Math 2173 Spring 2021 Recitation Handout 9 Solutions

Group Member 1: Sohail Farhangi

Group Member 2: ____________________

Group Member 3: ____________________

Group Member 4: ____________________

Group Member 5: ____________________

Group Member 6: ____________________

Below is a checklist of instructions to follow when completing this assignment. Failure to follow these directions will result in penalty on your final score and/or in some problems not being graded. If multiple directions are not followed, then it is also possible that the assignment will not be accepted for any credit at all. Please contact your TA or make a post on the discussion boards for this course if you have any questions about this assignment or these directions.

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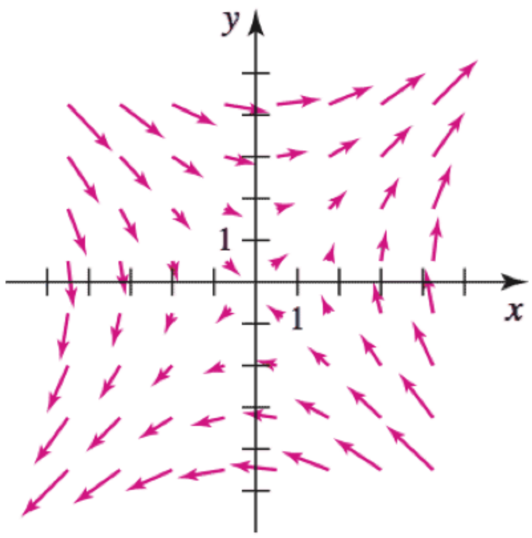
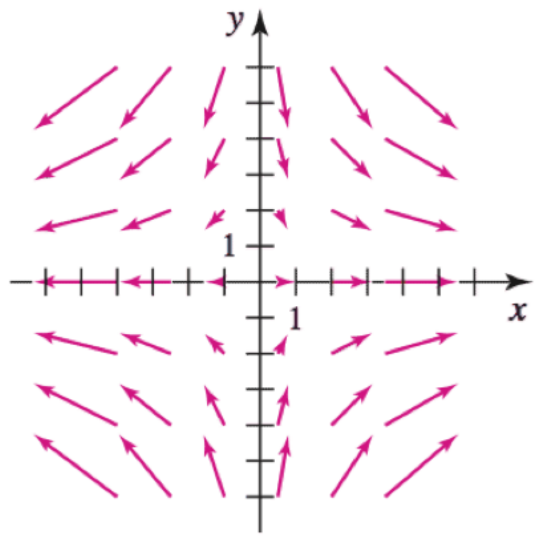
	Checklist of Instructions
	Please clearly write the names of all group members working on this assignment in the spaces allotted above.
	This assignment must be completed by a group of 3, 4, 5, or 6 members.
	This assignment is to be uploaded to gradescope as a pdf file no later than 11:59 PM EST on Sunday, March 21.
	The assignment will be uploaded by 1 group member, and that group member will be responsible for manually entering the names of all other collaborators into gradescope.
	This assignment must be completed using this template. You may either print this template to write on it and then scan it (pages ordered correctly) into a pdf file, or you may write directly on the template using programs such as notability.
	If you need more space than what is given to solve a given problem, then you will find blank pages provided at the end of this template. At the end of each problem section of this assignment you will find a space in which to indicate on what page your work is continued in case you used additional pages to complete your solution. You must provide the page number on which your work is continued in the alloted space, or write 'N/A' incase you did not use any additional pages.
	On the additional pages, you will also find space in which to indicate which problem the page is being used for, and if the page is used then that space must also be filled.
	To complete this handout, you may use your textbook, class notes, discussions with your TA and group members, and any resources that are available on Carmen. You should not receive any help from the MSLC or people outside of your group when solving these problems. You may discuss these problems on the Carmen discussion boards, but you should not provide your entire solution when answering a such question, you should only give a hint or a helpful idea.

Ungraded Optional Problem 15.1.16: Match vector fields (a)-(d) with graphs (A)-(D).

