MATH 2173 – Engineering Calculus B – Practice Midterm 2 Solutions

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Procedures for the exam: Please log on to the Main Lecture Zoom room by **10:15 am ET**. You will be split into a breakout room with one of the moderators.
https://osu.zoom.us/j/96249977566?pwd=Wk5jN1k5c2JmZkZEUTRVOHNhOUJDdz09
Zoom Meeting ID: 962 4997 7566, Password: 650807
At **10:20 am ET**, the exam will be made available on the Carmen Homepage. At this time, you may download the exam and immediately begin working. You may download, complete, and submit the assignment using an ipad or other tablet device. You may also do your work on paper and then scan and submit your work. If you choose the first option, please use the exam template for your work. If you choose the second option, be sure to clearly label your work. If you have questions throughout the exam, you can direct message your moderator using the chat feature in Zoom.
When you have completed the exam, you should send a message to your moderator letting them know that you are no longer writing and are beginning to submit. You may then submit to Gradescope. You must submit the exam to Gradescope by **11:15 am ET**. If you have not finished by 11:10 am ET, an announcement will be made letting you know that you must now begin submitting. In other words, you should spend approximately 50 minutes working on the exam with the remaining five minutes left for submitting the exam.

Exam rules: In order to get credit for the exam, you must be in the Main Lecture Zoom room for the duration of the exam with your webcam on. No books, no notes, no calculators, and no internet resources may be used to complete the exam.
You must show your work. Work on the scrap work page will not be graded unless you indicate otherwise. Your work must be legible, and your final answers must be reasonably simplified.
On some problems, you are asked to use a specific method to solve the problem. On all other problems, you may use any method we've covered. You may not use methods we have not covered.
If at any point you experience technical difficulties, you must immediately e-mail both Dr. Skipper and your recitation instructor. Your email **must** include a complete copy of your exam, even if that is just in the form of photographs taken on your phone.
Unlike the recitation handouts, you are permitted to append additional pages directly after the problem they are for instead of following the template. The template formatting is optional for this exam.
Good luck!

Modified Problem 14.5.49 (8+15 points): Let S be the solid region in the first octant outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos(\varphi)$.

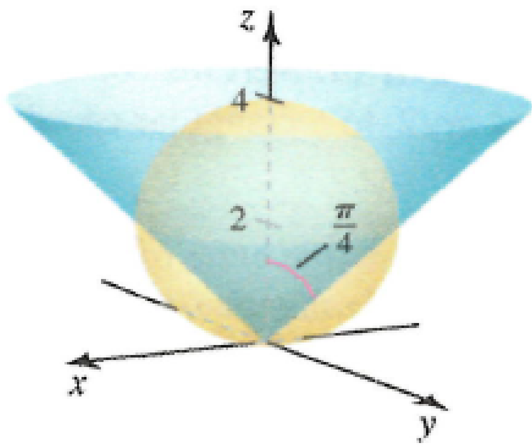


FIGURE 1. The unmodified picture of S from the textbook.

a. Set up but do not evaluate a triple integral in spherical coordinates to find volume of S.

b. Set up but do not evaluate a triple integral in cylindrical coordinates to find volume of S.

Solution to a: Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to polar coordinates. Remember that the angle φ is measured from the z-axis and satisfies $0 \leq \varphi \leq \pi$, not $0 \leq \varphi \leq 2\pi$. Also remember that this cross section is showing you the portions of the solid from $\theta = 0$ and $\theta = \pi$.

