8}}}{} \]

\[ T = 217.54 \text{ K} \]
Example 3:
Alfred the Elf is hired onto Santa’s flight team to help improve the aerodynamics on his sled. Alfred is tasked with building and attaching a manometer onto Santa’s sled so they can collect flight data. At a point during the flight the manometer reads as follows, calculate the pressure in Container A.

\[ \rho_{\text{hot coco}} = 710 \frac{kg}{m^3} \]
\[ \rho_{\text{air}} = 1.225 \frac{kg}{m^3} \]
\[ \rho_{\text{milk}} = 1035 \frac{kg}{m^3} \]

\[ \Delta P = -\rho_{\text{air}} g \Delta h \]

1. \[ P_i = P_{\text{amb}} - \rho_{\text{milk}} g \Delta h \]
2. \[ P_i = 101325 - 1035(9.81)(0.28 - 0.45 \sin(40)) \]
3. \[ P_i = 101418.9634 \text{Pa} \]

* ERROR IN REVIEW (VIDEO)

* ASSUME \( P_{\text{amb}} = 101325 \text{ Pa} \)
2. \( P_2 = P_1 - P_{\text{air}} g \Delta h \)
\[ P_2 = 101418 - 1.225(9.81)(0.11 - 0.28) \]
\[ P_2 = 101,420.05 \text{ Pa} \]

3. \( P_3 = P_2 - P_{\text{horiz}} g \Delta h \)
\[ P_3 = 101420.05 - 710(9.81)(0.30 - 0.11) \]
\[ P_3 = 100096.6 \text{ Pa} \]

Because no height ratio was given
For \( P_a \), assume equilibrium.
\[ P_3 = P_{\text{amb}} = 100096.6 \text{ Pa} \]