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Application to Diophantine approximation

On the dimension drop conjecture for diagonal flows on the space of lattices

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Introduction

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- Γ a lattice in G

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- μ the G-invariant probability measure on X.

Introduction

- For a subset F of G and a non-empty open subset U of X define the set

$$E(F,U) := \{x \in X : gx \notin U \ \forall\, g \in F\}$$

of points in X whose F-trajectory stays away from U.

Introduction

- If F is a subgroup or a subsemigroup of G acting ergodically on (X, μ), then the set $\{gx : g \in F\}$ is dense for μ-almost all $x \in X$, in particular $\mu\big(E(F, U)\big) = 0$.

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- If F is a subgroup or a subsemigroup of G acting ergodically on (X, μ), then the set $\{gx : g \in F\}$ is dense for μ-almost all $x \in X$, in particular $\mu(E(F, U)) = 0$.
- **Example:** $G = \mathrm{SL}_2(\mathbb{R})$, $g_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$, $u_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.

Introduction

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- **Question (Mirzakhani):** If $E(F, U)$ has measure zero, does it necessarily have less than full Hausdorff dimension?
- **Dimension drop conjecture:** If $F \subset G$ is a subsemigroup and U is an open subset of X, then either $E(F, U)$ has positive measure, or its dimension is less than the dimension of X.

Known results

- F consists of quasiunipotemt elements, that is, for each $g \in F$ all eigenvalues of $\operatorname{Ad} g$ have absolute value 1. This follows from Ratner's Measure Classification Theorem and the work of Dani and Margulis.

$$\overline{\{u_t x\}} = Hx,$$

where H is a closed subgroup of G.

$$\dim E(F, U) \leq \dim X - 1$$

- **(Einseidler–Kadyrov–Pohl)**: G is a simple Lie group of real rank 1.
- (**Kleinbock–Weiss, Kleinbock–M**): G is semisimple without compact factors, Γ is irreducible, F is a one-parameter Ad-diagonalizable subsemigroup of G, and the complement of U is compact.

Main results

- $$G = \mathrm{SL}_{m+n}(\mathbb{R}),\ \Gamma = \mathrm{SL}_{m+n}(\mathbb{Z}),\ X = G/\Gamma$$

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- $$G = \mathrm{SL}_{m+n}(\mathbb{R}),\ \Gamma = \mathrm{SL}_{m+n}(\mathbb{Z}),\ X = G/\Gamma$$
  $$g\Gamma \to g\mathbb{Z}^{m+n}$$
- $$F^+ := \{g_t : t \geq 0\},$$
  $$g_t := \mathrm{diag}(e^{nt}, \dots, e^{nt}, e^{-mt}, \dots, e^{-mt}).$$
- Choose $a > 0$ and consider
  $$F_a^+ := \left\{ \mathrm{diag}(e^{ant}, \dots, e^{ant}, e^{-amt}, \dots, e^{-amt}) : t \in \mathbb{Z}_+ \right\}.$$

Main results

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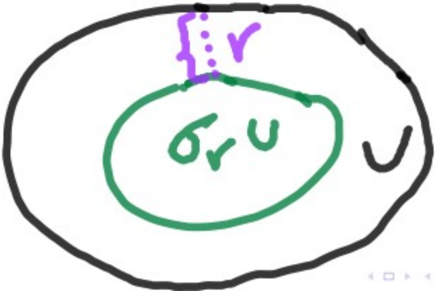
- $$d(g_1\Gamma, g_2\Gamma) = \inf_{\lambda \in \Gamma} d(g_1\lambda, g_2)$$

Main results

- For an open subset U of X and $r > 0$ denote by $\sigma_r U$ the *inner r-core* of U, defined as

$$\sigma_r U := \{x \in X : d(x, U^c) > r\}.$$



Main results

- $$\theta_U := \sup\left\{0 < \theta \leq 1 : \mu(\sigma_{2\sqrt{mn\theta}} U) \geq \frac{1}{2}\mu(U)\right\}$$

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- The notation $A \gg B$, where A and B are quantities depending on certain parameters, will mean $A \geq CB$, with C being a constant dependent only on m and n.

