A Categorification of Biquandle Brackets

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Biquandles

A *biquandle* is a set X with two binary operations $\underline{\triangleright}, \overline{\triangleright}$ such that for every $x, y, z \in X$,

(i) $x \,\underline{\triangleright}\, x = x \,\overline{\triangleright}\, x$

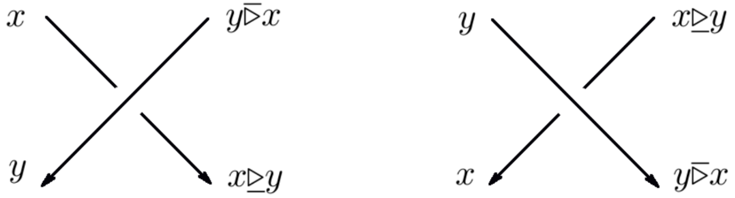
(ii) The maps $\alpha_y(x) = x \,\overline{\triangleright}\, y$, $\beta_y(x) = x \,\underline{\triangleright}\, y$, and $S(x, y) = (y \,\overline{\triangleright}\, x, x \,\underline{\triangleright}\, y)$ are invertible.

(iii) The following exchange laws are satisfied:

$$
\begin{aligned}
(x \,\underline{\triangleright}\, y) \,\underline{\triangleright}\, (z \,\underline{\triangleright}\, y) &= (x \,\underline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z) \\
(x \,\underline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\underline{\triangleright}\, y) &= (x \,\overline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z) \\
(x \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\overline{\triangleright}\, y) &= (x \,\overline{\triangleright}\, z) \,\overline{\triangleright}\, (y \,\underline{\triangleright}\, z).
\end{aligned}
$$

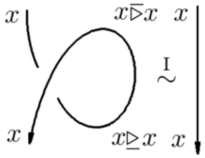
Why Biquandles?

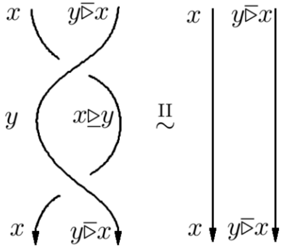
Suppose we color a link diagram with elements of a biquandle so that the following relations are satisfied at each crossing:

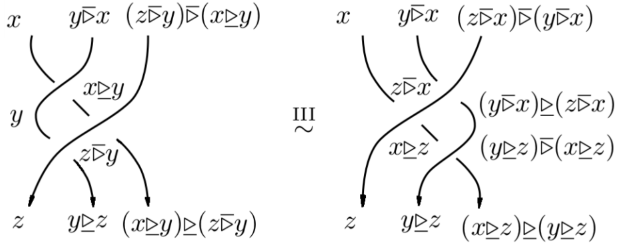


Then the biquandle axioms correspond to the invariance of such a coloring under the Reidemeister moves.

Why Biquandles?







Examples of Biquandles

- *The trivial biquandle*: $X = \{x\}$, with $x \,\overline{\triangleright}\, x = x \,\underline{\triangleright}\, x = x$.
- *Constant action biquandles*: X is any set, $\sigma : X \to X$ is any bijection. Let $x \,\underline{\triangleright}\, y = x \,\overline{\triangleright}\, y = \sigma(x)$ for all $x, y \in X$.
- *Alexander biquandles*: X is any $\mathbb{Z}\left[t^{\pm 1}, r^{\pm 1}\right]$-module. Let $x \,\underline{\triangleright}\, y = tx + (r - t)\, y$ and $x \,\overline{\triangleright}\, y = ry$.

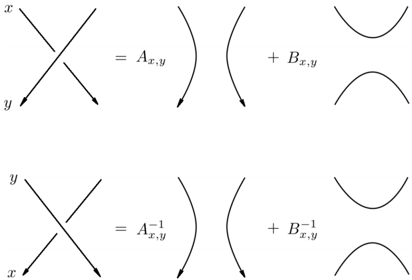
Biquandle Counting Invariant

Let X be a biquandle. The number of X-colorings of a link diagram is an invariant, called the biquandle counting invariant.

If L is a link, the X-counting invariant of L is $\Phi_X^{\mathbb{Z}}(L)$.

