Thermodynamic Formalism for Finite Horizon Sinai Billiards

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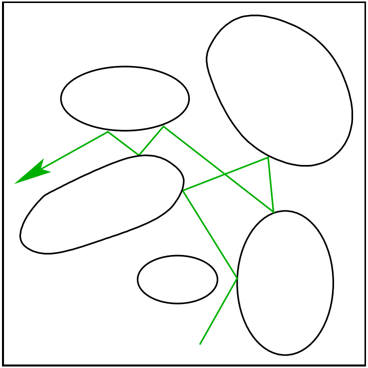
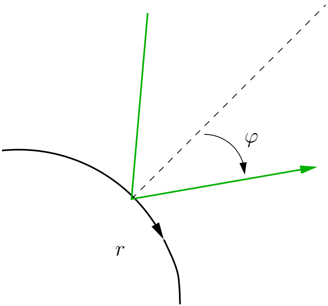
Finite Horizon Sinai Billiard

- Billiard table $\mathcal{Q} = \mathbb{T}^2 \backslash \cup_i B_i$; scatterers B_i.
- Boundaries of scatterers are $\mathcal{C}^3$ and have strictly positive curvature.
- Billiard flow is given by a point particle moving at unit speed with elastic collisions at the boundary

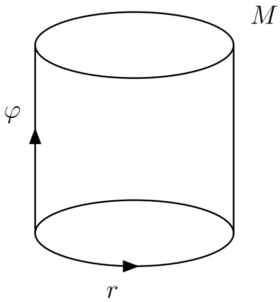


Assume a **finite horizon** condition: there is an upper bound on the free flight time between collisions.

The Associated Billiard Map



- $r =$ position coordinate oriented clockwise on boundary of scatterer ∂B_i
- $\varphi =$ angle outgoing trajectory makes with normal to scatterer



$M = \big(\cup_i \, \partial B_i \big) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, the natural "collision" cross-section for the billiard flow.

$T : (r, \varphi) \to (r', \varphi')$ is the first return map: the billiard map.

- a hyperbolic map with singularities

Statistical Properties

T preserves a smooth invariant measure on M, $\mu_{\mathsf{SRB}} = \cos\varphi \, dr \, d\varphi$

With respect to this measure, many statistical properties have been proved using a variety of techniques.

With respect to μ_{SRB}, T:

- is ergodic [Sinai '70] and Bernoulli [Gallavotti, Ornstein '74]
- enjoys exponential decay of correlations [L.S. Young '98]
- satisfies many limit theorems:
  - Central Limit Theorem [Bunimovich, Sinai '81]
  - Almost-sure invariance principle [Melbourne, Nicol '05],
  - Local moderate and large deviations [Melbourne, Nicol '08], [Young, Rey-Bellet '08]

Motivation for Present Work

Very few studies of other invariant measures for T.

One place to begin: **Equilibrium states**.

Given a function ϕ, an invariant measure μ for T is called an equilibrium state for ϕ if

$$h_\mu(T) + \int \phi \, d\mu = P(\phi) := \sup \left\{ h_\nu(T) + \int \phi \, d\nu : \nu \text{ invariant prob} \right\}$$

For Hölder continuous ϕ, the existence and uniqueness of equilibrium states has been established for

- uniformly hyperbolic systems (Anosov and Axiom A) [Sinai '72], [Bowen '74], [Ruelle '78]
- nonuniformly hyperbolic maps and flows
  - Markov partitions [Sarig '11], [Lima, Mattheus '18], [Buzzi, Crovisier, Sarig '19]
  - non-uniform specification [Climenhaga, Thompson '13], [Burns, Climenhaga, Fisher, Thompson '18]

For billiards, the theory is virtually unexplored.