Main results

- **Kleinbock–M**: There exist positive constants c, r_1, p_1, p_2, p_3 such that for any $a > 0$ and for any open subset U of X one has

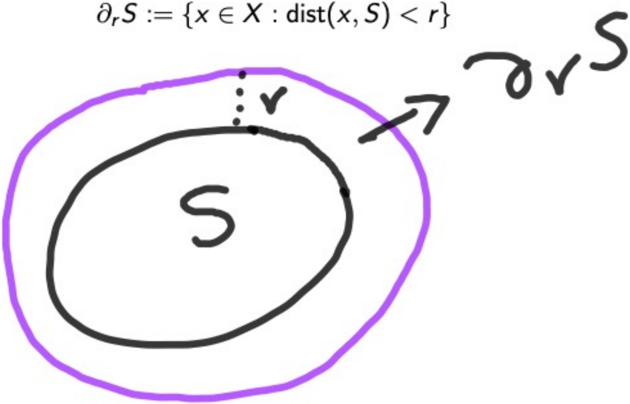
$$\operatorname{codim} E(F_a^+, U) \gg \frac{\mu(U)}{\log \frac{1}{r(U,a)}},$$

where

$$r(U, a) := \min\left(\mu(U)^{p_1}, \theta_U^{p_2}, ce^{-p_3 a}, r_1\right).$$

In particular, if U is non-empty we always have $\dim E(F_a^+, U) < \dim X$.

- $$\partial_r S := \{x \in X : \mathrm{dist}(x, S) < r\}$$



Main results

- $$\partial_r S := \{x \in X : \text{dist}(x, S) < r\}$$

- **Corollary:** If $S \subset X$ is a k-dimensional embedded smooth submanifold, then there exist $\varepsilon_S, c_S, p_S, C_S > 0$ such that for any $a > 0$ and any positive $\varepsilon < \min(\varepsilon_S, c_S e^{-ap_S})$ one has
$$\text{codim}\, E(F_a^+, \partial_\varepsilon S) \geq C_S \frac{\varepsilon^{\dim X - k}}{\log(1/\varepsilon)}$$

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- **Corollary:**
$$\operatorname{codim} E\big(F_a^+, B(z,\varepsilon)\big) \gg \frac{\mu\big(B(z,\varepsilon)\big)}{\log(1/\varepsilon)}$$

Sketch of proof

- Unstable horospherical subgroup with respect to F_a^+, defined as:

$$H = \{g \in G : g_t g g_{-t} \to \infty \text{ as } t \to \infty\}$$

- $$H := \left\{ \begin{bmatrix} I_m & s \\ 0 & I_n \end{bmatrix} : s \in M_{m,n} \right\}.$$

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Sketch of proof

- **Kleinbock–M**: For any $a > 0$, any $x \in X$, and for any open subset U of X one has

$$\operatorname{codim}\left(\{h \in H : hx \in E(F_a^+, U)\}\right) \gg \frac{\mu(U)}{\log \frac{1}{r(U,a)}}$$

Sketch of proof

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- There exists a nested family of compact subsets $\{Q_t\}_{t>0}$ of X and $t_0 > 0$ such that for all $k \in \mathbb{N}$ and all $t > t_0$

$$\operatorname{codim}\{h \in H : g_{Nkt}hx \in Q_t^c \ \ \forall N \in \mathbb{N}\} > 0$$

Sketch of proof

- **(Eskin–Margulis–Mozes)**: A subspace L of $\mathbb{R}^{m+n}$ is x-rational if $L \cap x$ is a lattice in L, and for any x-rational subspace L, denote by $d_x(L)$ the volume of $L/(L \cap x)$. Now for $1 \leq i \leq m+n$ define

$$\alpha_i(x) := \sup\left\{\frac{1}{d_x(L)} : L \in F_i(x)\right\},$$

where $F_i(x)$ is the set of i-dimensional x-rational subspaces of $\mathbb{R}^{m+n}$.