Biquandle Brackets (Nelson et. al., 2017)

Let X be a biquandle and R a commutative unital ring. We would like to choose elements $A_{x,y}, B_{x,y} \in R^\times$ (for each $x, y \in X$), $w \in R^\times$ and $\delta \in R$ such that the element of R determined by the skein relations



with δ the value of a circle and w the value of a positive kink is an invariant of X-colored links.

Biquandle Brackets (Nelson et. al., 2017)

Such a collection of elements of R is called a *biquandle bracket* and must satisfy the following axioms.

(i) For all $x \in X$, $\delta A_{x,x} + B_{x,x} = w$ and $\delta A_{x,x}^{-1} + B_{x,x}^{-1} = w^{-1}$.

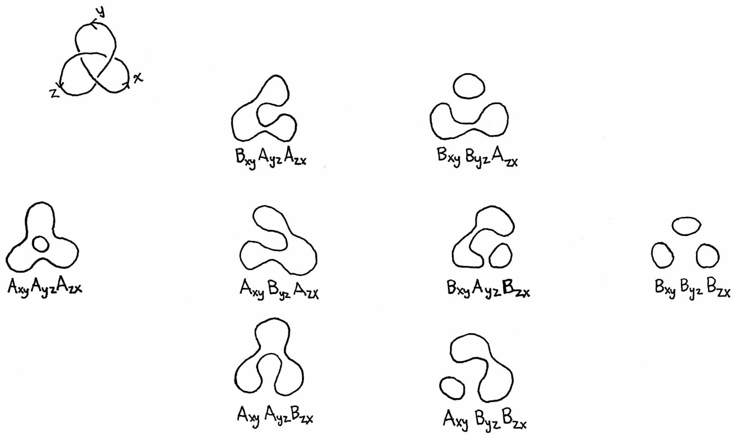
(ii) For all $x, y \in X$, $\delta = -A_{x,y} B_{x,y}^{-1} - A_{x,y}^{-1} B_{x,y}$.

(iii) For all $x, y, z \in X$, all of the following equations hold.

$$
\begin{aligned}
A_{x,y} A_{y,z} A_{x \underline{\triangleright} y, z \overline{\triangleright} y} &= A_{x,z} A_{y \overline{\triangleright} x, z \overline{\triangleright} x} A_{x \underline{\triangleright} z, y \underline{\triangleright} z}, \\
A_{x,y} B_{y,z} B_{x \underline{\triangleright} y, z \overline{\triangleright} y} &= B_{x,z} B_{y \overline{\triangleright} x, z \overline{\triangleright} x} A_{x \underline{\triangleright} z, y \underline{\triangleright} z}, \\
B_{x,y} A_{y,z} B_{x \underline{\triangleright} y, z \overline{\triangleright} y} &= B_{x,z} A_{y \overline{\triangleright} x, z \overline{\triangleright} x} B_{x \underline{\triangleright} z, y \underline{\triangleright} z}, \\
A_{x,y} A_{y,z} B_{x \underline{\triangleright} x, z \overline{\triangleright} y} &= A_{x,z} B_{y \overline{\triangleright} x, z \overline{\triangleright} z} A_{x \underline{\triangleright} z, y \underline{\triangleright} z} + A_{x,z} A_{y \overline{\triangleright} x, z \overline{\triangleright} x} B_{x \underline{\triangleright} z, y \underline{\triangleright} z} \\
&\quad + \delta A_{x,z} B_{y \overline{\triangleright} x, z \overline{\triangleright} x} B_{x \underline{\triangleright} z, y \underline{\triangleright} z} + B_{x,z} B_{y \overline{\triangleright} x, z \overline{\triangleright} x} B_{x \underline{\triangleright} z, y \underline{\triangleright} z}, \\
B_{x,z} A_{y \overline{\triangleright} x, z \overline{\triangleright} x} A_{x \underline{\triangleright} z, y \underline{\triangleright} z} &= B_{x,y} A_{y,z} A_{x \underline{\triangleright} y, z \overline{\triangleright} y} + A_{x,y} B_{y,z} A_{x \underline{\triangleright} y, z \overline{\triangleright} y} \\
&\quad + \delta B_{x,y} B_{y,z} A_{x \underline{\triangleright} y, z \overline{\triangleright} y} + B_{x,y} B_{y,z} B_{x \underline{\triangleright} y, z \overline{\triangleright} y}.
\end{aligned}
$$

Note that w and δ are determined by A and B.

