4.7: L'Hopital's Rule

Problem 1 *Evaluate each limit. Remember to state the form of the limit.*

(a) $\lim\limits_{x\to\infty} \left(\ln(1+e^{-x})\right)^x$

Solution: *Since* $\lim\limits_{x\to\infty}(1+e^{-x}) = 1+0 = 1$ *and* $\ln(1) = 0$, *this limit is of the form* 0^∞. *This is a determinate form which converges to* 0. *Thus,* $\lim\limits_{x\to\infty}\left(\ln(1+e^{-x})\right)^x = 0$.

(b) $\lim\limits_{x\to\infty}\left(\frac{1}{x}+1\right)^{\frac{1}{x}}$

Solution: *This limit is of the form* 1^0, *which is a determinate form that converges to 1. Thus,* $\lim\limits_{x\to\infty}\left(\frac{1}{x}+1\right)^{\frac{1}{x}} = 1$

(c) $\lim\limits_{x\to\infty}\left(\frac{\arctan x}{x}\right)$

Solution: *Since* $\lim\limits_{x\to\infty}\arctan x = \frac{\pi}{2}$, *this limit is of the form* $\frac{\#}{\infty} \to 0$. *Thus,* $\lim\limits_{x\to\infty}\left(\frac{\arctan x}{x}\right) = 0$

(d) $\lim\limits_{x\to\infty}(x - ln(x))$

Solution: *This limit is of the form* $\infty - \infty$, *which is an indeterminate form. We can rewrite this as:*

$$\lim_{x\to\infty}(x - ln(x)) = \lim_{x\to\infty}\left(x\left(1-\frac{ln(x)}{x}\right)\right)$$

We can see that $\lim\limits_{x\to\infty}\left(\frac{ln(x)}{x}\right)$ *is of the form* $\frac{\infty}{\infty}$ *so we can use L'Hopital's Rule.*

$$\overset{L.R.}{\Longrightarrow} \lim_{x\to\infty}\left(\frac{1/x}{1}\right) = \lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$$

Now we have:

$$\lim_{x\to\infty}\left(x\left(1-\frac{ln(x)}{x}\right)\right)$$

This limit has the form $\infty \cdot 1$. *This is a determinate form, and, therefore,* $\lim\limits_{x\to\infty}(x - ln(x)) = \infty$

(e) $\lim\limits_{x\to\infty}\left(x\ln\left(\frac{1}{x}\right)\right)$

Solution: *As x approaches ∞, $\frac{1}{x}$ approaches 0 from the right. So*

$$\lim_{x\to\infty} \ln\left(\frac{1}{x}\right) = -\infty$$

Therefore, the limit in question is of the form $\infty \cdot -\infty$, which converges to $-\infty$. Thus,

$$\lim_{x\to\infty} \left(x \ln\left(\frac{1}{x}\right)\right) = -\infty$$

(f) $\lim_{x\to 0^+} (\sin x \cot x)$

Solution: *Since $\lim_{x\to 0^+} \cot x = \infty$, this limit is of the form $0 \cdot \infty$. This is an indeterminate form.*

Note: $\cot x = \frac{\cos x}{\sin x}$. *So*

$$\lim_{x\to 0^+} (\sin x \cot x) = \lim_{x\to 0^+} \cos x = 1.$$

Problem 2 *True or False: You can use L'Hôpital's Rule to compute $\lim_{x\to 0} \frac{|x|}{x}$.*

Solution: *False. The function $|x|$ is not differentiable at $x = 0$, and so L'Hospital's Rule is not applicable.*

Problem 3 *Circle the correct answer in each part:*

(a) *Consider the limit $\lim_{x\to 0} (\cos x)^{\sin x}$.*

(i) *Evaluate the limit.*

Solution: ***The correct choice is (iii).***
Evaluation of limit:

$$\lim_{x\to 0} \underbrace{(\cos x)^{\sin x}}_{\text{form } 1^0} = 1$$

i. *the limit DNE*
ii. *e*
iii. 1
iv. ∞
v. $-\infty$
vi. 0
vii. *none of the previous answers is correct*