Geometric Potentials

Important family of potentials: geometric potentials,

$$t\phi = -t \log J^u T, \quad t \in \mathbb{R}\,.$$

- $t = 1$ gives the smooth invariant measure μ_{SRB}. This is an equilibrium state for ϕ and uniqueness is proved in a class of measures whose support decays sufficiently near singularities [Katok, Strelcyn '86].
- $t = 0$ yields the measure of maximal entropy [Baladi, D. '20]. This is Bernoulli (and hence mixing) and globally unique, but its rate of mixing is not known.
- $t < 0$ implies $P(t) = \infty$ since $J^u T$ is unbounded near tangential collisions. Restrict to $t > 0$.
- [Chen, Wang, Zhang] proves existence (but not uniqueness) of equilibrium state for t near 1 using Young towers.

Goals for Present Work

- Connect the results for $t = 0$ and $t = 1$ by proving existence and uniqueness of equilibrium states for $-t \log J^u T$ for $t \in (0, t_*)$ for some $t_* > 1$.
- Prove the analyticity of $P(t)$ on this interval.

Main Tool: Transfer operator

$$\mathcal{L}_t f = \frac{f \circ T^{-1}}{(J^s T)^{1-t} \circ T^{-1}}$$

Construct equilibrium state μ_t out of left and right eigenvectors of $\mathcal{L}_t$ corresponding to the eigenvalue of maximum modulus.

Sources of difficulty:

- T has discontinuities so a topological definition of pressure must overcome the effect of this cutting.
- The potential is not Hölder continuous
  - $J^s T \approx \cos \varphi$ so the potential is unbounded
  - $J^s T$ is not continuous on any open set

Weight Function for Topological Pressure

To control the evolution of $\mathcal{L}_t^n f$, must control integrals of the type,

$$\int_W \mathcal{L}_t^n f \, \psi \, dm_W = \int_{T^{-n}W} f \, \psi \circ T^n \, |J^s T^n|^t \, dm_{T^{-n}W} \, .$$

- $W \in \mathcal{W}^s$, the set of local stable manifolds with uniformly bounded curvature
- m_W is arclength measure on W
- $\psi \in C^\alpha(W)$ is a Hölder continuous test function
- f is an element of our Banach space (closure of $C^1(M)$ in some norm)

$T^{-n}W = \cup_i W_i$, smooth, connected, homogeneous components.

We need to estimate precisely how $\sum_{W_i} |J^s T^n|^t_{C^0(W_i)}$ grows as a function of n and W.

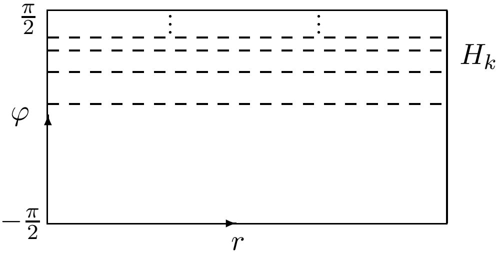
Hyperbolicity and Complexity for the Sinai Billiard

- Tangential collisions create singularity curves for T where DT blows up, $\|DT_x\| \sim \dfrac{1}{\cos\varphi(Tx)} \sim d(x, \mathcal{S}_1)^{-1/2}$
- T is uniformly hyperbolic away from singularity curves
  - Two invariant families of stable and unstable cones, uniformly transverse to one another
  - Unstable cones contract under DT; Stable cones contract under DT^{-1}
- To control distortion, a standard technique is to consider boundaries of **homogeneity strips** $H_{\pm k}$ as part of an expanded singularity set,

$$H_k = \left\{ \frac{\pi}{2} - \frac{1}{k^q} < \varphi < \frac{\pi}{2} - \frac{1}{(k+1)^q} \right\}$$

and similarly for H_{-k}, $|k| \geq k_0$. q will depend on $t > 0$.

Hyperbolicity and Complexity for the Sinai Billiard



- The singularity set becomes countable:
  $\mathcal{S}_0^{\mathbb{H}} := \mathcal{S}_0 \cup \left(\cup_{|k| \geq k_0} \partial H_k\right)$
- Need a **complexity bound**: expansion beats cutting due to singularities
- To achieve this, we choose q such that $qt \geq 2$.

Complexity Bound

Modification of the one-step expansion due to Chernov.

