MATH 2173 – Engineering Calculus B – Practice Midterm 1 Version 1 Solutions

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Procedures for the exam: Please log on to the Main Lecture Zoom room by **10:15 am ET**. You will be split into a breakout room with one of the moderators.
`https://osu.zoom.us/j/96249977566?pwd=Wk5jN1k5c2JmZkZEUTRVOHNhOUJDdz09`
Zoom Meeting ID: 962 4997 7566, Password: 650807
At **10:20 am ET**, the exam will be made available on the Carmen Homepage. At this time, you may download the exam and immediately begin working. You may download, complete, and submit the assignment using an ipad or other tablet device. You may also do your work on paper and then scan and submit your work. If you choose the first option, please use the exam template for your work. If you choose the second option, be sure to clearly label your work. If you have questions throughout the exam, you can direct message your moderator using the chat feature in Zoom.
When you have completed the exam, you should send a message to your moderator letting them know that you are no longer writing and are beginning to submit. You may then submit to Gradescope. You must submit the exam to Gradescope by **11:15 am ET**. If you have not finished by 11:10 am ET, an announcement will be made letting you know that you must now begin submitting. In other words, you should spend approximately 50 minutes working on the exam with the remaining five minutes left for submitting the exam.

Exam rules: In order to get credit for the exam, you must be in the Main Lecture Zoom room for the duration of the exam with your webcam on. No books, no notes, no calculators, and no internet resources may be used to complete the exam.
You must show your work. Work on the scrap work page will not be graded unless you indicate otherwise. Your work must be legible, and your final answers must be reasonably simplified.
On some problems, you are asked to use a specific method to solve the problem. On all other problems, you may use any method we've covered. You may not use methods we have not covered.
If at any point you experience technical difficulties, you must immediately e-mail both Dr. Skipper and your recitation instructor. Your email **must** include a complete copy of your exam, even if that is just in the form of photographs taken on your phone.
Unlike the recitation handouts, you are permitted to append additional pages directly after the problem they are for instead of following the template. The template formatting is optional for this exam.
Good luck!

Problem 1 (6 points): Below is a contour plot of some function $z = f(x, y)$ along with 4 vectors.