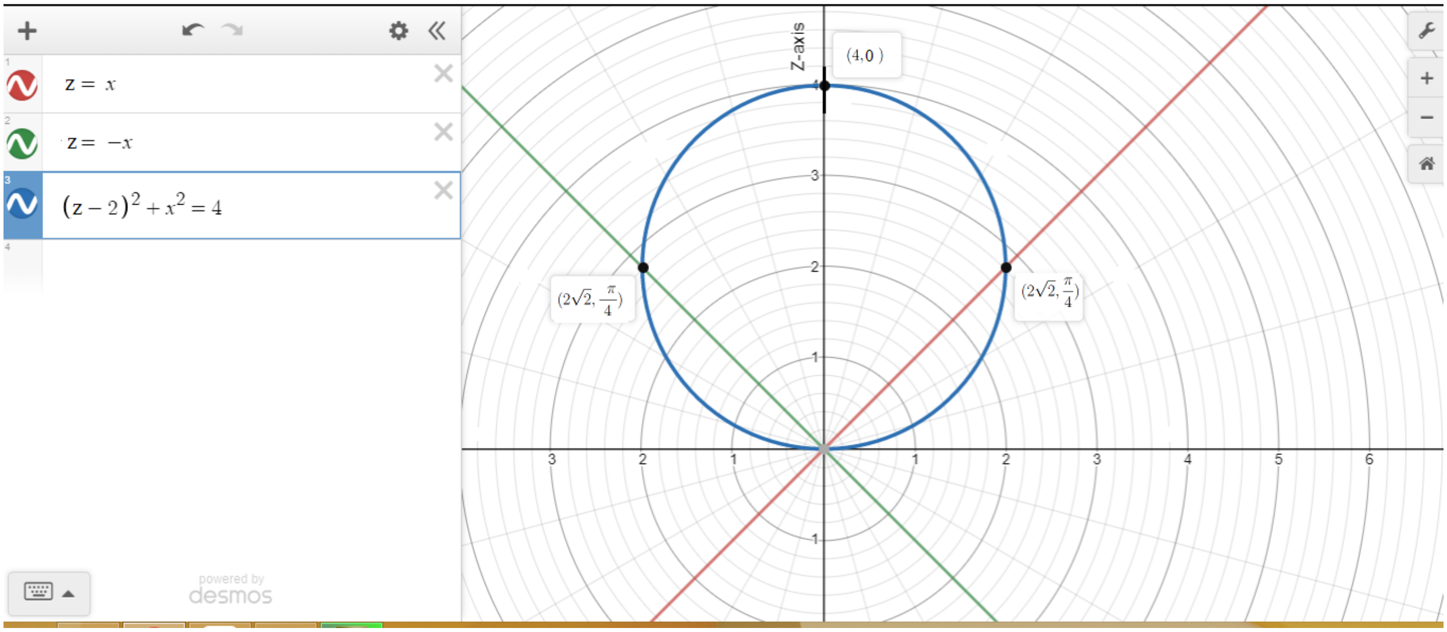


FIGURE 2. The xz-plane cross section in spherical coordinates.

Since we are only working in the first octant, we have $0 \leq \theta \leq \frac{\pi}{2}$. We now see that for any $\theta \in [0, \frac{\pi}{2})$ we have that $\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$. Recalling that the blue circle is defined by $\rho = 4\cos(\varphi)$, we see that once φ is also chosen we have that $0 \leq \rho \leq 4\cos(\varphi)$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \boxed{\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4\cos(\varphi)} \rho^2\sin(\varphi)d\rho d\varphi d\theta} \tag{1}$$

$$= \int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{3}\rho^3\sin(\varphi)\Big|_{\rho=0}^{4\cos(\varphi)} d\varphi d\theta = \int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{64}{3}\underbrace{\cos^3(\varphi)}_{u^3}\underbrace{\sin(\varphi)d\varphi}_{-du} d\theta \tag{2}$$

$$= -\frac{64}{3}\int_0^{\frac{\pi}{2}}\int_{\varphi=\frac{\pi}{4}}^{\frac{\pi}{2}} u^3 du d\theta = -\frac{64}{3}\int_0^{\frac{\pi}{2}} \frac{1}{4}u^4\Big|_{\varphi=\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \tag{3}$$

$$= -\frac{64}{3}\int_0^{\frac{\pi}{2}} \frac{1}{4}\cos^4(\varphi)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta = -\frac{64}{3}\int_0^{\frac{\pi}{2}} -\frac{1}{16}d\theta = -\frac{64}{3}\cdot\frac{\pi}{2}\cdot\frac{-1}{16} = \boxed{\frac{2\pi}{3}}. \tag{4}$$

Solution to b: Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to Cartesian coordinates with (r, z) taking the place of (x, y). Remember that this cross section is also showing you the portions of the solid from $\theta = 0$ and $\theta = \pi$.