For $V \in \mathcal{W}^s$, let V_i denote the homogeneous connected components of $T^{-1}V$.

Lemma 1

There exists $\theta < 1$ such that if $qt \geq 2$, then for all $V \in \mathcal{W}^s$,

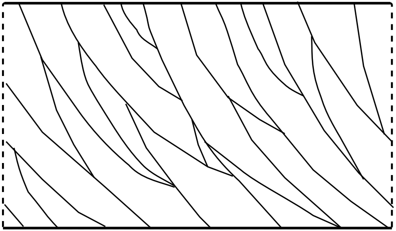
$$\limsup_{\delta \downarrow 0} \sup_{|V| \leq \delta} \sum_{V_i} |J^s T|^t_{C^0(V_i),*} < \theta^t$$

*The * denotes an adapted metric for the contraction.*

The lemma holds since a short stable curve can be cut by at most $\tau_{\max}/\tau_{\min}$ tangential collisions under T^{-1}. Then k_0 can be chosen large enough to make θ^{-1} arbitrarily close to the hyperbolicity constant $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$.

A Definition of Topological Pressure

- Define $\mathcal{S}_n = \cup_{i=0}^n T^{-i}\mathcal{S}_0$,
  $\mathcal{S}_n^{\mathbb{H}} = \cup_{i=0}^n T^{-i}\mathcal{S}_0^{\mathbb{H}}$
- Let $\mathcal{M}_0^n$ = connected components of $M \setminus \mathcal{S}_n$,
- $\mathcal{M}_0^{n,\mathbb{H}}$ = connected components of $M \setminus (\mathcal{S}_{n-1}^{\mathbb{H}} \cup T^{-n}\mathcal{S}_0)$



$M \setminus \mathcal{S}_n$

Define for $t > 0$,

- $$Q_n(t) := \sum_{A \in \mathcal{M}_0^{n,\mathbb{H}}} \sup_A |J^s T^n|^t$$
- $$P_*(t) := \lim_{n\to\infty} \frac{1}{n} \log Q_n(t)$$
- The limit exists since the sequence $\log Q_n(t)$ is subadditive: $Q_{n+k}(t) \le Q_n(t)Q_k(t)$. It follows, $Q_n(t) \ge e^{nP_*(t)}$.

Properties of $P_*(t)$ and Variational Inequality

Theorem 1

For a finite horizon Sinai billiard:

- $P_*(t)$ is a convex, continuous, decreasing function of t for all $t > 0$;
- $P_*(t)$ satisfies a variational inequality,

$$P_*(t) \geq P(t) = \sup \left\{ h_\mu(T) - t \int \log J^u T \, d\mu : \mu \ T\text{-inv. prob.} \right\}$$

To prove that in fact $P_*(t) = P(t)$ for $t \in (0, t_*)$ for some $t_* > 1$, we will:

- Construct an appropriate Banach space $\mathcal{B}$ on which the transfer operator $\mathcal{L}_t$ has spectral radius $e^{P_*(t)}$;
- Use its maximal eigenvectors to construct an invariant measure μ_t whose pressure equals $P_*(t)$.

Definition of Norms: Weak Norm

Norms similar to those used in [D., Zhang '13], modified for t.
$\mathcal{W}^s$ = set of local stable manifolds for T

Fix $0 < \alpha \le 1/(q+1)$.

For $f \in C^1(M)$, define the **weak norm** of f by

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^\alpha(W) \\ |\psi|_{\mathcal{C}^\alpha(W)} \le 1}} \int_W f \, \psi \, dm_W \, .$$

Define $\mathcal{B}_w$ to be the completion of $C^1(M)$ in the $|\cdot|_w$ norm.

Remark: Norms that integrate against Hölder test functions on local stable manifolds are in spirit dual to the technique of standard pairs developed by Dolgopyat and Chernov.

Definition of Norms: Strong Norm

Choose $p > q+1$, $\beta \in (1/p, \alpha)$ and $\gamma < \min\{1/p, \alpha - \beta\}$.