$(a)\ \vec{F} = \langle 0, x^2 \rangle \quad (b)\ \vec{F} = \langle x - y, x \rangle \quad (c)\ \vec{F} = \langle 2x, -y \rangle \quad (d)\ \vec{F} = \langle y, x \rangle$



(A)



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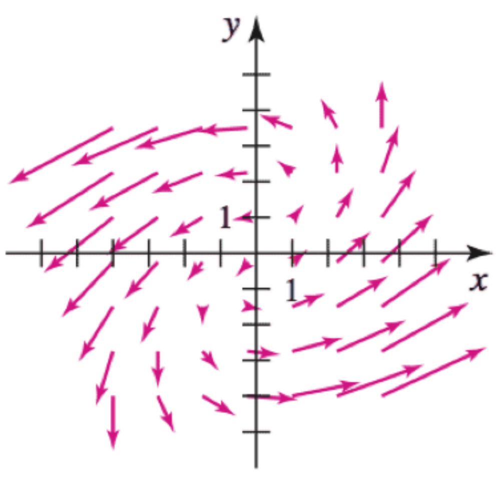

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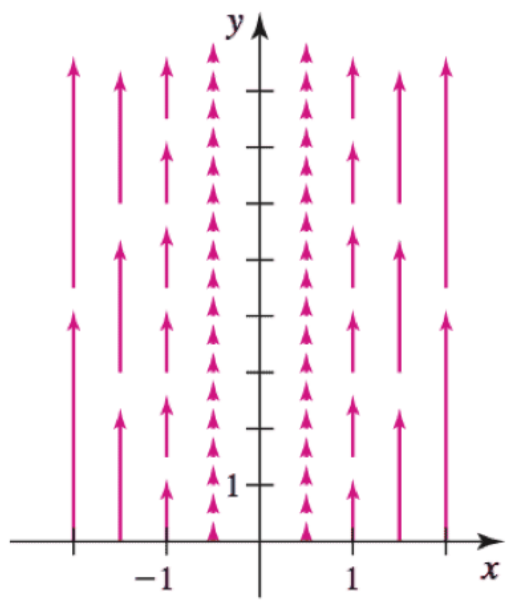


(D)

Ungraded Optional Problem 15.2.22: Find the average value of

$$f(x, y) = x^2 + 4y^2 \tag{1}$$

on the circle of radius 9 centered at the origin.

Our personal notes for this Ungraded Optional Problem are continued on page

Problem 15.1.26 (5 points): Find the gradient field $\vec{F} = \nabla\varphi$ for the potential function

$$\varphi(x,y) = \sqrt{x^2+y^2}, \quad \text{for } x^2+y^2 \leq 9, (x,y) \neq (0,0). \tag{2}$$

Sketch two level curves of φ and two vectors of $\vec{F}$ of your choice.

Solution: Firstly, we see that

$$\vec{F} = \nabla\varphi = \langle \varphi_x, \varphi_y \rangle = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \rangle. \tag{3}$$

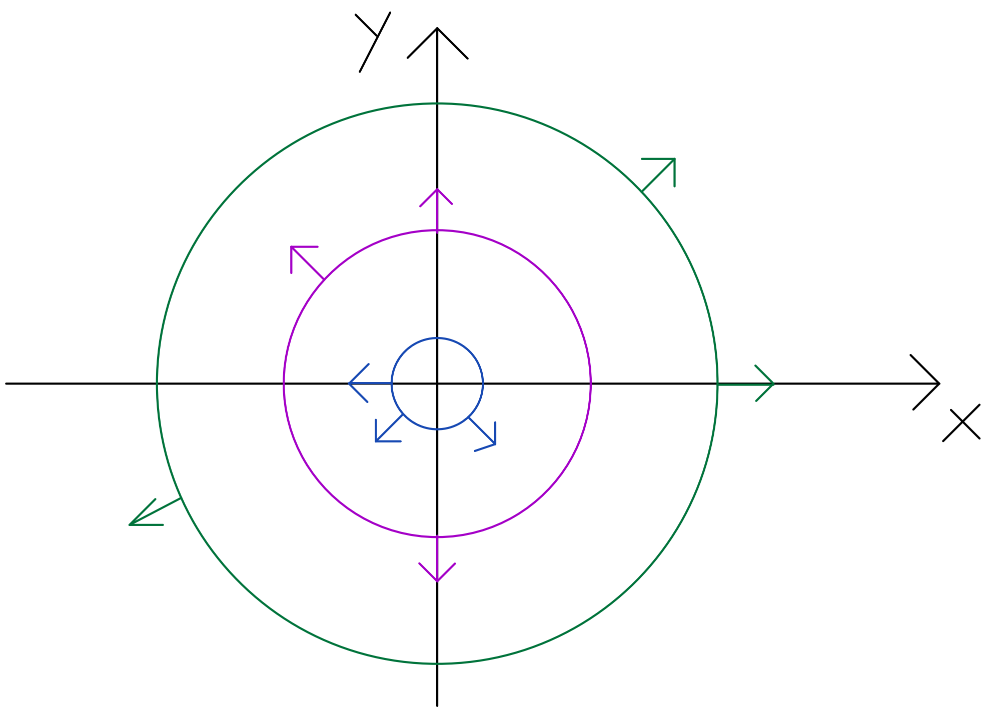
Next, we recall that the level curves of ϕ are the curves of the form $\phi(x,y) = c$ for some constant c. We see that