(ii) *What Limit Law, rule or technique did you use to find this limit?*

Solution: *The correct choice is (iv).*

i. *The Squeeze Theorem;*
ii. *L'Hôpital's Rule;*

iii. *The Product Law;*

iv. *evaluated the function at $x = 0$, since the function is continuous at $x = 0$;*

v. *none of the previous answers is correct*

(b) *Evaluate the limit* $\lim_{x \to 4^-} \frac{\ln x}{x - 4}$.

Solution: *The correct choice is (v).*

Evaluation of limit:

$$\lim_{x \to 4^-} \underbrace{\frac{\ln x}{x - 4}}_{\text{form } \ln(4)/0^-} = -\infty$$

(i) *the limit DNE*

(ii) e

(iii) 1

(iv) ∞

(v) $-\infty$

(vi) 0

(vii) *none of the previous answers is correct*

(c) *Evaluate the limit* $\lim_{x \to \infty} \frac{\ln x}{x - 4}$.

Solution: *The correct choice is (vi).*

Evaluation of limit:

$$\lim_{x \to \infty} \underbrace{\frac{\ln x}{x - 4}}_{\text{form } \infty/\infty} \overset{L.H.}{=} \lim_{x \to \infty} \frac{1/x}{1}$$
$$= 0$$

(i) *the limit DNE*

(ii) e

(iii) 1

(iv) ∞

(v) $-\infty$

(vi) 0

(vii) *none of the previous answers is correct*

(d) *Consider the limit* $\lim_{h \to 0} \frac{(2 + h)^3 - 8}{h} = f'(2)$. *Determine the function* f.

Solution: *The correct choice is (ii).*

(i) *such a function DNE;*

(ii) $f(x) = x^3$;

(iii) $f(x) = (2+x)^3$;

(iv) $f(x) = \dfrac{(2+x)^3}{x}$;

(v) *none of the previous answers is correct*

(e) *Consider the limit* $\lim_{x\to 0^+}\left(\dfrac{\sin x}{x}\right)^{|\ln x|}$. *Determine the form of this limit.*

Solution: *The correct choice is (v).*

(i) $\dfrac{0}{0}$;

(ii) $\dfrac{\infty}{\infty}$;

(iii) 1^0;

(iv) 0^0;

(v) 1^∞;

(vi) ∞^∞;

(vii) *none of the previous answers is correct*

Problem 4 *Determine the following limits. Use L'Hospital's Rule if applicable.*

(a) $\lim_{x\to\infty}\dfrac{x}{\sqrt{x^2+1}}$

Solution: *This limit is of the form:* $\dfrac{\infty}{\infty}$

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} &= \lim_{x\to\infty}\frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}\\ &= \lim_{x\to\infty}\frac{x}{|x|\sqrt{1+\frac{1}{x^2}}}\\ &= \lim_{x\to\infty}\frac{x}{x\sqrt{1+\frac{1}{x^2}}}\\ &= \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}}\\ &= \frac{1}{\sqrt{1+0}} = 1\end{aligned}$$

(b) $\lim_{x\to-\infty} x^2e^x$

Solution: *This limit is of the form:* $\infty\cdot 0$

$$\begin{aligned}\lim_{x\to-\infty} x^2e^x &= \lim_{x\to-\infty}\frac{x^2}{e^{-x}}\ \left(\text{of the form } \frac{\infty}{\infty}\right)\\ &\overset{L.R.}{=} \lim_{x\to-\infty}\frac{2x}{-e^{-x}}\ \left(\text{of the form } \frac{\infty}{\infty}\right)\\ &\overset{L.R.}{=} \lim_{x\to-\infty}\frac{2}{e^{-x}}\\ &= 0\end{aligned}$$

where "L.R." above an equals sign means that that equality is due to "L'Hospital's Rule".