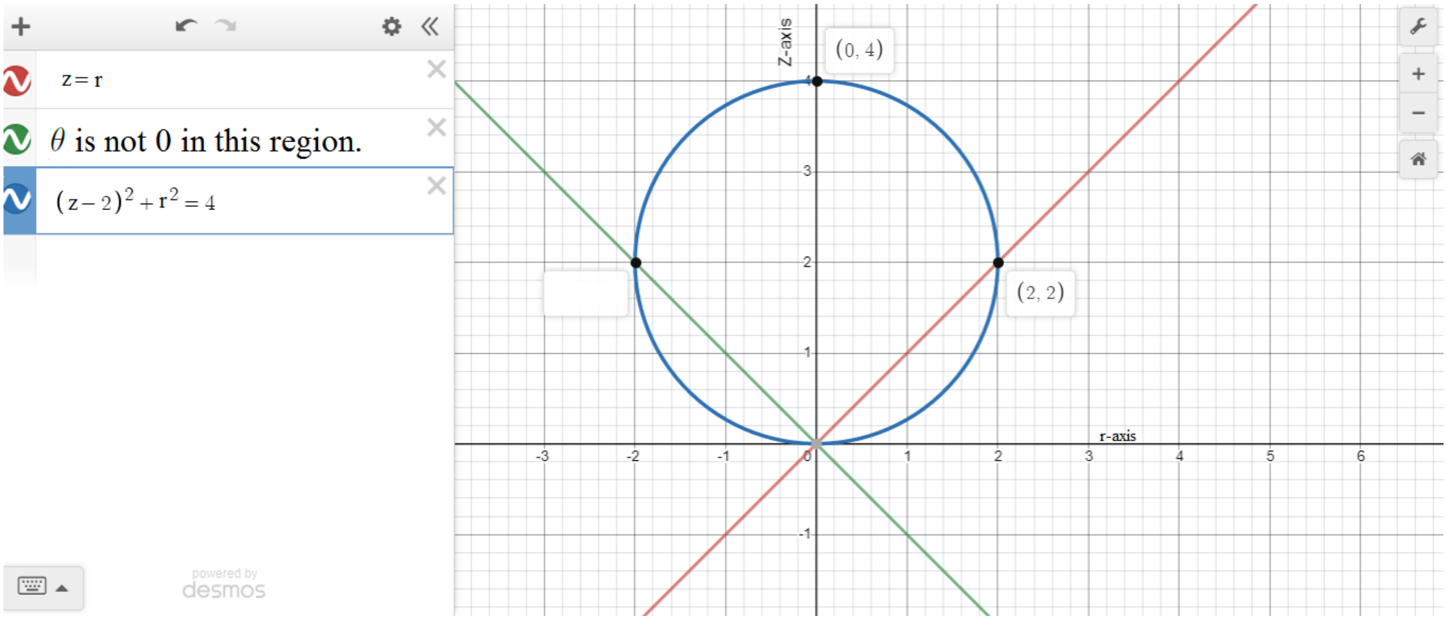


FIGURE 3. The xz-plane cross section in cylindrical coordinates.

Since we are only working in the first octant, we have $0 \leq \theta \leq \frac{\pi}{2}$. We now see that for any $0 \leq \theta < \frac{\pi}{2}$ we have that $0 \leq z \leq 2$. Noting that we have $r = \sqrt{4-(z-2)^2} = \sqrt{4z - z^2}$ on the blue circle, we see that once z is chosen we have $z \leq r \leq \sqrt{4z - z^2}$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \boxed{\int_0^{\frac{\pi}{2}} \int_0^2 \int_z^{\sqrt{4z-z^2}} rdrdzd\theta} \tag{5}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 \frac{1}{2}r^2\Big|_z^{\sqrt{4z-z^2}} dzd\theta = \int_0^{\frac{\pi}{2}} \int_0^2 (2z - z^2)dzd\theta \tag{6}$$

$$\int_0^{\frac{\pi}{2}} (z^2 - \frac{1}{3}z^3)\Big|_0^2 d\theta = \int_0^{\frac{\pi}{2}} \frac{4}{3}d\theta = \boxed{\frac{2\pi}{3}}. \tag{7}$$

