On the dimension drop conjecture for diagonal flows on the space of lattices

Shahriar Mirzadeh

Michigan State University

Ergodic Theory Seminar
Ohio State University
Joint work with Dmitry Kleinbock
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Introduction

- $G$ a Lie group
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- $\Gamma$ a lattice in $G$
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- $X$ is the homogeneous space $G/\Gamma$
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- $X$ is the homogeneous space $G/\Gamma$
- **Example:** $G = \text{SL}_n(\mathbb{R})$, $\Gamma = \text{SL}_n(\mathbb{Z})$

\[ G/\Gamma = \text{SL}_n(\mathbb{R})/\text{SL}_n(\mathbb{Z}) \]
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- $\Gamma$ a lattice in $G$
- $X$ is the homogeneous space $G/\Gamma$
- **Example**: $G = \text{SL}_n(\mathbb{R})$, $\Gamma = \text{SL}_n(\mathbb{Z})$

$$G/\Gamma = \text{SL}_n(\mathbb{R})/\text{SL}_n(\mathbb{Z})$$

- $\mu$ the $G$-invariant probability measure on $X$. 

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• For a subset $F$ of $G$ and a non-empty open subset $U$ of $X$ define the set

$$E(F, U) := \{ x \in X : gx \notin U \ \forall \ g \in F \}$$

of points in $X$ whose $F$-trajectory stays away from $U$. 
If $F$ is a subgroup or a subsemigroup of $G$ acting ergodically on $(X, \mu)$, then the set $\{gx : g \in F\}$ is dense for $\mu$-almost all $x \in X$, in particular $\mu(E(F, U)) = 0$. 
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• If $F$ is a subgroup or a subsemigroup of $G$ acting ergodically on $(X, \mu)$, then the set \( \{gx : g \in F\} \) is dense for $\mu$-almost all $x \in X$, in particular $\mu(E(F, U)) = 0$.

• Example: $G = \text{SL}_2(\mathbb{R})$, $g_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$, $u_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. 
Introduction

• **Question (Mirzakhani):** If $E(F, U)$ has measure zero, does it necessarily have less than full Hausdorff dimension?
**Introduction**

- **Question (Mirzakhani):** If \( E(F, U) \) has measure zero, does it necessarily have less than full Hausdorff dimension?
- **Dimension drop conjecture:** If \( F \subset G \) is a subsemigroup and \( U \) is an open subset of \( X \), then either \( E(F, U) \) has positive measure, or its dimension is less than the dimension of \( X \).
Known results

- **$F$** consists of quasiunipotent elements, that is, for each $g \in F$ all eigenvalues of $\text{Ad} \, g$ have absolute value 1. This follows from Ratner’s Measure Classification Theorem and the work of Dani and Margulis.

$$\overline{\{u_t x\}} = Hx,$$

where $H$ is a closed subgroup of $G$.

$$\dim E(F, U) \leq \dim X - 1$$

- **(Einseidler–Kadyrov–Pohl):** $G$ is a simple Lie group of real rank 1.

- **(Kleinbock–Weiss, Kleinbock–M):** $G$ is semisimple without compact factors, $\Gamma$ is irreducible, $F$ is a one-parameter $\text{Ad}$-diagonalizable subsemigroup of $G$, and the complement of $U$ is compact.
Main results

\[ G = \text{SL}_{m+n}(\mathbb{R}), \quad \Gamma = \text{SL}_{m+n}(\mathbb{Z}), \quad X = G/\Gamma \]

\[ g\Gamma \rightarrow g\mathbb{Z}^{m+n} \]
Main results

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\[ F^+ := \{ g_t : t \geq 0 \}, \]

\[ g_t := \text{diag}(e^{nt}, \ldots, e^{nt}, e^{-mt}, \ldots, e^{-mt}). \]
Main results

\begin{itemize}
  \item \( G = \text{SL}_{m+n}(\mathbb{R}) \), \( \Gamma = \text{SL}_{m+n}(\mathbb{Z}) \), \( X = G/\Gamma \)

  \[ g\Gamma \to g\mathbb{Z}^{m+n} \]

  \item \( F^+ := \{ g_t : t \geq 0 \} \),

  \[ g_t := \text{diag}(e^{nt}, \ldots, e^{nt}, e^{-mt}, \ldots, e^{-mt}) \).