Define the **strong stable norm** of f by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^\beta(W) \\ |\psi|_{\mathcal{C}^\beta(W)} \le |W|^{-1/p}}} \int_W f \, \psi \, dm_W$$

Define the **strong unstable norm** of f by

$$\|f\|_u = \sup_{\varepsilon \le \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d(W_1, W_2) \le \varepsilon}} \sup_{\substack{|\psi_i|_{\mathcal{C}^\alpha(W_i)} \le 1 \\ d(\psi_1, \psi_2) = 0}} \varepsilon^{-\gamma} \left| \int_{W_1} f \psi_1 - \int_{W_2} f \psi_2 \right|$$

The **strong norm** of f is defined to be $\|f\|_{\mathcal{B}} = \|f\|_s + c_u \|f\|_u$,

Define $\mathcal{B}$ to be the completion of $C^1(M)$ in the $\| \cdot \|_{\mathcal{B}}$ norm.

Regularity of J^sT

Since $\mathcal{B}$ is defined as the completion of $C^1(M)$ in $\|\cdot\|_{\mathcal{B}}$, a priori, it is not clear that $\mathcal{L}_t$ acts continuously on $\mathcal{B}$.

Lemma 2 ([Chernov, Markarian '06])

For $W \in \mathcal{W}^s$ and $\eta > 0$, let $W_u(\eta) = \{$ points in W whose unstable manifold extends a length at least η on both sides of $W\}$. Then $m_W(W \setminus W_u(\eta)) \le C\eta$ for some $C > 0$ indep. of W and η.

Lemma 3

$\exists\, C_1, C_2 > 0$ such that for any homogeneous unstable curve U and any $\rho > 0$, there exists $U' \subset U$ with $m_U(U \setminus U') \le C_1\rho$ such that

$$\left| \frac{J^sT(x)}{J^sT(y)} - 1 \right| \le C_2 \left(\rho^{-\frac{q}{q+1}} d(x,y) + d(x,y)^{1/(q+1)} \right) .$$

These two lemmas allow us to approximate $\mathcal{L}_t f$ by C^1 functions in the $\|\cdot\|_{\mathcal{B}}$ norm.

Banach Spaces

Theorem 2

- We have a sequence of continuous inclusions,

$$\mathcal{C}^1(M) \subset \mathcal{B} \subset \mathcal{B}_w \subset (\mathcal{C}^\alpha(M))^*.$$

- The embedding of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ is compact.
- Lasota-Yorke inequalities: There exists $\sigma < 1$, $C, C_n > 0$ such that for all $f \in \mathcal{B}$, $n \geq 0$,

$$\begin{aligned} |\mathcal{L}_t^n f|_w &\leq C Q_n(t) |f|_w \,, \\ \|\mathcal{L}_t^n f\|_s &\leq C Q_n(t) (\sigma^n \|f\|_s + C_n |f|_w) \\ \|\mathcal{L}_t^n f\|_u &\leq C Q_n(t) (\sigma^n \|f\|_u + C_n \|f\|_s) \,. \end{aligned}$$

This implies the spectral radius of $\mathcal{L}_t$ on $\mathcal{B}$ is at most $e^{P_*(t)}$ and its essential spectral radius is at most $\sigma e^{P_*(t)}$. To prove $\mathcal{L}_t$ is quasi-compact, we need a lower bound on the spectral radius.

Definition of $t_* > 1$.

To establish upper and lower bounds on $\|\mathcal{L}_t^n 1\|_{\mathcal{B}}$, need precise control of the growth of $\sum_{W_i \subset T^{-n}W} |J^s T^n|^t_{C^0(W_i)}$ and $Q_n(t)$.

This requires us to restrict values of t for which we can prove the exact exponential growth of $Q_n(t)$.

Definition

Let $\chi = \lim_{t \to 1^-} \dfrac{P_*(t)}{1-t}$ and define $t_* := \dfrac{\chi}{\chi + \log \theta} > 1$.