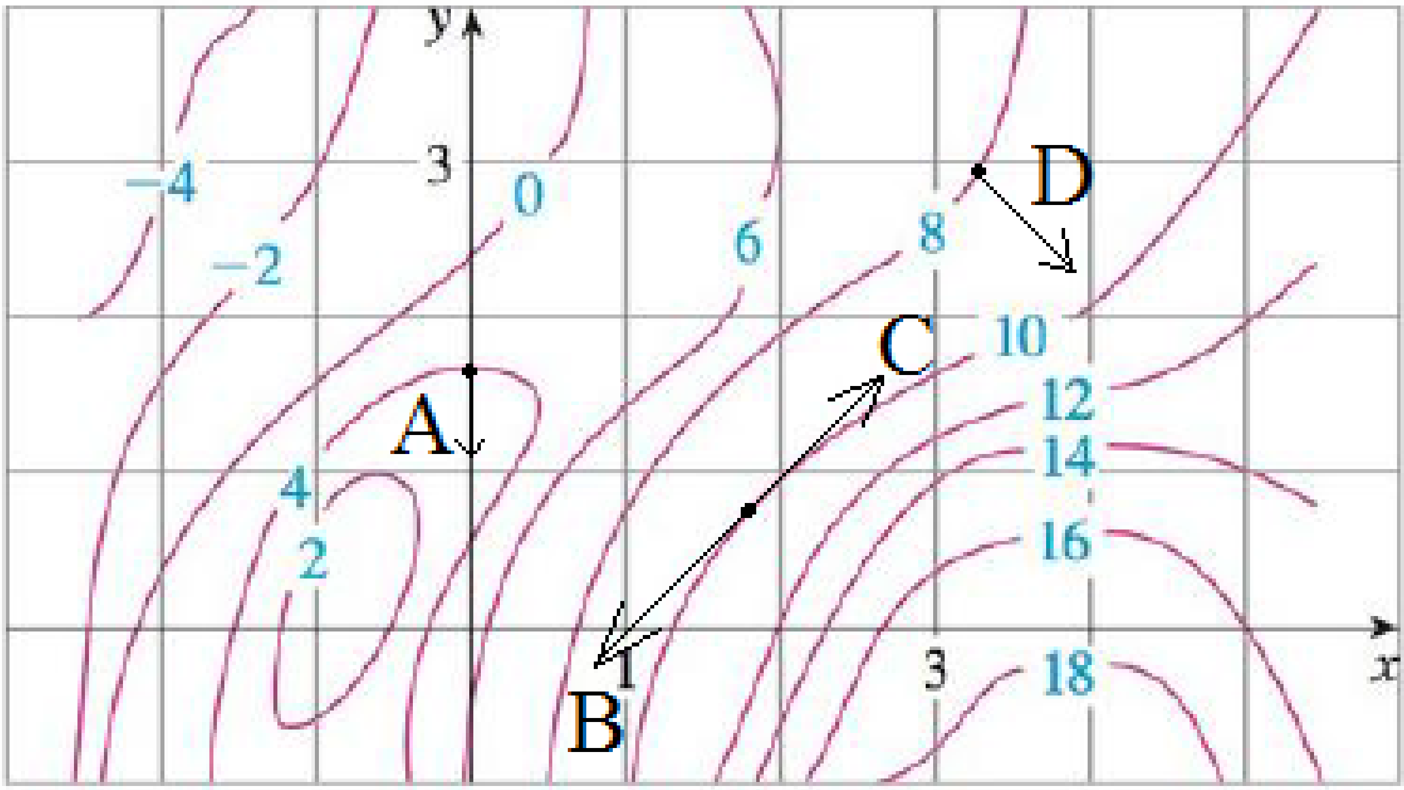


FIGURE 1. Contour plot of $z = f(x, y)$.

Which of the vectors in the above plot could possibly be a gradient vector of the function $f(x, y)$? Please circle all that apply.

(A) (B) (C) (D)
None of the vectors could possibly be a gradient vector for $f(x, y)$.

Explanation: The gradient vector of a function $f(x, y)$ is normal to the level curves (the curves of the form $f(x, y) = c$, with c a constant) and points in the direction of maximum increase. We see that vector A is normal to a level curve of f, but points in the direction of decrease and is therefore not a gradient vector. We see that vectors B and C are tangent to a level curve, not normal to the level curve, so neither of them can be a gradient vector. We see that vector D is normal to a level curve of f and points in the direction of increase, so D could be a gradient vector of f.

Problem 2 (14 points): Consider the function $f(x,y) = x^2 + y^2$ and and the point $P = (2,3)$.

(a) Find the unit vector that points in direction of maximum decrease of the function f at the point P.

(b) Calculate the directional derivative of f at the point P in the direction of the vector $\vec{u} = \langle 3, 2\rangle$.

Solution to (a): We see that $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle 2x, 2y\rangle$. We see that $-\nabla f(2,3) = \langle -4, -6\rangle$ is a vector that points in the direction of maximum decrease of f at the point P. Since $|\langle -4,-6\rangle| = \sqrt{52} = 2\sqrt{13}$, we see that

(1) $$\frac{\langle -4,-6\rangle}{|\langle -4,-6\rangle|} = \frac{1}{2\sqrt{13}}\langle -4,-6\rangle = \boxed{\langle \frac{-2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}\rangle}$$

is the direction of maximum decrease of f at the point P.

Solution to (b): We see that $|\vec{u}| = \sqrt{13}$, so

(2) $$\vec{w} = \frac{\vec{u}}{|\vec{u}|} = \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle$$

is the unit vector that points in the same direction as $\vec{u}$, so

(3) $$d_{\vec{w}}f(2,3) = \nabla f(2,3)\cdot\vec{w} = \langle 4, 6\rangle \cdot \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle = \boxed{\frac{24}{\sqrt{13}}}.$$

Problem 3 (14 points): Determine all critical points of the function $f(x,y) = x^3 - y^3 + xy$, then classify each of the critical points as a local maximum, local minimum, or saddle point.

