Introduction

Exponential random graph models (ERGMs) are families of distributions defined by a set of network statistics and, thus, give rise to interesting graph theoretic questions. Our research focuses on the ERGMs where the edge, 2-path, and triangle counts are the sufficient statistics. These models are useful for modeling networks with a transitivity effect such as social networks. For statistical tests, given an observed network G, one would like to understand the set of simple graphs with the same edge, 2-path, and triangle counts as G. This set is called the fiber of G and are the 0-1 points on an algebraic variety, which we refer to as the reference variety. The goal of this project is to understand the geometry of the reference variety.

In particular, our poster focuses on understanding graphs that are singular points of the reference variety. This direction has led us to discover a connection between graph automorphisms and graphs with deficient Jacobian.

Background and Notation

Let G be a graph with n vertices. Let A be the adjacency matrix of the graph, i.e.

\[ A_{ij} = \begin{cases} x_{ij} & \text{if } i < j \\
0 & \text{if } i = j \\
x_{ji} & \text{if } i > j. \end{cases} \]

The sufficient statistics of the triangle-2-star-edge model are:
- The number of edges of G: \( y_1 = \sum_{i<j} A_{ij} \)
- The number of 2-paths in G: \( y_2 = \sum_{i<j} (A^2)_{ij} \)
- The number of triangles in G: \( y_3 = \frac{1}{6} \tr(A^3) \)

Given a graph G and sufficient statistic t(G) = \( (y_1, y_2, y_3) \), the reference ideal of t(G) is

\[ I_t = \left( \sum_{i<j} x_{ij} - y_1, \sum_{i<j<k} x_{ijk} - 2y_2, \sum_{i<j<k<l} x_{ijkl} - 6y_3 \right) \subseteq \mathbb{k}[x_{ij}]_{1 \leq i < j \leq n} \]

Definition 1.1: Let k be a field, and let \( f_1, \ldots, f_s \) be polynomials in \( k[x_{ij}], \) then we define

\[ V(I) := \{ (a_1, \ldots, a_s) \in k^s : f_i(a_1, \ldots, a_s) = 0 \text{ for all } f_i \in I \}. \]

We call \( V(f_1, \ldots, f_t) \) the affine variety defined by \( f_1, \ldots, f_t. \)

We will call the variety defined by \( I_t \) the reference variety:

\[ V_t = V(I_t) \]

Example 1 Consider the graph G pictured.

1. The number of edges of G:
\[ y_1 = \sum_{i<j} A_{ij} = 4. \]
2. The number of 2-paths in G:
\[ y_2 = \sum_{i<j} (A^2)_{ij} = 5. \]
3. The number of triangles:
\[ y_3 = \frac{1}{6} \tr(A^3) = 1. \]

The reference ideal of G is:
\[ I_t = \langle x_{12} + x_{13} + x_{23} + x_{14} + x_{24} + x_{34} - 4, \]
\[ x_{12}x_{34} + x_{12}x_{23} + x_{12}x_{24} + x_{13}x_{24} + x_{13}x_{23} + x_{13}x_{14} + x_{14}x_{24} + x_{14}x_{23} + x_{14}x_{13} + x_{24}x_{23} + x_{24}x_{23} + x_{24}x_{13} + x_{24}x_{13} + x_{24}x_{13} + x_{24}x_{13} + x_{24}x_{13} - 10, \]
\[ x_{12}x_{34}x_{23} + x_{12}x_{13}x_{24} + x_{12}x_{13}x_{14} + x_{12}x_{23}x_{14} - 6 \rangle. \]

Using Macaulay2, we get the following results:
1. The dimension of the reference ideal \( I_t \) is: \( \dim(I_t) = 3. \)
2. The degree of the reference ideal \( I_t \) is: \( \deg(I_t) = 6. \)
3. The variety \( V_t \) defined by \( I_t \) is irreducible.

Dimension

Theorem: Let G be a graph with sufficient statistic t(G) = \( (y_1, y_2, y_3) \). If \( y_1^2y_2 - 4y_1^3 - 4y_2^3 - 18y_1y_2^2 - 27y_3^2 \neq 0, \) then \( \dim(V_t) = \left( \begin{array}{c} n \\ 2 \end{array} \right) - 3. \)

One way of understanding the dimension is with the Jacobian of \( V_t. \)

Definition 2.1: Given a graph \( G = (V, E) \) with adjacency matrix \( A \) and two vertices \( r, s \in V \), let the joint degree of \( r \) and \( s \) in \( G \) be \( \text{jointdeg}_{rs} := \deg r + \deg s - 2A_{rs}. \)

The Jacobian of \( V_t \), denoted \( Jac(V_t) \), is an \( (n^2 - n) \times 3 \) matrix whose rows are indexed by pairs \( (r, s) \) with \( 1 \leq r, s \leq n \) and \( r \neq s \). The \( (r, s) \)-th row of \( Jac(V_t) \) has the following form:

\[ [1, \text{jointdeg}_{rs}, \# \text{ of 2-paths between } r \text{ and } s]. \]

Understanding Singularities

Question: When is \( G \) a singular point of \( V_t \)? We know that \( G \) is a singular when the rank \( Jac(V_t)_{|G} < 3 \). Can we describe such graphs?

Definition 3.1: An automorphism of a graph \( G \) is an isomorphism from \( G \) to itself.

Proposition 3.2: Let \( G \) be a graph. If the automorphism group of \( G \) is the full symmetric group, then rank \( Jac(V_t)_{|G} = 1. \)

Proof: Suppose not. Suppose rank \( Jac(V_t)_{|G} \neq 1 \), thus there are at least two distinct rows, \( s \) and \( u, v \) in \( Jac(V_t)_{|G} \).

Case 1: The number of 2-paths between \( r, s \) and \( u, v \) are different. After applying the permutation, the number of 2-paths between \( r, s \) and \( u, v \) changes. This is a contradiction, thus the automorphism group of \( G \) is the full symmetric group.

Case 2: The joint degrees between \( r, s \) and \( u, v \) are different. Show that the number of 2-paths between \( r, s \) and \( u, v \) change after applying the permutation. Since graph automorphism preserves the degree of vertices and the edges between vertices, this is a contradiction.

In both cases, \( \sigma \) is not a graph automorphism. This is a contradiction to our first assumption thus \( r, s \) and \( u, v \) must be the same. Therefore, \( Jac(V_t)_{|G} = 1. \)

Irreducibility

Conjecture: \( V_t \) is irreducible.

We have shown this is true for all 4, 5 and some 6 vertex graphs. We have not shown that all graphs with 6 vertex are irreducible; however, based on our observations we believe they will also be irreducible.

References