Modified Problem 14.R.77 (25 points): Let R be the region in the first quadrant bounded by the hyperbolas $xy = 1$ and $xy = 4$ and the lines $y = x$ and $y = 3x$. Evaluate

$$\iint_R y^4 dA. \tag{8}$$

Solution: Noting that the line $y = x$ can be rewritten as $\frac{y}{x} = 1$ and the line $y = 3x$ can be rewritten as $\frac{y}{x} = 3$, we decide to use the change of variables $u = xy$ and $v = \frac{y}{x}$ in order to make our new region of integration in the uv-place a rectangle. In particular, we see that $R' = \{(u, v) : 1 \le u \le 4, 1 \le v \le 3\}$ is the new region of integration.

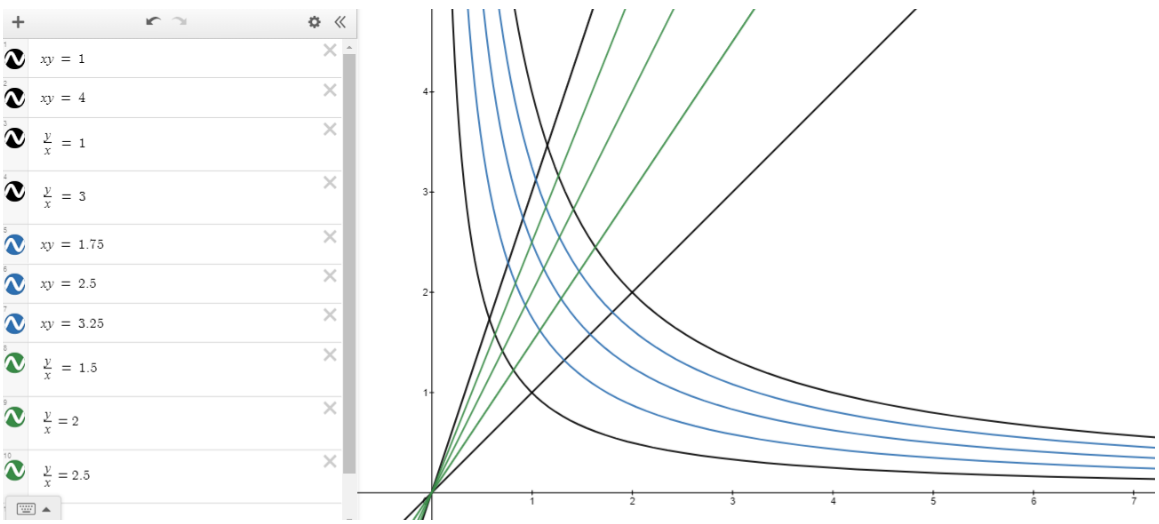


FIGURE 4. The original region of integration in the xy-plane R.