(c) $\lim_{x\to\infty} x^{\frac{1}{x}}$

Solution: *This limit is of the form:* ∞^0

$$\begin{aligned}
\lim_{x\to\infty} x^{\frac{1}{x}} &= \lim_{x\to\infty} e^{\ln\left(x^{\frac{1}{x}}\right)} \\
&= \lim_{x\to\infty} e^{\frac{1}{x}\ln x} \\
&= e^{\lim_{x\to\infty} \frac{\ln x}{x}} \quad \left(\textit{limit is of the form } \frac{\infty}{\infty}\right) \\
&\overset{L.R.}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{x}}{1}} \\
&= e^{\lim_{x\to\infty} \frac{1}{x}} \\
&= e^0 = 1
\end{aligned}$$

(d) $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x$

Solution: *This limit is of the form:* 1^∞

$$\begin{aligned}
\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x &= \lim_{x\to\infty} e^{\ln\left(\left(1+\frac{2}{x}\right)^x\right)} \\
&= \lim_{x\to\infty} e^{x\ln\left(1+\frac{2}{x}\right)} \\
&= e^{\lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{1/x}} \quad \left(\textit{limit is of the form } \frac{0}{0}\right) \\
&\overset{L.R.}{=} e^{\lim_{x\to\infty} \frac{-2x^{-2}\cdot\frac{1}{(1+2/x)}}{-x^{-2}}} \\
&= e^{\lim_{x\to\infty} \left(2\left(\frac{1}{1+\frac{2}{x}}\right)\right)} \\
&= e^2
\end{aligned}$$

(e) $\lim_{x\to 0^+} (\sin\theta)^{\tan\theta}$

Solution: *This limit is of the form:* 0^0

$$\begin{aligned}
\lim_{x\to 0^+}(\sin\theta)^{\tan\theta} &= \lim_{x\to 0^+} e^{\ln\left((\sin\theta)^{\tan\theta}\right)} \\
&= \lim_{x\to 0^+} e^{\tan\theta\ln(\sin\theta)} \\
&= e^{\lim\limits_{x\to 0^+}\frac{\ln(\sin\theta)}{\cot\theta}} \quad \left(\textit{limit is of the form } \frac{\infty}{\infty}\right) \\
&\overset{L.R.}{=} e^{\lim\limits_{x\to 0^+}\frac{\cos\theta\cdot\frac{1}{\sin\theta}}{-\csc^2\theta}} \\
&= e^{\lim\limits_{x\to 0^+}\left(\frac{\cos\theta}{\sin\theta}\cdot\frac{\sin^2\theta}{1}\right)} \\
&= e^{\lim\limits_{x\to 0^+}(\cos\theta\cdot\sin\theta)} \\
&= e^0 = 1
\end{aligned}$$

Problem 5 *Use limits to compare growth rates. State which function grows faster or that they have comparable growth rates.*

(a) $b^x; x^x; b > 1$

Solution: $\lim\limits_{x\to\infty}\frac{b^x}{x^x} = \lim\limits_{x\to\infty}\left(\frac{b}{x}\right)^x = 0$. *Since this limit is of the form* 0^∞ *which is a determinate form and equals* 0. *Therefore,* x^x *grows faster.*

(b) $x^x; \left(\frac{x}{e}\right)^x$

Solution: $\lim\limits_{x\to\infty}\frac{x^x}{(x/e)^x} = \lim\limits_{x\to\infty}\frac{x^x}{\frac{x^x}{e^x}} = \lim\limits_{x\to\infty} x^x\cdot\frac{e^x}{x^x} = \lim\limits_{x\to\infty} e^x = \infty$ *Therefore,* x^x *grows faster.*

(c) $x^3; x^3\cdot\ln(x)$

Solution: $\lim\limits_{x\to\infty}\frac{x^3}{x^3\cdot\ln(x)} = \lim\limits_{x\to\infty}\frac{1}{\ln(x)} = 0$. *Therefore,* $x^3\cdot\ln(x)$ *grows faster.*