  \item Choose \( a > 0 \) and consider

  \[ F_a^+ := \{ \text{diag}(e^{ant}, \ldots, e^{ant}, e^{-amt}, \ldots, e^{-amt}) : t \in \mathbb{Z}_+ \} \].
\end{itemize}
Main results

• Fix a right-invariant Riemannian structure on $G$, and denote by $d$ the corresponding Riemannian metric, using the same notation for the induced metric on $X$. 
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$$d(g_1 \Gamma, g_2 \Gamma) = \inf_{\lambda \in \Gamma} d(g_1 \lambda, g_2)$$
Main results

• For an open subset $U$ of $X$ and $r > 0$ denote by $\sigma_r U$ the \textit{inner r-core} of $U$, defined as

$$\sigma_r U := \{x \in X : d(x, U^c) > r\}.$$
Main results

\[ \theta_U := \sup \left\{ 0 < \theta \leq 1 : \mu(\sigma_{2\sqrt{mn}}U) \geq \frac{1}{2}\mu(U) \right\} \]
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- The notation \( A \gg B \), where \( A \) and \( B \) are quantities depending on certain parameters, will mean \( A \geq CB \), with \( C \) being a constant dependent only on \( m \) and \( n \).
Main results

- **Kleinbock–M**: There exist positive constants $c, r_1, p_1, p_2, p_3$ such that for any $a > 0$ and for any open subset $U$ of $X$ one has

$$\text{codim } E(F_a^+, U) \gg \frac{\mu(U)}{\log \frac{1}{r(U,a)}},$$

where

$$r(U, a) := \min \left( \mu(U)^{p_1}, \theta_U^{p_2}, ce^{-p_3a}, r_1 \right).$$

In particular, if $U$ is non-empty we always have $\dim E(F_a^+, U) < \dim X$. 
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\[ \partial_r S := \{ x \in X : \text{dist}(x, S) < r \} \]
Main results

\[ \partial_r S := \{ x \in X : \text{dist}(x, S) < r \} \]

- **Corollary:** If \( S \subset X \) is a \( k \)-dimensional embedded smooth submanifold, then there exist \( \varepsilon_S, c_S, p_S, C_S > 0 \) such that for any \( a > 0 \) and any positive \( \varepsilon < \min(\varepsilon_S, c_S e^{-ap_S}) \) one has

\[
\text{codim } E(F_a^+, \partial_\varepsilon S) \geq C_S \varepsilon^{\text{dim } X-k} \frac{1}{\log(1/\varepsilon)}
\]
Main results

• $\partial_r S := \{ x \in X : \text{dist}(x, S) < r \}$

• **Corollary:** If $S \subset X$ is a $k$-dimensional embedded smooth submanifold, then there exist $\varepsilon_S, c_S, p_S, C_S > 0$ such that for any $a > 0$ and any positive $\varepsilon < \min(\varepsilon_S, c_S e^{-a p_S})$ one has

$$\text{codim } E(F_a^+, \partial_{\varepsilon} S) \geq C_S \varepsilon^{\text{dim } X - k} \frac{\varepsilon}{\log(1/\varepsilon)}$$

• **Corollary:**

$$\text{codim } E(F_a^+, B(z, \varepsilon)) \gg \frac{\mu(B(z, \varepsilon))}{\log(1/\varepsilon)}$$
Sketch of proof