$$\phi(x,y) = c \Leftrightarrow \sqrt{x^2+y^2} = c \Leftrightarrow x^2+y^2 = c^2, \tag{4}$$

so the level curves of ϕ are circles centered at the origin. We recall that at a given point (x,y) the vector $\nabla\varphi(x,y)$ is perpendicular to the level curve that passes through (x,y), and we also observe that for any (x,y) we have

$$|\nabla\varphi(x,y)| = \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2} = 1, \tag{5}$$

so we obtain the sketch below of some vectors from the gradient field and some level curves.



Problem 15.2.23. (6 points): Find the average value of

$$f(x,y)=\sqrt{4+9y^{2/3}} \tag{6}$$

on the curve $y=x^{3/2}$, for $0\le x\le 5$.

Solution: For curves of the form $y=f(x)$, $a\le x\le b$, we have the parameterization $\mathbf{r}(t)=\langle t,f(t)\rangle$, $a\le t\le b$. In this particular problem, we see that $\mathbf{r}(t)=\langle t,t^{\frac{3}{2}}\rangle$, $0\le t\le 5$. It follows that

$$\mathbf{r}'(t)=\langle 1,\frac{3}{2}t^{\frac{1}{2}}\rangle\to|\mathbf{r}'(t)|=\sqrt{1^2+(\frac{3}{2}t^{\frac{1}{2}})^2}=\sqrt{1+\frac{9}{4}t}. \tag{7}$$

Recall that the average value of f over a curve C is given by

$$\text{Av}(f)=\frac{\int_C f ds}{\text{Arclength}(C)}=\frac{\int_C f ds}{\int_C 1 ds}. \tag{8}$$

We begin by calculating $\int_C f ds$ and see that

$$\int_C f ds=\int_0^5 f(\mathbf{r}(t))|\mathbf{r}'(t)|dt=\int_0^5 f(t,t^{\frac{3}{2}})\sqrt{1+\frac{9}{4}t}dt \tag{9}$$

$$=\int_0^5\sqrt{4+9(t^{\frac{3}{2}})^{\frac{2}{3}}}\sqrt{1+\frac{9}{4}t}dt=\int_0^5\sqrt{4+9t}\sqrt{1+\frac{9}{4}t}dt \tag{10}$$

$$=\int_0^5 2\sqrt{1+\frac{9}{4}t}\sqrt{1+\frac{9}{4}t}dt=2\int_0^5(1+\frac{9}{4}t)dt=2(t+\frac{9}{8}t^2\Big|_0^5)=\frac{265}{4}. \tag{11}$$

We now calculate the arclength of our given curve and see that

$$\text{Arclength}(C)=\int_C 1ds=\int_0^5|\mathbf{r}'(t)|dt=\int_0^5\sqrt{1+\frac{9}{4}t}dt \tag{12}$$

$$\overset{u=1+\frac{9}{4}t}{=}\int_{t=0}^5\sqrt{u}\frac{4}{9}du=\frac{4}{9}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{t=0}^5=\frac{8}{27}(1+\frac{9}{4}t)^{\frac{3}{2}}\Big|_0^5=\frac{335}{27}. \tag{13}$$

It follows that the final answer is

(14) $$\mathrm{Av}(f) = \frac{\int_C f ds}{\mathrm{Arclength}(C)} = \frac{\int_C f ds}{\int_C 1 ds} = \frac{\frac{265}{4}}{\frac{335}{27}} = \boxed{\frac{1431}{268}}.$$

Problem 15.2.16. (7 points): Consider

$$\int_{\mathcal{C}} (x^2 + y^2) ds,$$

where $\mathcal{C}$ is the line segment from $(0,0)$ to $(5,5)$.