Biquandle Brackets (Nelson et. al., 2017)



$$\beta(f) = w^{-3}\left(\delta^2 A_{x,y}A_{y,z}A_{z,x} \begin{array}{l} +\delta B_{x,y}A_{y,z}A_{z,x} \\ +\delta A_{x,y}B_{y,z}A_{z,x} \\ +\delta A_{x,y}A_{y,z}B_{z,x} \end{array} \begin{array}{l} +\delta^2 B_{x,y}B_{y,z}A_{z,x} \\ +\delta^2 B_{x,y}A_{y,z}B_{z,x} \\ +\delta^2 A_{x,y}B_{y,z}B_{z,x} \end{array} +\delta^3 B_{x,y}B_{y,z}B_{z,x} \right)$$

Biquandle Brackets (Nelson et. al., 2017)

If $\beta = (A, B)$ is an X-bracket taking values in a ring R, the value on a link L of the link invariant defined by β is the multiset

$$\Phi_X^\beta(L) = \{\beta(f) : f \text{ is an } X\text{-coloring of } L\}.$$

Note that Φ_X^β enhances $\Phi_X^{\mathbb{Z}}$ since $\left|\Phi_X^\beta(L)\right| = \Phi_X^{\mathbb{Z}}(L)$.

Examples of Biquandle Brackets

- *The Jones Polynomial*: Let $X = \{x\}$ be the trivial biquandle, let $R = \mathbb{Z}\left[q^{\pm 1}\right]$, and let $A_{x,x} = q$ and $B_{x,x} = q^{-1}$.
- Let X be the constant action biquandle with set $X = \{1, 2\}$ and action $\sigma = (1\ 2)$. Then the following matrix defines a biquandle bracket taking values in $\mathbb{Z}_5$:

$$[A \mid B] = \left[\begin{array}{cc|cc} 1 & 3 & 4 & 2 \\ 4 & 1 & 1 & 4 \end{array}\right]$$

- There are a lot of other biquandle brackets!

A Brief Digression...

If H is a group and S is a commutative ring, an H-graded S-module is a direct sum $M = \bigoplus_{h \in H} M_h$, where M_h is an S-module for each $h \in H$. For any $a \in M_h$, we say $\deg(a) = h$.

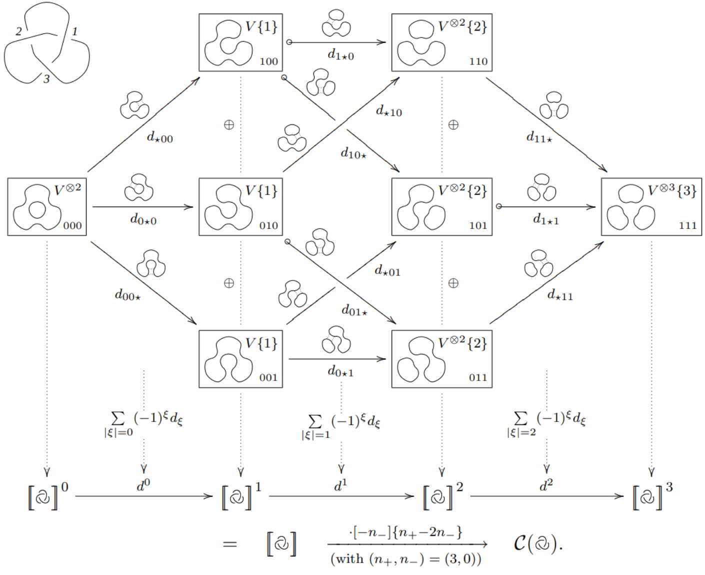
The graded dimension of M is $\operatorname{rdim}(M) = \sum_{h \in H} h \operatorname{rank}(M_h)$.

A cochain complex of H-graded S-modules is a sequence $C = \left(C^i\right)_{i \in \mathbb{Z}}$ of H-graded S-modules along with differentials $d^i : C^i \to C^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all $i \in \mathbb{Z}$.

The Euler Characteristic of C is $\chi(C) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{rdim}\left(C^i\right)$.

Note: if d is degree-preserving, then $\chi(\mathcal{H}) = \chi(C)$, where $\mathcal{H}$ is the cohomology of C.

Khovanov Homology (Khovanov, 2000)



(image from Bar-Natan, 2001)

A "Categorification" of Biquandle Brackets

Begin with $\beta = (A, B)$, an X-bracket taking values in R.