- Unstable horospherical subgroup with respect to $F_a^+$, defined as:

$$H = \{ g \in G : g_t g g_{-t} \to \infty \text{ as } t \to \infty \}$$

- 

$$H := \left\{ \begin{bmatrix} l_m & s \\ 0 & l_n \end{bmatrix} : s \in M_{m,n} \right\}.$$
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Sketch of proof

- Unstable horospherical subgroup with respect to $F^+_a$, defined as:

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\]

- 

\[
g^t \begin{bmatrix} l_m & s \\ 0 & l_n \end{bmatrix} g^{-t} = \begin{bmatrix} l_m & e^{(m+n)t}s \\ 0 & l_n \end{bmatrix}
\]
• **Kleinbock–M**: For any $a > 0$, any $x \in X$, and for any open subset $U$ of $X$ one has

\[
\text{codim} \left( \{ h \in H : hx \in E(F_a^+, U) \} \right) \gg \frac{\mu(U)}{\log \frac{1}{r(U,a)}}
\]
(KKLM): The set of points in $X$ whose orbit diverges (leaves every compact subset of $X$) has codimension at least $\frac{mn}{m+n}$.
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• There exists a nested family of compact subsets $\{Q_t\}_{t>0}$ of $X$ and $t_0 > 0$ such that for all $k \in \mathbb{N}$ and all $t > t_0$

$$\text{codim} \{ h \in H : g_{Nkt}hx \in Q^c_t \quad \forall N \in \mathbb{N} \} > 0$$
Sketch of proof

- (Eskin–Margulis–Mozes): A subspace $L$ of $\mathbb{R}^{m+n}$ is $x$-rational if $L \cap x$ is a lattice in $L$, and for any $x$-rational subspace $L$, denote by $d_x(L)$ the volume of $L/(L \cap x)$. Now for $1 \leq i \leq m + n$ define

$$\alpha_i(x) := \sup \left\{ \frac{1}{d_x(L)} : L \in F_i(x) \right\},$$

where $F_i(x)$ is the set of $i$-dimensional $x$-rational subspaces of $\mathbb{R}^{m+n}$. 
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Sketch of proof

• (KKLM, EMM): There exists $c_0 \geq 1$ depending only on $m, n$ with the following property: for any $t \geq 1$, any $x \in X$, and for any $i \in \{1, \ldots, m + n - 1\}$ one has

$$\int_H \alpha_i^{1/2} (g_t h x) \, d\rho_1(h) \leq$$

$$c_0 \left( e^{-t/2} \alpha_i(x)^{1/2} + e^{mnt} \max_{0 < j \leq \min(m+n-i,i)} \sqrt{\alpha_{i+j}(x)^{1/2} \alpha_{i-j}(x)^{1/2}} \right)$$
• (EMM, KKLM): 'Convexity trick': For any $t \geq 1$ there exist positive constants $\omega_0 = \omega_0(t), \ldots, \omega_{m+n} = \omega_{m+n}(t)$ and $C_0$ such that the linear combination

$$\tilde{\alpha} := \sum_{i=0}^{m+n} \omega_i \alpha_i^{1/2}$$

satisfies

$$\int_H \tilde{\alpha}(g_t h x) d\rho_1(h) \leq 2c_0 e^{-t/2} \tilde{\alpha}(x) + C_0$$

for all $x \in X$. 
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Sketch of proof

• (Kleinbock–M):

\[
\text{codim} \left( \{ h \in H : hx \in E(F^+_a, U \cup Q^c_t) \} \right) > 0
\]
Sketch of proof

• Fix a basis \(\{Y_1, \ldots, Y_n\}\) for the Lie algebra \(\mathfrak{g}\) of \(G\), and, given a smooth function \(h \in C^\infty(X)\) and \(\ell \in \mathbb{Z}_+\), define the \(“L^p, \text{order } \ell”\) Sobolev norm \(\|h\|_{\ell,p}\) of \(h\) by

\[
\|h\|_{\ell,p} := \sum_{|\alpha| \leq \ell} \|D^\alpha h\|_p,
\]

where \(\alpha = (\alpha_1, \ldots, \alpha_n)\) is a multiindex, \(|\alpha| = \sum_{i=1}^n \alpha_i\), and \(D^\alpha\) is a differential operator of order \(|\alpha|\) which is a monomial in \(Y_1, \ldots, Y_n\), namely \(D^\alpha = Y_1^{\alpha_1} \cdots Y_n^{\alpha_n}\).