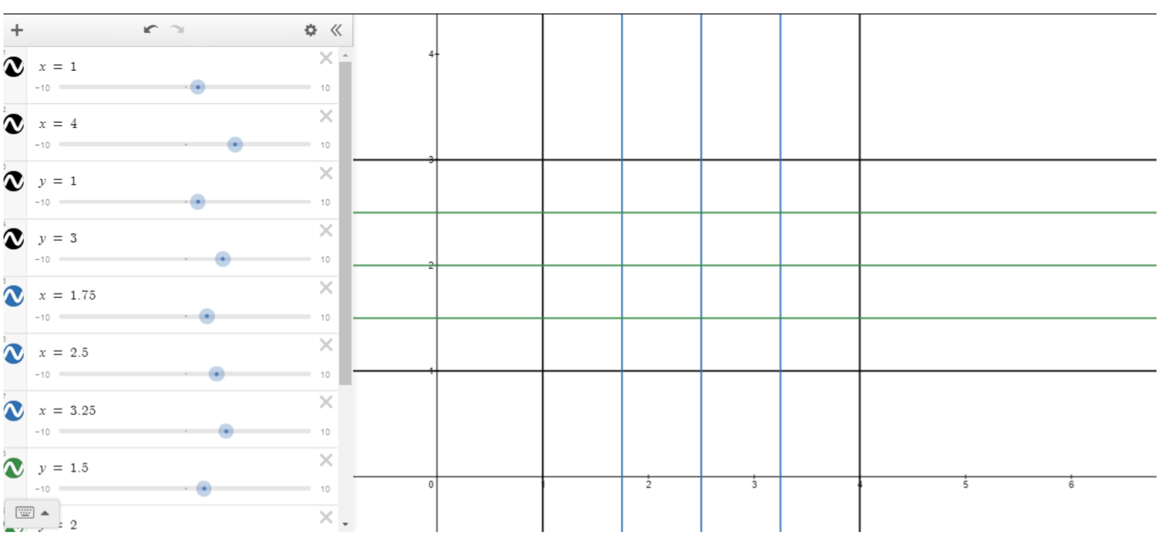


FIGURE 5. The new region of integration in the uv-plane R'.

In order to calculate the Jacobian $J(u,v)$ we must first solve for x and y in terms of u and v. To that end, we see that

$$\begin{array}{rcl} u &=& xy \\ v &=& \frac{y}{x} \end{array} \rightarrow x = (x^2)^{\frac{1}{2}} = (\frac{u}{v})^{\frac{1}{2}} = u^{\frac{1}{2}}v^{-\frac{1}{2}} \text{ and } y = (y^2)^{\frac{1}{2}} = u^{\frac{1}{2}}v^{\frac{1}{2}}. \tag{9}$$

We note that we took the positive square roots above since we are working in the first quadrant of the xy-place, so x and y are both nonnegative. We now see that

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}u^{-\frac{1}{2}}v^{-\frac{1}{2}} & -\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}} \\ \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}} & \frac{1}{2}u^{\frac{1}{2}}v^{-\frac{1}{2}} \end{vmatrix} \tag{10}$$

$$= \frac{1}{2}u^{-\frac{1}{2}}v^{-\frac{1}{2}} \cdot \frac{1}{2}u^{\frac{1}{2}}v^{-\frac{1}{2}} - (-\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}}) \cdot \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}} = \frac{1}{2}v^{-1}. \tag{11}$$

Since $1 \leq v \leq 3$ in our new region of integration R', we see that $\frac{1}{2}v^{-1} \geq 0$ on R', so $|J(u,v)| = J(u,v)$ on R'. We now see that

$$\iint_R y^4 dA = \iint_{R'} (u^{\frac{1}{2}}v^{\frac{1}{2}})^4 |J(u,v)| dA = \int_1^4 \int_1^3 u^2v^2 \cdot \frac{1}{2}v^{-1} dv du \tag{12}$$

$$= \frac{1}{2}\int_1^4 \int_1^3 u^2 v dv du = \frac{1}{2}\int_1^4 \frac{1}{2}u^2v^2\Big|_{v=1}^3 du \tag{13}$$

$$= \int_1^4 2u^2 du = \frac{2}{3}u^3\Big|_1^4 = \boxed{42}. \tag{14}$$

Problem 15.2.21 (18 points): Find the average value of the function $f(x,y) = x + 2y$ on the line segment from $(1,1)$ to $(2,5)$.

Solution: Firstly, we recall that the average value of a function f over a curve C is given by

$$\text{Av}(f) = \frac{\int_C f\,ds}{\text{Arclength}(c)} = \frac{\int_C f\,ds}{\int_C 1ds}. \tag{15}$$

In order to calculate the relevant line integrals, we begin by parameterizing the line segment from $(1,1)$ to $(2,5)$. We see that

$$\vec{r}(t) = \langle 1,1\rangle + t(\langle 2,5\rangle - \langle 1,1\rangle) = \langle 1+t, 1+4t\rangle, 0 \le t \le 1, \tag{16}$$

is a parameterization of the line segment from $(1,1)$ to $(2,5)$. It follows that

$$\vec{r}\,'(t) = \langle 1,4\rangle \to |\vec{r}\,'(t)| - \sqrt{1^2+4^2} = \sqrt{17}. \tag{17}$$

We are now able to calculate both of the relevant line integrals.