Solution: To find the criticial points of f, we simply have to find all (x,y) for which both partial derivatives of f are 0.

$$\begin{array}{rcl} f_x(x,y) & = & 0 \\ f_y(x,y) & = & 0 \end{array} \Leftrightarrow \begin{array}{rcl} 3x^2 + y & = & 0 \\ -3y^2 + x & = & 0 \end{array} \Leftrightarrow \begin{array}{rcl} -3x^2 & = & y \\ 3y^2 & = & x \end{array} \tag{4}$$

$$\rightarrow x = 3(-3x^2)^2 = 27x^4 \rightarrow x = 0, \frac{1}{3} \rightarrow (x,y) = \boxed{(0,0), (\frac{1}{3}, -\frac{1}{3})}. \tag{5}$$

We now proceed to calculate all of the second derivatives of f as well as the discriminant function so that we can apply the second derivative test.

$$\begin{array}{rcl} f_{xx}(x,y) & = & 6x \\ f_{yy}(x,y) & = & -6y \\ f_{xy}(x,y) & = & 1 \end{array} \tag{6}$$

$$\rightarrow D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = -36xy - 1. \tag{7}$$

Since $D(0,0) = -1 < 0$, we see that $\boxed{(0,0) \text{ is a saddle point}}$.

Since $D(\frac{1}{3}, -\frac{1}{3}) = 3 > 0$ and $f_{xx}(\frac{1}{3}, -\frac{1}{3}) = 2 > 0$ we see that $\boxed{(\frac{1}{3}, -\frac{1}{3}) \text{ is a local minimum}}$.

Problem 4 (22 points): Find the absolute minimum and absolute maximum values of the function $f(x, y) = xy$ over the region $R = \{(x, y) \mid (x - 1)^2 + y^2 \leq 1\}$.

Solution: Since R is a closed and bounded region, and f is a continuous function, the Extreme Value Theorem tells us that f will attain its absolute minimum and absolute maximum values over the region R. Furthermore, we know that the extreme values of f will either be attained on the boundary of R, or at a critical point of f in the interior of R.

We will begin by finding all critical points in the interior of R. Since $f_x(x, y) = y$ and $f_y(x, y) = x$, we immediately see that $(0, 0)$ is the only critical point of f, and it is on the boundary (not interior) of the region R, but it is still a candidate for where f can attain one of its extreme values. We note that $f(0, 0) = 0$.

We will now proceed to find the absolute minimum and absolute maximum values of f on the boundary of R. Since the boundary of R is given by $\partial R = \{(x, y) \mid (x - 1)^2 + y^2 = 1\}$, we will use the method of Lagrange Multipliers to optimize the function $f(x, y) = xy$ subject to the constraint $g(x, y) = (x - 1)^2 + y^2 - 1 = 0$. We note that

$$\nabla f(x, y) = \langle y, x \rangle \text{ and } \nabla g(x, y) = \langle 2x - 2, 2y \rangle, \tag{8}$$

so the method of Lagrange Multipliers results in the following system of equations for us to solve:

$$\begin{aligned} g(x, y) &= 0 \\ \nabla f(x, y) &= \lambda \nabla g(x, y) \end{aligned} \Leftrightarrow \begin{aligned} (x - 1)^2 + y^2 &= 1 \\ y &= \lambda(2x - 2) \\ x &= \lambda 2y \end{aligned} \tag{9}$$

$$\rightarrow \lambda x(2x - 2) = xy = \lambda 2y^2 \rightarrow 0 = 2\lambda(y^2 - x^2 + x). \tag{10}$$

By the zero-product property, we see that we must have $\lambda = 0$ or $y^2 - x^2 + x = 0$, so we will consider both cases separately.