Since $P_*(1) = 0$, $-\chi$ is the one-sided derivative of $P_*(t)$ at $t = 1$ (exists by convexity of $P_*(t)$).

$t_* > 1$ is the value of t where the tangent line to $P_*(t)$ at $t = 1$ intersects the line $t \log \theta$.

Key property: If $t < t_*$, then $\theta^t < e^{P_*(t)}$.

Exact Exponential Growth of $Q_n(t)$

Fix $t_0 > 0$ and $t_1 \in (1, t_*)$. Restrict to $t \in [t_0, t_1]$.
There exists $c_1 > 0$ such that for all $t \in [t_0, t_1]$,

$$e^{nP_*(t)} \leq Q_n(t) \leq c_1 e^{nP_*(t)} \quad \text{for all } n \geq 1.$$

This, in turn, relies on several growth/fragmentation lemmas.

$\mathcal{G}_n(W) =$ homogeneous components of $T^{-n}W$, $W \in \mathcal{W}^s$.

$Sh_n^\delta(W) = \{W_i \in G_n(W) : |W_i| < \delta\}$

Lemma (Growth Lemma)

For all $\varepsilon > 0$, there exists $\delta > 0$ and $n_\delta \geq 1$ such that

- $\forall W \in \mathcal{W}^s$,
$$\sum_{W_i \in Sh_n^\delta(W)} |J^s T^n|^t_{C^0(W_i)} \leq \varepsilon \sum_{W_i \in \mathcal{G}_n(W)} |J^s T^n|^t_{C^0(W_i)}$$
- The set of $A \in \mathcal{M}_0^{n,\mathbb{H}}$ such that $T^n A$ has unstable diameter longer than δ contribute 'most' mass to $Q_n(t)$.

Uniform Growth of Stable Manifolds and Spectral Radius

The growth lemma implies we can choose $\delta > 0$ and then find $c_0 > 0$ such that for any $W \in \mathcal{W}^s$ with $|W| \geq \delta$,

$$c_0 Q_n(t) \leq \sum_{W_i \in \mathcal{G}_n(W)} |J^s T^n|^t_{C^0(W_i)} \leq c_0^{-1} Q_n(t).$$

Together with the Lasota-Yorke inequalities, this implies that the spectral radius of $\mathcal{L}_t$ on $\mathcal{B}$ is $e^{P_*(t)}$, and more precisely,

$$c_0 e^{nP_*(t)} \leq \|\mathcal{L}_t^n 1\|_{\mathcal{B}} \leq c_0^{-1} e^{nP_*(t)}.$$

This implies something stronger: that the peripheral spectrum of $\mathcal{L}_t$ contains no Jordan blocks.

A Spectral Gap for $\mathcal{L}_t$

Theorem 3

For each $t_0 > 0$ and $t_1 < t_*$, there exists a Banach space $\mathcal{B} = \mathcal{B}(t_0, t_1)$ such that $\mathcal{L}_t$ has a spectral gap:

- $e^{P_*(t)}$ is the eigenvalue of maximum modulus, it is simple, and the remainder of the spectrum of $\mathcal{L}_t$ is contained in a disk of radius $\bar{\sigma} e^{P_*(t)}$, where $\bar{\sigma} < 1$ is uniform for $t \in [t_0, t_1]$.

Letting ν_t and $\tilde{\nu}_t$ denote the right and and left eigenvectors for $\mathcal{L}_t$, define

$$\mu_t(\psi) = \frac{\langle \nu_t, \psi \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle}, \quad \psi \in C^\alpha(M).$$

Then μ_t is an invariant probability measure for T, and enjoys exponential decay of correlations against Hölder observables.

μ_t has no atoms, gives 0 weight to any C^1 curve and is positive on open sets. Moreover, $\int |\log d(x, \mathcal{S}_{\pm 1})| \, d\mu_t < \infty$.

Entropy of μ_t and a Variational Principle

Define $B(x, n, \varepsilon) = \{y \in M : d(T^{-i}x, T^{-i}y) \le \varepsilon, \forall i \in [0, n]\}$.