$$\int_C f\,ds = \int_0^1 f(\vec{r}(t))|\vec{r}\,'(t)|dt = \int_0^1 f(1+t, 1+4t)\sqrt{17}dt \tag{18}$$

$$= \int_0^1 \Big((1+t) + 2(1+4t)\big)\sqrt{17}dt = \sqrt{17}\int_0^1 (3+9t)dt \tag{19}$$

$$= \sqrt{17}(3t + \frac{9}{2}t^2\Big|_0^1) = \frac{15\sqrt{17}}{2}. \tag{20}$$

$$\int_C 1ds = \int_0^1 1\cdot|\vec{r}\,'(t)|dt = \int_0^1 \sqrt{17}dt = \sqrt{17} \tag{21}$$

$$\to \text{Av}(f) = \frac{\int_C f\,ds}{\int_C 1ds} = \frac{\frac{15\sqrt{17}}{2}}{\sqrt{17}} = \boxed{\frac{15}{2}}. \tag{22}$$

Problem 15.2.43 (14 points): Given the force field $\mathbf{F} = \langle x, y, z\rangle$, find the work required to move an object around the tilted ellipse that is parameterized by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 4\cos(t)\rangle$, $0 \le t \le 2\pi$.

Solution: We see that

$$\text{Work} = \int_C \mathbf{F}\cdot\hat{T}ds = \int_0^{2\pi} \mathbf{F}(4\cos(t), 4\sin(t), 4\cos(t))\cdot\vec{r}\,'(t)dt \tag{23}$$

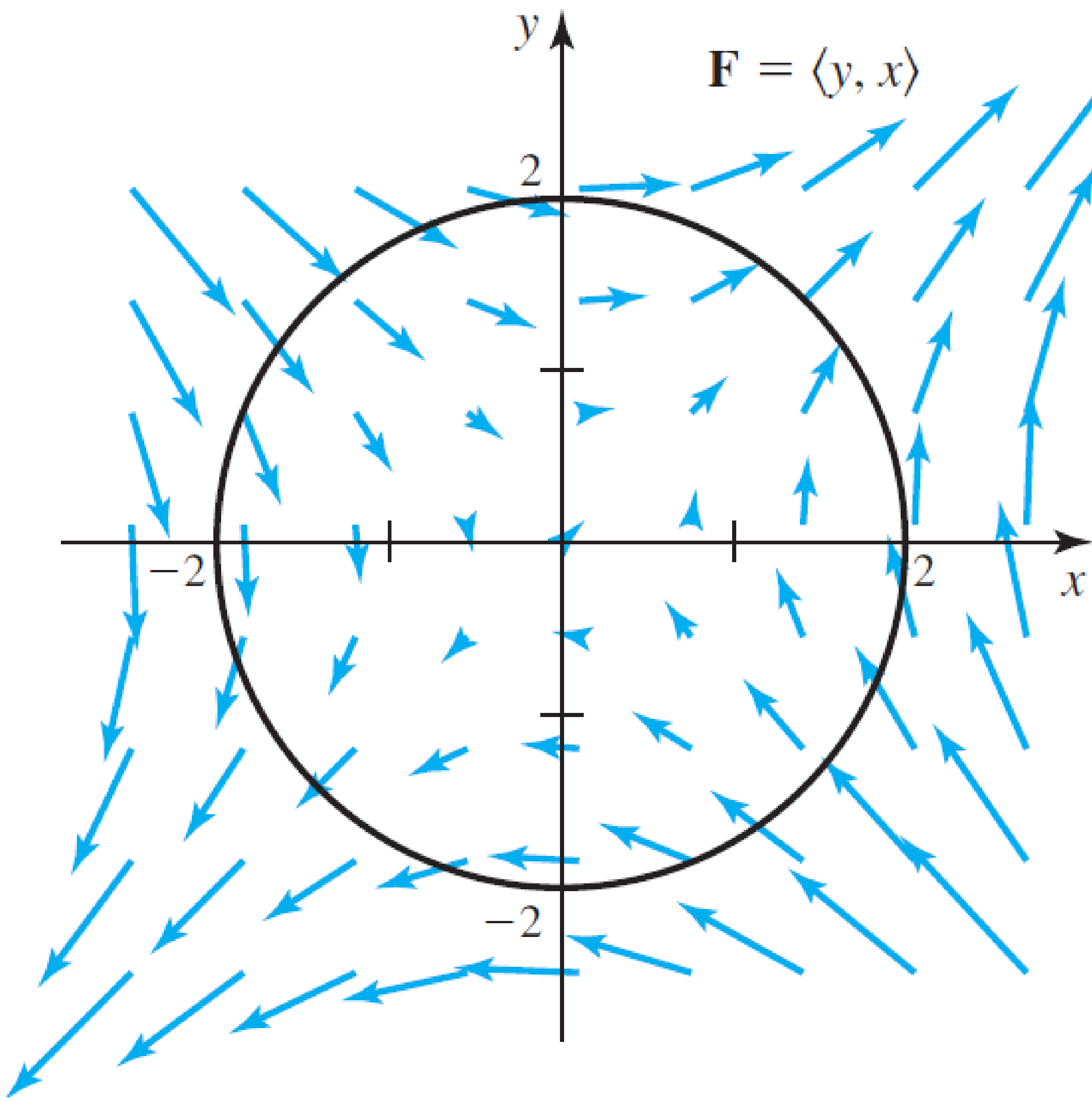
$$= \int_0^{2\pi} \underbrace{\langle 4\cos(t), 4\sin(t), 4\cos(t)\rangle}_{\mathbf{F}(\vec{r}(t))=\vec{r}(t)\text{ coincidentally}}\cdot\langle -4\sin(t), 4\cos(t), -4\sin(t)\rangle dt \tag{24}$$