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Sketch of proof

- **(KKLM, EMM):** There exists $c_0 \geq 1$ depending only on m, n with the following property: for any $t \geq 1$, any $x \in X$, and for any $i \in \{1, \ldots, m+n-1\}$ one has

$$\int_H \alpha_i^{1/2}(g_t hx)\, d\rho_1(h) \leq$$

$$c_0 \left(e^{-t/2} \alpha_i(x)^{1/2} + e^{mnt} \max_{0<j\leq \min(m+n-i,i)} \sqrt{\alpha_{i+j}(x)^{1/2} \alpha_{i-j}(x)^{1/2}} \right)$$

Sketch of proof

- **(EMM, KKLM):** 'Convexity trick': For any $t \geq 1$ there exist positive constants $\omega_0 = \omega_0(t), \ldots, \omega_{m+n} = \omega_{m+n}(t)$ and C_0 such that the linear combination

$$\tilde{\alpha} := \sum_{i=0}^{m+n} \omega_i {\alpha_i}^{1/2}$$

satisfies

$$\int_H \tilde{\alpha}(g_t hx)\, d\rho_1(h) \leq 2c_0 e^{-t/2} \tilde{\alpha}(x) + C_0$$

for all $x \in X$.

Sketch of proof

- **(Kleinbock–M):**

$$\operatorname{codim}\left(\{h \in H : hx \in E(F_a^+, U \cup Q_t^c)\}\right) > 0$$

Sketch of proof

- Fix a basis $\{Y_1, \dots, Y_n\}$ for the Lie algebra $\mathfrak{g}$ of G, and, given a smooth function $h \in C^\infty(X)$ and $\ell \in \mathbb{Z}_+$, define the *"L^p, order ℓ" Sobolev norm* $\|h\|_{\ell,p}$ of h by

$$\|h\|_{\ell,p} := \sum_{|\alpha| \le \ell} \|D^\alpha h\|_p,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex, $|\alpha| = \sum_{i=1}^n \alpha_i$, and D^α is a differential operator of order $|\alpha|$ which is a monomial in $Y_1, \dots, Y_n$, namely $D^\alpha = Y_1^{\alpha_1} \cdots Y_n^{\alpha_n}$.

- $$C_2^\infty(X) = \{h \in C^\infty(X) : \|h\|_{\ell,2} < \infty \text{ for any } \ell = \mathbb{Z}_+\}.$$

- $$\|f\|_{C^\ell} := \sup_{x \in X,\ |\alpha| \le \ell} |D^\alpha f(x)|.$$

Sketch of proof

- $B^P(r)$: Ball of radius r centered at identity in P.
-

$$r_0(x) = \sup\{r > 0 : \text{the map }, g \mapsto gx \text{ is injective on } B^G(r)\}$$

Sketch of proof

- **Definition:** Say that a subgroup P of G has *Effective Equidistribution Property* (EEP) with respect to the flow (X, F^+) if P is normalized by F^+, and there exists $\lambda > 0$ and $\ell \in \mathbb{N}$ such that for any $x \in X$ and $t > 0$ with

$$t \gg \log \frac{1}{r_0(x)},$$

any $f \in C^\infty_{comp}(P)$ with $\operatorname{supp} f \subset B^P(1)$ and any $\psi \in C^\infty_2(X)$ it holds that

$$\left| I_{f,\psi}(g_t, x) - \int_P f\, d\nu \int_X \psi\, d\mu \right| \ll \max(\|\psi\|_{C^1}, \|\psi\|_{\ell,2}) \cdot \|f\|_{C^\ell} \cdot e^{-\lambda t},$$

where

$$I_{f,\psi}(g_t, x) := \int_P f(p)\psi(g_t p x)\, d\nu(p)\,.$$

Sketch of proof

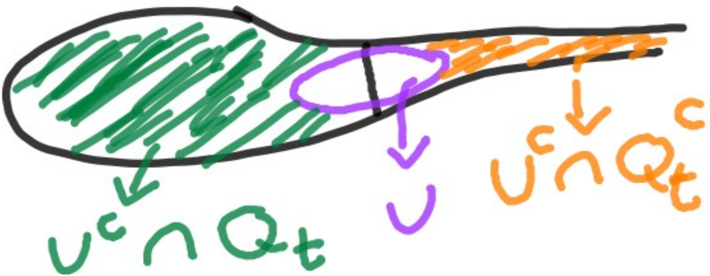
- (**Kleinbock–Margulis, Kleinbock–M**): H satisfies (EEP).