(1) Find a parametric description for $\mathcal{C}$ in the form $\vec{r}(t) = \langle x(t), y(t) \rangle$. (*Remember to state the domain of the parameter.*)

(2) Evaluate $|\vec{r'}(t)|$.

(3) Convert the line integral to an ordinary integral with respect to the parameter and evaluate it.

Solution to (1): We recall that

$$\vec{r}(t) = \vec{P} + t(\vec{Q} - \vec{P}), 0 \leq t \leq 1 \tag{15}$$

is one way in which to parameterize the line segment that starts at the point P and ends at the point Q. In this particular problem, we see that

$$\vec{r}(t) = \langle 0,0 \rangle + t(\langle 5,5 \rangle - \langle 0,0 \rangle) = \langle 5t, 5t \rangle, 0 \leq t \leq 1 \tag{16}$$

is a parameterization for the line segment from $(0,0)$ to $(5,5)$.

Solution to (2): We see that

$$\vec{r'}(t) = \langle 5,5 \rangle \to |\vec{r'}(t)| = \sqrt{5^2 + 5^2} = \sqrt{50} = 5\sqrt{2}. \tag{17}$$

Solution to (3): We have

$$\int_C (x^2 + y^2) ds = \int_0^1 ((5t)^2 + (5t)^2) 5\sqrt{2} dt = 5\sqrt{2} \int_0^1 (25t^2 + 25t^2) dt \tag{18}$$

$$= 250\sqrt{2} \int_0^1 t^2 dt = \frac{250\sqrt{2}}{3} t^3 \Big|_0^1 = \boxed{\frac{250\sqrt{2}}{3}}. \tag{19}$$

Problem 15.2.30. (5 points): Compute

$$\int_{\mathcal{C}} xe^{yz}ds, \tag{20}$$

where $\mathcal{C}$ is $\vec{r}(t) = \langle t, 2t, -4t\rangle$ for $1 \le t \le 2$.

Solution: We observe that

$$\vec{r}\,'(t) = \langle 1, 2, -4\rangle \to |\vec{r}\,'(t)| = \sqrt{1^2 + 2^2 + (-4)^2} = \sqrt{21}, \text{ so} \tag{21}$$

$$\int_{\mathcal{C}} xe^{yz}ds = \int_1^2 te^{2t(-4t)}\sqrt{21}dt = \sqrt{21}\int_1^2 te^{-8t^2}dt \tag{22}$$

$$\overset{u=-8t^2}{=} \sqrt{21}\int_{t=1}^2 e^u(-\frac{1}{16})du = \sqrt{21}(-\frac{1}{16}e^u\Big|_{t=1}^2) \tag{23}$$

$$= -\frac{\sqrt{21}}{16}e^{-8t^2}\Big|_1^2 = \boxed{\frac{\sqrt{21}}{16}(e^{-8} - e^{-32})}. \tag{24}$$

Problem 15.2.55. (5+5+2 points): Let $f(x, y) = x$ and consider the segment of the parabola $y = x^2$ joining $O(0, 0)$ and $P(1, 1)$.

(1) Let $\mathcal{C}_1$ be the segment from O to P. Find a parameterization of $\mathcal{C}_1$, then evaluate $\int_{\mathcal{C}_1} f ds$.

(2) Let $\mathcal{C}_2$ be the segment from P to O. Find a parameterization of $\mathcal{C}_2$, then evaluate $\int_{\mathcal{C}_2} f ds$.

(3) Compare the results of (1) and (2).

Solution to (1): As mentioned in the solution to problem 15.2.23, we see that $\mathbf{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le 1$ is a parameterization of the segement of the parabola $y = x^2$ from $O(0, 0)$ to $P(1, 1)$. We see that

$$\mathbf{r}'(t) = \langle 1, 2t \rangle \to |\mathbf{r}'(t)| = \sqrt{1^2 + 4t^2} = \sqrt{1 + 4t^2}, \text{ so} \tag{25}$$

$$\int_C f ds = \int_0^1 f(\mathbf{r}(t))|\mathbf{r}'(t)|dt - \int_0^1 f(t, t^2)\sqrt{1 + 4t^2}dt = \int_0^1 t\sqrt{1 + 4t^2}dt \tag{26}$$

$$\overset{u=1+4t^2}{=} \int_{t=0}^1 \sqrt{u}\frac{1}{8}du = \frac{1}{12}u^{\frac{3}{2}}\Big|_{t=0}^1 = \frac{1}{12}(1 + 4t^2)^{\frac{3}{2}}\Big|_0^1 = \boxed{\frac{5^{\frac{3}{2}} - 1}{12}}. \tag{27}$$