$$= \int_0^{2\pi} (-16\cos(t)\sin(t) + 16\sin(t)\cos(t) - 16\cos(t)\sin(t))dt \tag{25}$$

$$= \int_0^{2\pi} -16\sin(t)\cos(t)dt = \int_0^{2\pi} -8\sin(2t)dt \tag{26}$$

$$= 4\cos(2t)\Big|_0^{2\pi} = \boxed{0}. \tag{27}$$

Problem 15.2.66 (20 points): Consider the flow field $\mathbf{F} = \langle y, x \rangle$ shown in the figure below.



a. Compute the outward flux across the quarter circle C:$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$, $0 \leq t \leq \frac{\pi}{2}$.

b. Compute the outward flux across the quarter circle C:$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$, $\frac{\pi}{2} \leq t \leq \pi$.

c. Explain why the flux across the quarter circle in the third quadrant equals the flux computed in part **a**.

d. Explain why the flux across the quarter circle in the fourth quadrant equals the flux computed in part **b**.

e. What is the outward flux across the full circle?

Solution to a: We begin by calculating the unit normal vector $\hat{n}(t)$ at any point on the circle (as opposed to only on the first quadrant). We see that

$$\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t)\rangle \to |\mathbf{r}'(t)| = \sqrt{(-2\sin(t))^2 + (2\cos(t))^2} = 2 \tag{28}$$

$$\to \hat{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)} = \frac{\langle -2\sin(t), 2\cos(t)\rangle}{2} = \langle -\sin(t), \cos(t)\rangle \tag{29}$$

$$\to \hat{n}(t) = \hat{T}(t) \times \hat{k} = \langle \cos(t), -(-\sin(t))\rangle = \langle \cos(t), \sin(t)\rangle. \tag{30}$$

We now are able to calculate the desired flux as

$$\text{Flux}(C) = \int_C \mathbf{F}\cdot\hat{n}ds = \int_0^{\frac{\pi}{2}} \mathbf{F}(2\cos(t), 2\sin(t)) \cdot \langle \cos(t), \sin(t)\rangle \underbrace{2dt}_{ds} \tag{31}$$

$$= \int_0^{\frac{\pi}{2}} \langle 2\sin(t), 2\cos(t)\rangle \cdot \langle 2\cos(t), 2\sin(t)\rangle dt \tag{32}$$

$$= \int_0^{\frac{\pi}{2}} (4\sin(t)\cos(t) + 4\cos(t)\sin(t))dt \tag{33}$$

$$= \int_0^{\frac{\pi}{2}} 8\sin(t)\cos(t)dt = \int_0^{\frac{\pi}{2}} 4\sin(2t)dt = -2\cos(2t)\Big|_0^{\frac{\pi}{2}} = \boxed{4}. \tag{34}$$

Solution to b: Since we have already found $\hat{n}(t)$ in part **a**, we proceed directly to the calculation of the flux, which is also similar to the calculation that we did in part **a**.