Proposition (Measure of Bowen Balls)

There exists $C > 0$ s.t. for all $x \in M$, $n \ge 1$, and $y \in B(x, n, \varepsilon)$,

$$\mu_t(B(x, n, \varepsilon)) \le C e^{-nP_*(t) + t \log J^s T^n(y)}.$$

- [Brin, Katok '81] $\implies$ for μ_t-a.e. $x \in M$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_t(B(x, n, \varepsilon)) = h_{\mu_t}(T).$$

- This plus the Proposition implies

$$h_{\mu_t}(T) \ge P_*(t) - t \int \log J^s T \, d\mu_t = P_*(t) + t \int \log J^u T \, d\mu_t$$

- But $P_*(t) \ge h_{\mu_t}(T) - t \int \log J^u T \, d\mu_t$ by Theorem 1.
- Conclude: $P_*(t) = h_{\mu_t}(T) - t \int \log J^u T \, d\mu_t = P(t)$.

Uniqueness of Equilibrium State

We prove uniqueness using the concept of tangent measure.

We say μ is a C^1-tangent measure at t if

$$P(-t\log J^uT + \phi) \geq P(t) + \int \phi \, d\mu \,, \quad \text{for all } \phi \in C^1(M)$$

If μ is an equilibrium state for $-t\log J^uT$, then μ is a tangent measure [Walters '82].

We show there can be only one tangent measure for each t by showing that for each $\phi \in C^1(M)$, the perturbed transfer operator defined by

$$\mathcal{L}_{t,z\phi} f = \frac{f \circ T^{-1}}{(J^sT)^{1-t} \circ T^{-1}} e^{z\phi\circ T^{-1}} \,, \quad z \in \mathbb{C} \,,$$

is an analytic perturbation of $\mathcal{L}_t$.

Generalized Variational Principle

Theorem 4

Let $t \in [t_0, t_1]$ and $\phi \in C^1(M)$. For $|z|$ sufficiently small,

- $\mathcal{L}_{t,z\phi}$ has a spectral gap on $\mathcal{B}$;
- the spectral radius of $\mathcal{L}_{t,z\phi}$ is $e^{P(-t \log J^u T + z\phi)}$;
- restricting to $z \in \mathbb{R}$,

$$\frac{d}{dz} e^{P(-t \log J^u T + z\phi)} \Big|_{z=0} = e^{P(t)} \int \phi \, d\mu_t \, ;$$

- finally, $P_*(t \log J^s T + z\phi) = P(-t \log J^u T + z\phi)$ and there exists a unique equilibrium measure attaining the supremum.

The derivative formula for $e^{P(-t \log J^u T + z\phi)}$ implies that any tangent measure μ must satisfy $\int \phi d\mu = \int \phi d\mu_t$ for all C^1 functions ϕ. Thus $\mu = \mu_t$, so μ_t is unique.

Analyticity of $P(t)$

Since J^sT is not piecewise Hölder, a separate set of arguments is needed to show that $\mathcal{L}_t$ is analytic as a function of t, $t \in [t_0, t_1]$.

Theorem 5

The function $t \mapsto P(t)$ is analytic on $(0, t_*)$, with

$$P'(t) = \int \log J^sT \, d\mu_t = -\int \log J^uT \, d\mu_t < 0 \,,$$

$$P''(t) = \sum_{k \geq 0} \left[\int (\log J^sT \circ T^k) \log J^sT \, dmu_t - (P'(t))^2 \right] \geq 0 \,.$$

Moreover, $P''(t) = 0$ if and only if $\log J^sT = f - f \circ T + P'(t)$ for some $f \in L^2(\mu_t)$.

If there exists $t_1 \neq t_2$ in $(0, t_*)$ such that $\mu_{t_1} = \mu_{t_2}$, then $P(t)$ is affine on $(0, t_*)$ and $\log J^sT$ is μ_t-a.e. cohomologous to a constant for all $t \in (0, t_*)$.