Let $q_{x,y} = -\frac{B_{x,y}}{A_{x,y}}$ for all $x, y \in X$.

Let x_0 be some distinguished element of X, and let $q = q_{x_0,x_0}$.

Let G be the group $\left\langle qq_{x,y}^{-1} : x, y \in X \right\rangle \leq R^\times$.

Let S be the $R^\times$-graded group algebra $\mathbb{Z}[G]$, with the $R^\times$-grading given by $\deg(g) = g$ for all $g \in G$.

A "Categorification" of Biquandle Brackets

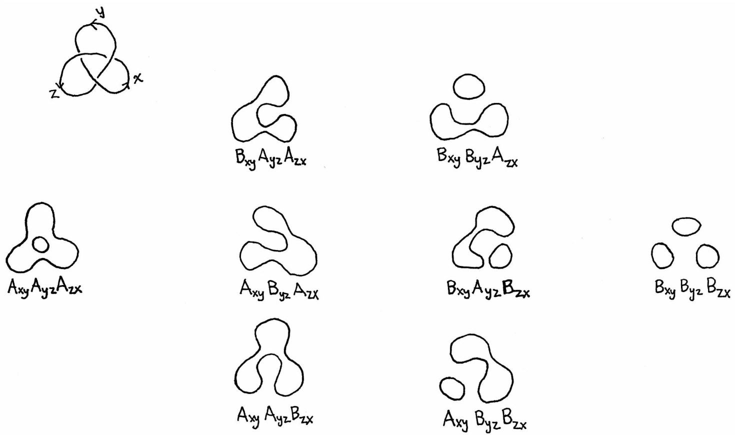
Let M be the $R^{\times}$-graded S-module $S\left[t\right]/\left(t^2\right)$ with the additional grading given by $\deg(1) = q$ and $\deg(t) = q^{-1}$.

M is a Frobenius algebra with the following multiplication and comultiplication operations:

$$\begin{aligned} m &: M \otimes M \to M \\ m &: 1 \otimes 1 \mapsto 1, \qquad 1 \otimes t \mapsto t, \\ &\quad t \otimes 1 \mapsto t, \qquad t \otimes t \mapsto 0 \end{aligned}$$

$$\begin{aligned} \Delta &: M \to M \otimes M \\ \Delta &: 1 \mapsto 1 \otimes t + t \otimes 1, \qquad t \mapsto t \otimes t \end{aligned}$$

A "Categorification" of Biquandle Brackets

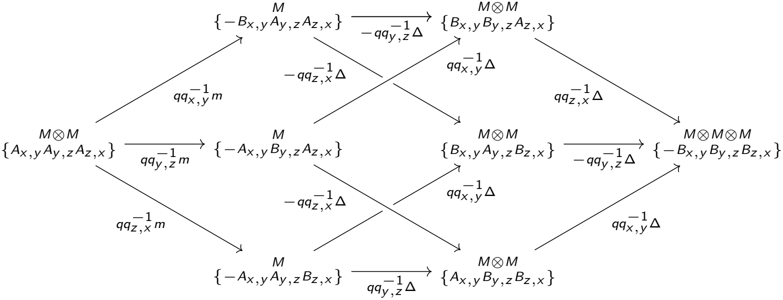


Let L be a link and let f be an X-coloring of L...

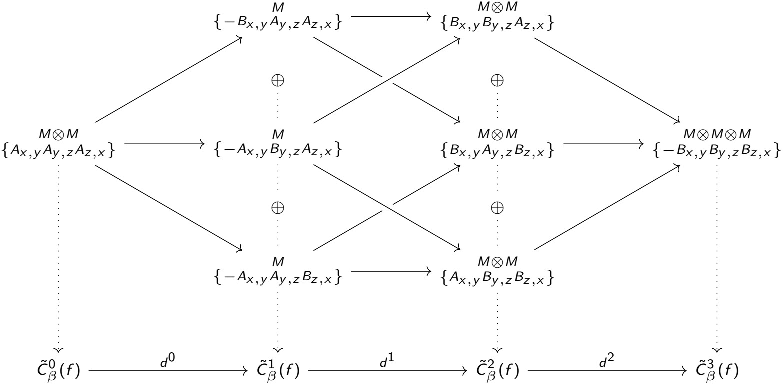
A "Categorification" of Biquandle Brackets