\[
C^\infty_2(X) = \{h \in C^\infty(X) : \|h\|_{\ell,2} < \infty \text{ for any } \ell = \mathbb{Z}_+\}.
\]

•

\[
\|f\|_{C^\ell} := \sup_{x \in X, |\alpha| \leq \ell} |D^\alpha f(x)|.
\]
Sketch of proof

- $B^P(r)$: Ball of radius $r$ centered at identity in $P$.
- $r_0(x) = \sup\{r > 0 : \text{the map } g \mapsto gx \text{ is injective on } B^G(r)\}$
Sketch of proof

**Definition:** Say that a subgroup $P$ of $G$ has *Effective Equidistribution Property* (EEP) with respect to the flow $(X, F^+)$ if $P$ is normalized by $F^+$, and there exists $\lambda > 0$ and $\ell \in \mathbb{N}$ such that for any $x \in X$ and $t > 0$ with

$$t \gg \log \frac{1}{r_0(x)},$$

any $f \in C^\infty_{comp}(P)$ with $\text{supp } f \subset B^P(1)$ and any $\psi \in C_2^\infty(X)$ it holds that

$$\left| I_{f,\psi}(g_t, x) - \int_P f \, d\nu \int_X \psi \, d\mu \right| \ll \max(\|\psi\|_{C^1}, \|\psi\|_{\ell,2}) \cdot \|f\|_{C^\ell} \cdot e^{-\lambda t},$$

where

$$I_{f,\psi}(g_t, x) := \int_P f(p) \psi(g_t px) \, d\nu(p).$$
Sketch of proof

- \((\text{Kleinbock–Margulis, Kleinbock–M})\): \(H\) satisfies (EEP).
Sketch of proof

\[ \nu \left( \left\{ h \in B^H_r : g_t h x \in U^c \right\} \right) \]
\[ = \int_{\mathcal{H}} 1_{B^H_r}(h) 1_{U^c}(g_t h x) \, d\nu(h) \]
\[ \approx \int_{\mathcal{H}} f(h) \psi(g_t h x) \, d\nu(h) \]
\[ \approx \int_{\mathcal{H}} f \, d\nu \int_X \psi \, d\mu + C(f, \psi) e^{-\lambda t} \]
\[ \approx \nu \left( B^H_r \right) \mu(U^c) + C' e^{-\lambda' t} \]
• If \( g_{Nkt}hx \in U^c \), then \( g_{Nkt}hx \in U^c \cap Q_t \) or \( g_{Nkt}hx \in Q_t^c \)
Sketch of proof

\[ g_{kt}hx \in U^c \cap Q_t, \ g_{2kt}hx \in U^c \cap Q_t \]
\[ g_{3kt}hx \in Q^c_t, \ g_{4kt}hx \in Q^c_t, \ g_{5kt}hx \in Q^c_t \]
\[ g_{6kt}hx \in U^c \cap Q_t \]
\[ \vdots \]
Application to Diophantine approximation

- **(Dirichlet’s approximation theorem):** For any \( s \in M_{m,n} \) and any \( N > 0 \),

  there exists \( p \in \mathbb{Z}^m \) and \( q \in \mathbb{Z}^n \setminus \{0\} \) such that

  \[
  \|sq - p\| < \frac{1}{N^{n/m}} \quad \text{and} \quad 0 < \|q\| \leq N.
  \]
Application to Diophantine approximation