Sketch of proof

- $$\begin{aligned}
&\nu\left(\left\{h \in B^H(r) : g_t hx \in U^c\right\}\right) \\
&= \int_H 1_{B^H(r)}(h) 1_{U^c}(g_t hx)\, d\nu(h) \\
&\approx \int_H f(h)\psi(g_t hx)\, d\nu(h) \\
&\approx \int_H f\, d\nu \int_X \psi\, d\mu + C(f,\psi)e^{-\lambda t} \\
&\approx \nu\left(B^H(r)\right)\mu(U^c) + C' e^{-\lambda' t}
\end{aligned}$$

Sketch of proof

- If $g_{Nkt}hx \in U^c$, then $g_{Nkt}hx \in U^c \cap Q_t$ or $g_{Nkt}hx \in Q_t^c$



Sketch of proof

- $$g_{kt}hx \in U^c \cap Q_t, g_{2kt}hx \in U^c \cap Q_t$$
  $$g_{3kt}hx \in Q_t^c, g_{4kt}hx \in Q_t^c, g_{5kt}hx \in Q_t^c$$
  $$g_{6kt}hx \in U^c \cap Q_t$$
  $$\vdots$$

Application to Diophantine approximation

- **(Drichlet's approximation theorem):** For any $s \in M_{m,n}$ and any $N > 0$,

$$\text{there exists } \mathbf{p} \in \mathbb{Z}^m \text{ and } \mathbf{q} \in \mathbb{Z}^n \setminus \{0\} \text{ such that}$$
$$\|s\mathbf{q} - \mathbf{p}\| < \frac{1}{N^{n/m}} \text{ and } 0 < \|\mathbf{q}\| \leq N.$$

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  $$\|s\mathbf{q} - \mathbf{p}\| < \frac{1}{N^{n/m}} \text{ and } 0 < \|\mathbf{q}\| \leq N.$$

- $s \in M_{m,n}$ is *Dirichlet improvable* if there exists a constant $c < 1$ such that, for all sufficiently large N

  there exists $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ such that

  $$\|s\mathbf{q} - \mathbf{p}\| < \frac{c}{N^{n/m}} \text{ and } 0 < \|\mathbf{q}\| \leq N.$$

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- $\mathbf{DI}_{m,n} = \bigcup_{c<1} \mathbf{DI}_{m,n}(c)$
- $$\dim \mathbf{DI}_{m,n}(c)?$$

Application to Diophantine approximation

- $s \in \mathrm{DI}_{m,n}$ if and only if there exists $\varepsilon > 0$ such that for large enough $t > 0$ the lattice $g_t h_s \mathbb{Z}^{m+n}$ has a vector of (supremum) norm less than $1 - \varepsilon$, where $h_s = \begin{bmatrix} I_m & s \\ 0 & I_n \end{bmatrix}$.

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- $s \in \mathrm{DI}_{m,n}(c)$, if and only if for large enough t there exists $\mathrm{v} = \begin{pmatrix} -\mathrm{p} \\ \mathrm{q} \end{pmatrix} \in \mathbb{Z}^{m+n} \smallsetminus \{0\}$ such that the vector

$$g_t h_s \mathrm{v} = \begin{pmatrix} e^{nt}(s\mathrm{q} - \mathrm{p}) \\ e^{-mt}\mathrm{q} \end{pmatrix}$$

belongs to

$$\mathcal{R}_c := \left\{ \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} \in \mathbb{R}^{m+n} : \|\mathrm{x}\| < c, \ \|\mathrm{y}\| \le 1 \right\}.$$

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- $h_s\mathbb{Z}^{m+n} \in E(F^+, U_c)$.
- **(Kleinbock–M)**: $\dim\left(\mathbf{DI}_{m,n}(c)\right) < mn$ for any $c < 1$.