(d) $a^x; b^x; 0 < a < b$

Solution: $\lim\limits_{x\to\infty}\frac{a^x}{b^x} = \lim\limits_{x\to\infty}\left(\frac{a}{b}\right)^x = 0$ *Therefore,* b^x *grows faster*

(e) $\log_a(x); \log_b(x); 1 < a < b$

Solution: $\lim\limits_{x\to\infty}\frac{\log_a(x)}{\log_b(x)}$

By L'Hopital's Rule: $\lim\limits_{x\to\infty}\frac{\frac{1}{x\ln(a)}}{\frac{1}{x\ln(b)}} = \lim\limits_{x\to\infty}\left(\frac{1}{x\ln(a)}\cdot\frac{x\ln(b)}{1}\right) = \frac{\ln b}{\ln a}$

Therefore, $\log_a(x)$ *and* $\log_b(x)$ *grow at comparable rates.*

(f) $\ln^3(x); x^{1/2}$

Solution: $\lim_{x\to\infty} \frac{\ln^3(x)}{x^{1/2}}$

By L'Hopital's Rule: $\lim_{x\to\infty} \frac{\ln^3(x)}{x^{1/2}} = \lim_{x\to\infty} \frac{3\ln^2(x)\cdot(1/x)}{(1/2)x^{-1/2}} = \lim_{x\to\infty} \frac{6\cdot\ln^2(x)}{x^{1/2}}$

By L'Hopital's Rule: $\lim_{x\to\infty} \frac{12\ln(x)\cdot(1/x)}{(1/2)x^{-1/2}} = \lim_{x\to\infty} \frac{24\cdot\ln(x)}{x^{1/2}}$

By L'Hopital's Rule: $\lim_{x\to\infty} \frac{48}{x^{1/2}} = 0$ *Therefore,* $x^{1/2}$ *grows faster*

(g) $x; \ln(x)\sqrt{x}$

Solution: $\lim_{x\to\infty} \frac{x}{\ln(x)\sqrt{x}} = \lim_{x\to\infty} \frac{x^{1/2}}{\ln x}$

By L'Hopital's Rule: $\lim_{x\to\infty} \frac{(1/2)x^{-1/2}}{x^{-1}} = \lim_{x\to\infty} (1/2)x^{1/2} = \infty$

Therefore, x grows faster

(h) *Challenge:* $x^{40}; 1.004^x$ *(Hint: Use the substitution $x = lnt$.)*

Solution:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^{40}}{1.004^x} &= \lim_{t\to\infty} \frac{\ln(t)^{40}}{1.004^{\ln(t)}} \\
&= \lim_{t\to\infty} \frac{\ln(t)^{40}}{e^{\ln(1.004)^{\ln t}}} \\
&= \lim_{t\to\infty} \frac{\ln(t)^{40}}{t^{\ln(1.004)}} \\
&= \lim_{t\to\infty} \left(\frac{\ln(t)}{t^{1.004/40}}\right)^{40} \\
&= \left[\lim_{t\to\infty} \left(\frac{\ln(t)}{t^{1.004/40}}\right)\right]^{40} \\
&\overset{L.R.}{=} \left[\lim_{t\to\infty} \frac{\frac{1}{t}}{(1.004/40)t^{(1.004/40)-1}}\right]^{40} \\
&= \left[\lim_{t\to\infty} \frac{1}{(1.004/40)t^{((1.004/40)-1+1)}}\right]^{40} \\
&= 0^{40} \\
&= 0
\end{aligned}$$

Therefore, 1.004^x *grows faster.*

(i) *Use Briggs Theorem 4.15 to put the following functions in order of growth rate.*

$$x^3\cdot\ln(x), \ln^3(x), x^x, 1.004^x, x^{40}, x^3$$

Solution:

$$\ln^3(x) << x^3 << x^3\cdot\ln(x) << x^{40} << 1.004^x << x^x$$