$$\text{Flux}(C) = \int_C \mathbf{F}\cdot\hat{n}ds = \int_{\frac{\pi}{2}}^{\pi} \mathbf{F}(2\cos(t), 2\sin(t)) \cdot \langle \cos(t), \sin(t)\rangle 2dt \tag{35}$$

$$= \int_{\frac{\pi}{2}}^{\pi} 8\sin(t)\cos(t)dt = -2\cos(2t)\Big|_{\frac{\pi}{2}}^{\pi} = \boxed{-4}. \tag{36}$$

Solution to c: The symmetry in the given picture shows us that the flux through the circle in quadrant 1 is the same as the flux through the circle in quadrant 3. To be more detailed, we can observe that the map $(x, y) \mapsto$

$(-x,-y)$ will send the first quadrant to the third quadrant, and the map $\theta \mapsto \theta+\pi$ (which is basically the same map) also maps the first quadrant to the third quadrant. It follows that for each $0 \le t \le \frac{\pi}{2}$ (remembering that t is essentially the angle θ in this situation) we have

$$\mathbf{F}(\mathbf{r}(t+\pi)) = \mathbf{F}(-\mathbf{r}(t)) = -\mathbf{F}(\mathbf{r}(t)), \text{ and} \tag{37}$$

$$\hat{n}(t+\pi) = -\hat{n}(t), \text{ so} \tag{38}$$

$$\text{Flux(Third Quadrant)} = \int_{\pi}^{\frac{3\pi}{2}} \mathbf{F}\cdot\hat{n}ds = \int_{\pi}^{\frac{3\pi}{2}} \mathbf{F}(\mathbf{r}(t))\cdot\hat{n}ds = \tag{39}$$

$$= \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{r}(t+\pi))\cdot\hat{n}(t+\pi)ds = \int_0^{\frac{\pi}{2}} (-\mathbf{F}(\mathbf{r}(t)))\cdot(-\hat{n}(t))ds \tag{40}$$

$$= \int_0^{\frac{\pi}{2}} \mathbf{F}\cdot\hat{n}(t)ds = \text{Flux(First Quadrant)}. \tag{41}$$

Solution to d: Once again the symmetry in the given picture shows us that the flux through the circle in quadrant 2 is the same as the flux through the circle in quadrant 4. To be more detailed, we perform calculations similar to those of part **c** to see that

$$\text{Flux(Fourth Quadrant)} = \int_{\frac{3\pi}{2}}^{2\pi} \mathbf{F}\cdot\hat{n}ds = \int_{\frac{3\pi}{2}}^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\hat{n}ds \tag{42}$$

$$= \int_{\frac{\pi}{2}}^{\pi} \mathbf{F}(\mathbf{r}(t+\pi))\cdot\hat{n}(t+\pi)ds = \int_{\frac{\pi}{2}}^{\pi} (-\mathbf{F}(\mathbf{r}(t)))\cdot(-\hat{n}(t))ds \tag{43}$$

$$= \int_{\frac{\pi}{2}}^{\pi} \mathbf{F}\cdot\hat{n}(t)ds = \text{Flux(Second Quadrant)}. \tag{44}$$

Solution to e: We could calculate the total flux directly, but to make use of parts **a**–**d**, we observe that

$$\text{Total Flux} = \int_0^{2\pi} \mathbf{F} \cdot \hat{n} ds \tag{45}$$

$$= \underbrace{\int_0^{\frac{\pi}{2}} \mathbf{F} \cdot \hat{n} ds}_{\text{Q1 Flux}} + \underbrace{\int_{\frac{\pi}{2}}^{\pi} \mathbf{F} \cdot \hat{n} ds}_{\text{Q2 Flux}} + \underbrace{\int_{\pi}^{\frac{3\pi}{2}} \mathbf{F} \cdot \hat{n} ds}_{\text{Q3 Flux}} + \underbrace{\int_{\frac{3\pi}{2}}^{2\pi} \mathbf{F} \cdot \hat{n} ds}_{\text{Q4 Flux}} \tag{46}$$

$$= 4 + (-4) + 4 + (-4) = \boxed{0}. \tag{47}$$