$$\begin{matrix}
 & M\{-B_{x,y}A_{y,z}A_{z,x}\} & M\otimes M\{B_{x,y}B_{y,z}A_{z,x}\} & \\
M\otimes M\{A_{x,y}A_{y,z}A_{z,x}\} & M\{-A_{x,y}B_{y,z}A_{z,x}\} & M\otimes M\{B_{x,y}A_{y,z}B_{z,x}\} & M\otimes M\otimes M\{-B_{x,y}B_{y,z}B_{z,x}\} \\
 & M\{-A_{x,y}A_{y,z}B_{z,x}\} & M\otimes M\{A_{x,y}B_{y,z}B_{z,x}\} &
\end{matrix}$$

A “Categorification” of Biquandle Brackets



A “Categorification” of Biquandle Brackets



A "Categorification" of Biquandle Brackets

Now we have $\tilde{C}_\beta(f)$, an $R^\times$-graded cochain complex.

Let $C_\beta(f)$ be the shifted cochain complex,

$$C_\beta(f) = \tilde{C}_\beta(f)\,[n_-]\left\{(-1)^{n_-}\, w^{-n_+} w^{n_-}\right\},$$

where $n_\pm$ is the number of positive/negative crossings in the link.

Finally, take the cohomology of $C_\beta(f)$ to obtain a complex $\mathcal{H}_\beta(f)$.

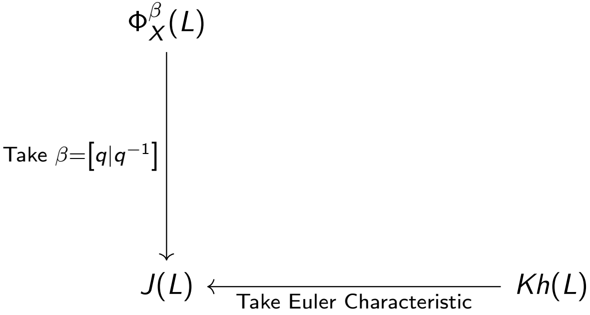
The multiset

$$Bh_\beta(L) = \{\mathcal{H}_\beta(f) : f \text{ is an } X\text{-coloring of } L\}$$

is an invariant of links.

Results

What we had:



Results

What we want:

$$
\begin{array}{ccc}
\Phi_X^\beta(L) & \xleftarrow{\text{Take Euler Characteristic}} & \bigstar_\beta(L) \\
\Big\downarrow {\scriptstyle \text{Take } \beta=[q|q^{-1}]} & & \Big\downarrow {\scriptstyle \text{Take } \beta=[q|q^{-1}]} \\
J(L) & \xleftarrow[\text{Take Euler Characteristic}]{} & Kh(L)
\end{array}
$$

Results

What we got:

$$
\begin{array}{ccc}
\Phi_X^{\beta}(L) & \xleftarrow{\text{Take Euler Characteristic}} & Bh_{\beta}(L) \\
\Big\downarrow \text{Take } \beta=[q|q^{-1}] & & \Big\downarrow \text{Take } \beta=[q|q^{-1}] \\
J(L) & \xleftarrow[\text{Take Euler Characteristic}]{} & Kh(L)
\end{array}
$$

Results

The Euler characteristic of $Bh_\beta(L)$ is actually $\mathrm{rdim}(S) \cdot \Phi_X^\beta(L)$.

This factor of $\mathrm{rdim}(S)$ cannot always be removed to yield $\Phi_X^\beta(L)$.

In fact, $Bh_\beta(L)$ is isomorphic to a quotient of $Kh(L)$ shifted by

$$\mathrm{rdim}(S) \cdot \prod_{\tau^+} \left(A_{x,y} A_{x_0,x_0}^{-1}\right) \cdot \prod_{\tau^-} \left(B_{x,y}^{-1} B_{x_0,x_0}\right).$$

This shift is the value of a *biquandle 2-cocycle invariant* which takes values in $R^\times/G$.

Further Questions

Does the $\bigstar_\beta(L)$ invariant exist?

How does the power of $Bh_\beta(L)$ compare to $Kh(L)$ and $\Phi_X^\beta(L)$?

More specifically, how does the power of the biquandle 2-cocycle invariant discussed above compare to $\Phi_X^\beta(L)$?