Case 1: For our first case let us assume that $\lambda = 0$. In this case we see that the last 2 equations from (9) tell us that $x = y = 0$, since $g(0, 0) = 0$, we see that we reobtain the critical point $(x, y) = (0, 0)$.

Case 2: For our next case let us assume that $y^2 - x^2 + x = 0$, so $y^2 = x^2 - x$. We see that

$$1 = y^2 + (x-1)^2 = x^2 - x + (x-1)^2 = 2x^2 - 3x + 1 \tag{11}$$

$$\to 2x^2 - 3x = 0 \to x = 0, \frac{3}{2} \to (x,y) = (0,0), (\frac{3}{2}, \frac{\sqrt{3}}{2}), (\frac{3}{2}, -\frac{\sqrt{3}}{2}). \tag{12}$$

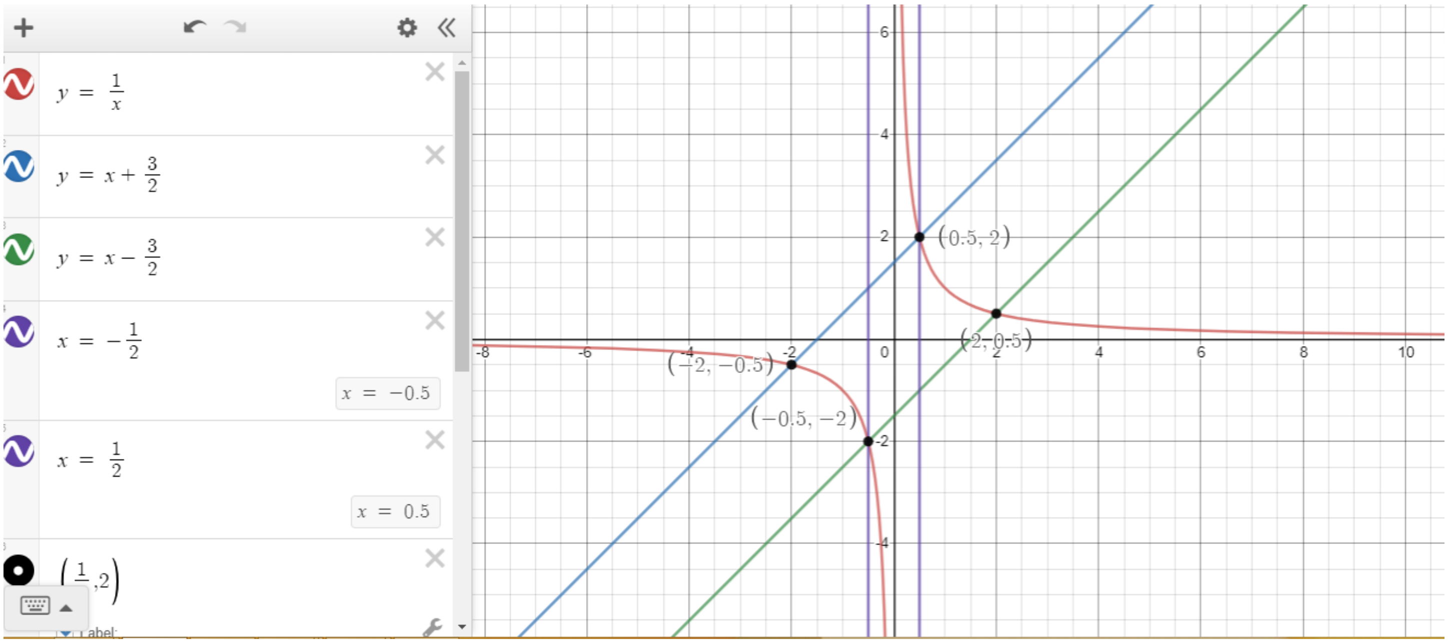
Making a table of our critical points and corresponding values of f, we see that

(x,y)	$f(x,y)$
$(0,0)$	0
$(\frac{3}{2}, \frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{4}$
$(\frac{3}{2}, -\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{4}$

so f attains its absolute maximum value of $\frac{3\sqrt{3}}{4}$ at the point $(\frac{3}{2}, \frac{\sqrt{3}}{2})$ and f attains its absolute minimum value of $-\frac{3\sqrt{3}}{4}$ at the point $(\frac{3}{2}, -\frac{\sqrt{3}}{2})$.