Solution to (2): We see that if we replace t by $1 - t$, then the parameterization starts at $P(1, 1)$ and ends at $O(0, 0)$, so $\mathbf{r}(t) = \langle 1 - t, (1 - t)^2 \rangle$ is a parameterization of the segment of the parabola $y = x^2$ from $P(1, 1)$ to $O(0, 0)$. We see that

$$\mathbf{r}'(t) = \langle -1, -2(1 - t) \rangle \tag{28}$$

$$\to |\mathbf{r}'(t)| = \sqrt{(-1)^2 + (-2(1 - t))^2} = \sqrt{1 + 4(1 - t)^2}, \text{ so} \tag{29}$$

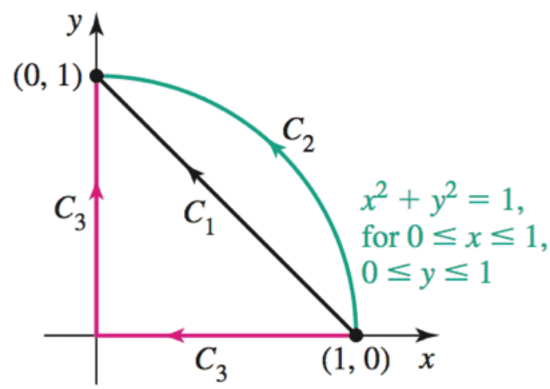
$$\int_C f ds = \int_0^1 f(\mathbf{r}(t))|\mathbf{r}'(t)|dt = \int_0^1 f(1 - t, (1 - t)^2)\sqrt{1 + 4(1 - t)^2}dt \tag{30}$$

$$= \int_0^1 (1-t)\sqrt{1+4(1-t)^2}dt \overset{u=1+4(1-t)^2}{=} \int_{t=0}^1 \sqrt{u}(-\frac{1}{8})du = -\frac{1}{12}u^{\frac{3}{2}}\Big|_{t=0}^1 \tag{31}$$

$$= -\frac{1}{12}(1+4(1-t)^2)^{\frac{3}{2}}\Big|_{t=0}^1 = \boxed{\frac{5^{\frac{3}{2}}-1}{12}}. \tag{32}$$

Solution to (3): We see that the answers to parts (1) and (2) are the same. This makes sense because we are integrating the same function values over the same region. This should be compared to the fact that $\int_a^b f(x)dx = -\int_b^a f(x)dx$. Note that the reason that we do not obtain a negative sign in part (2) is because $ds = |\mathbf{r}'(t)|dt$, and the absolute values absorb the negative sign. To see this fact in action back in the one dimensional case, we note that $\mathbf{r_1}(t) = \langle (b-a)t + a\rangle, 0 \le t \le 1$ is a parameterization of the line segment from $x = a$ to $x = b$, and $\mathbf{r_2}(t) = \langle (a-b)t + b\rangle, 0 \le t \le 1$ is a parameterization of the line segment from $x = b$ t $x = a$. We see that $\mathbf{r_1}'(t) = \langle b-a\rangle = -\langle a-b\rangle = \mathbf{r_2}'(t)$, so $ds = |\mathbf{r_1}'(t)| = |\mathbf{r_2}'(t)| = b-a$ is the same for both parameterizations.

Ungraded Optional Problem 15.2.60: Consider the rotation field $\vec{F} = \langle -y, x \rangle$, and the three paths shown in the figure.



(1) Compute the line integral $\int_{\mathcal{C}_1} \vec{F} \cdot \vec{T} ds$.
(2) Compute the line integral $\int_{\mathcal{C}_2} \vec{F} \cdot \vec{T} ds$.
(3) Compute the line integral $\int_{\mathcal{C}_3} \vec{F} \cdot \vec{T} ds$.
(4) Does it appear that the line integral $\int_{\mathcal{C}} \vec{F} \cdot \vec{T} ds$ is independent of the path, where $\mathcal{C}$ is any path from $(1, 0)$ to $(0, 1)$?