• (Dirichlet’s approximation theorem): For any $s \in M_{m,n}$ and any $N > 0$,

  there exists $p \in \mathbb{Z}^m$ and $q \in \mathbb{Z}^n \setminus \{0\}$ such that

  $$\|sq - p\| < \frac{1}{N^{n/m}} \text{ and } 0 < \|q\| \leq N.$$  

• $s \in M_{m,n}$ is Dirichlet improvable if there exists a constant $c < 1$ such that, for all sufficiently large $N$

  there exists $p \in \mathbb{Z}^m$ and $q \in \mathbb{Z}^n \setminus \{0\}$ such that

  $$\|sq - p\| < \frac{c}{N^{n/m}} \text{ and } 0 < \|q\| \leq N.$$
Application to Diophantine approximation

- $\text{DI}_{m,n}$: The set of Dirichlet improvable matrices $s \in M_{m,n}$
Application to Diophantine approximation

- **DI\(_{m,n}\)**: The set of Dirichlet improvable matrices \( s \in M_{m,n} \)
- (Davenport and Schmidt): \( \text{DI}_{m,n} \) has zero Lebesgue measure and has full Hausdorff dimension.
Application to Diophantine approximation

- \( \text{DI}_m,n \): The set of Dirichlet improvable matrices \( s \in M_{m,n} \)
- (Davenport and Schmidt): \( \text{DI}_m,n \) has zero Lebesgue measure and has full Hausdorff dimension.
- \( \text{DI}_m,n = \bigcup_{c<1} \text{DI}_m,n(c) \)
Application to Diophantine approximation

- **$\text{DI}_{m,n}$**: The set of Dirichlet improvable matrices $s \in M_{m,n}$
- **(Davenport and Schmidt)**: $\text{DI}_{m,n}$ has zero Lebesgue measure and has full Hausdorff dimension.
- $\text{DI}_{m,n} = \bigcup_{c<1} \text{DI}_{m,n}(c)$
- $\dim \text{DI}_{m,n}(c)$?
Application to Diophantine approximation

- $s \in \text{DI}_m,n$ if and only if there exists $\varepsilon > 0$ such that for large enough $t > 0$ the lattice $g_t h_s \mathbb{Z}^{m+n}$ has a vector of (supremum) norm less than $1 - \varepsilon$, where $h_s = \begin{bmatrix} I_m & s \\ 0 & I_n \end{bmatrix}$. 
Application to Diophantine approximation

- \( s \in \text{Di}_m,n \) if and only if there exists \( \varepsilon > 0 \) such that for large enough \( t > 0 \) the lattice \( g_t h_s \mathbb{Z}^{m+n} \) has a vector of (supremum) norm less than \( 1 - \varepsilon \), where \( h_s = \begin{bmatrix} l_m & s \\ 0 & l_n \end{bmatrix} \).

- \( s \in \text{Di}_m,n(c) \), if and only if for large enough \( t \) there exists \( v = \begin{pmatrix} -p \\ q \end{pmatrix} \in \mathbb{Z}^{m+n} \setminus \{0\} \) such that the vector

\[
g_t h_s v = \begin{pmatrix} e^{nt}(sq - p) \\ e^{-mt}q \end{pmatrix}
\]

belongs to

\[
\mathcal{R}_c := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{m+n} : \|x\| < c, \|y\| \leq 1 \right\}.
\]
Application to Diophantine approximation

\[ U_c = \{ x \in X : x \cap R_c = \{0\} \} \]
Application to Diophantine approximation

\[ U_c = \{ x \in X : x \cap \mathcal{R}_c = \{0\} \} \]

- \( h_s \mathbb{Z}^{m+n} \in E(F^+, U_c). \)
Application to Diophantine approximation

\[ U_c = \{ x \in X : x \cap R_c = \{0\} \} \]

- \( h_s \mathbb{Z}^{m+n} \in E(F^+, U_c) \).
- (Kleinbock–M): \( \dim (D_{m,n}(c)) < mn \) for any \( c < 1 \).