Problem 5 (22 points): Let R be the region that is bounded by both branches of $y = \frac{1}{x}$, the line $y = x + \frac{3}{2}$, and the line $y = x - \frac{3}{2}$. Sketch a picture of the region R, then write a sum of double integrals that would calculate the area of R. DO NOT EVALUATE THE INTEGRALS.

Solutions: We first sketch a picture of the region R.



We now solve for the intersection points of the curves $y = \frac{1}{x}$ and $y = x + \frac{3}{2}$ to see that

$$\begin{matrix} y = \frac{1}{x} \\ y = x + \frac{3}{2} \end{matrix} \rightarrow \frac{1}{x} = x + \frac{3}{2} \rightarrow x^2 + \frac{3}{2}x - 1 = 0 \tag{13}$$

$$\rightarrow x = -2, \frac{1}{2} \rightarrow (x, y) = (-2, -\frac{1}{2}), (\frac{1}{2}, 2). \tag{14}$$

Similarly, we solve for the intersection points of the curves $y = \frac{1}{x}$ and $y = x - \frac{3}{2}$ to see that

$$\begin{matrix} y = \frac{1}{x} \\ y = x - \frac{3}{2} \end{matrix} \rightarrow \frac{1}{x} = x - \frac{3}{2} \rightarrow x^2 - \frac{3}{2}x - 1 = 0 \tag{15}$$

$$\rightarrow x = -\frac{1}{2}, 2 \rightarrow (x, y) = (-\frac{1}{2}, -2), (2, \frac{1}{2}). \tag{16}$$

We now see that the area of R is given by

$$\iint_R 1dA = \iint_R 1dydx \tag{17}$$

$$= \boxed{\int_{-2}^{-\frac{1}{2}} \int_{\frac{1}{x}}^{x+\frac{3}{2}} 1dydx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{x-\frac{3}{2}}^{x+\frac{3}{2}} 1dydx + \int_{\frac{1}{2}}^{2} \int_{x-\frac{3}{2}}^{\frac{1}{x}} 1dydx} \tag{18}$$

$$= \boxed{\int_{-2}^{-\frac{1}{2}} \int_{\frac{1}{y}}^{y+\frac{3}{2}} 1dxdy + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{y-\frac{3}{2}}^{y+\frac{3}{2}} 1dxdy + \int_{\frac{1}{2}}^{2} \int_{y-\frac{3}{2}}^{\frac{1}{y}} 1dxdy} \tag{19}$$

Note: You only need one of (18) *or* (19) *in order to receive full credit.*

Problem 6 (22 points): Let R be the region in the xy-plane that is bounded by the spiral $r = \theta$ for $0 \leq \theta \leq \pi$ and the x-axis. Find the volume of the 3-dimensional solid S that lies above the region R and underneath the surface $z = x^2 + y^2$.

Solution: Below is a picture of the region R, which is the base of our solid S.

$$\text{Volume}(S) = \iint_R (z_{\text{top}} - z_{\text{bot.}})dA = \iint_R \underbrace{(x^2 + y^2) - 0}_{r^2} \underbrace{dA}_{r dr d\theta} \tag{20}$$

. . . .

$$= \int_0^{\pi} \int_0^{\theta} r^3 dr d\theta = \int_0^{\pi} \frac{1}{4} r^4 \Big|_{r=0}^{\theta} d\theta = \int_0^{\pi} \frac{1}{4}\theta^4 d\theta \tag{21}$$

. . . .

$$= \frac{1}{20}\theta^5 \Big|_0^{\pi} = \boxed{\frac{\pi^5}{20}}. \tag{22}$$