# Math 2173 Spring 2021 Recitation Handout 10 Solutions

Group Member 1: Sohail Farhangi

Group Member 2: ______________________________

Group Member 3: ______________________________

Group Member 4: ______________________________

Group Member 5: ______________________________

Group Member 6: ______________________________

Below is a checklist of instructions to follow when completing this assignment. Failure to follow these directions will result in penalty on your final score and/or in some problems not being graded. If multiple directions are not followed, then it is also possible that the assignment will not be accepted for any credit at all. Please contact your TA or make a post on the discussion boards for this course if you have any questions about this assignment or these directions.

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| | Checklist of Instructions |
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| | Please clearly write the names of all group members working on this assignment in the spaces allotted above. |
| | This assignment must be completed by a group of 3, 4, 5, or 6 members. |
| | This assignment is to be uploaded to gradescope as a pdf file no later than 11:59 PM EST on Sunday, March 28. |
| | The assignment will be uploaded by 1 group member, and that group member will be responsible for manually entering the names of all other collaborators into gradescope. |
| | This assignment must be completed using this template. You may either print this template to write on it and then scan it (pages ordered correctly) into a pdf file, or you may write directly on the template using programs such as notability. |
| | If you need more space than what is given to solve a given problem, then you will find blank pages provided at the end of this template. At the end of each problem section of this assignment you will find a space in which to indicate on what page your work is continued in case you used additional pages to complete your solution. You must provide the page number on which your work is continued in the alloted space, or write 'N/A' incase you did not use any additional pages. |
| | On the additional pages, you will also find space in which to indicate which problem the page is being used for, and if the page is used then that space must also be filled. |
| | To complete this handout, you may use your textbook, class notes, discussions with your TA and group members, and any resources that are available on Carmen. You should not receive any help from the MSLC or people outside of your group when solving these problems. You may discuss these problems on the Carmen discussion boards, but you should not provide your entire solution when answering a such question, you should only give a hint or a helpful idea. |

**Ungraded Optional Problem:** Let $a$ be a positive number. Compute the circulation of the vector field

$$\vec{F} = \langle -y, x \rangle$$

on the circle $\mathcal{C}$ of radius $a$ centered at the origin with counterclock-wise orientation.

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**Ungraded Optional Problem:** Let $\mathcal{C}$ be the unit circle with counterclockwise orientation. Compute the flux of $\vec{F} = \langle x, y \rangle$ across $\mathcal{C}$.

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**Problem 15.2.47. (5 points):** Compute the circulation of

$$\vec{F} = \langle y - x, x \rangle$$

on the curve $\mathcal{C}$ which is given by $\vec{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$ for $0 \leq t \leq 2\pi$.

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**Solution:** We see that

(1) $$\text{Circulation} = \int_C \vec{F} \cdot \hat{T} ds = \int_C \vec{F} \cdot \vec{r}\,'(t) dt$$

(2) $$= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \langle -2\sin(t), 2\cos(t) \rangle dt$$

(3) $$= \int_0^{2\pi} \langle \langle \underbrace{2\sin(t)}_{y} - \underbrace{2\cos(t)}_{x}, \underbrace{2\cos(t)}_{x} \rangle \cdot \langle -2\sin(t), 2\cos(t) \rangle dt$$

(4) $$= \int_0^{2\pi} \left( -4\sin^2(t) + 4\cos(t)\sin(t) + 4\cos^2(t) \right) dt$$

(5) $$=^1 \int_0^{2\pi} \left( 4\cos(2t) + 2\sin(2t) \right) dt$$

(6) $$= 2\sin(2t) - \cos(2t) \Big|_0^{2\pi} = \boxed{0}.$$

[1] $\cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1 = 1 - 2\sin^2(t)$ and $\sin(2t) = 2\sin(t)\cos(t)$.

**(Modified) Problem 15.2.66 (5 points):** Let $a$ be a positive number. Consider the vector field $\vec{F} = \langle y, x\rangle$ and the curve $\mathcal{C}$ given by $\vec{r}(t) = \langle a\cos(t), a\sin(t)\rangle$ for $0 \le t \le 2\pi$. Compute the flux of $\vec{F}$ across $\mathcal{C}$. (Your answer should be in terms of $a$.)

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**Solution:** We see that

$$\vec{r}\,'(t) = \langle -a\sin(t), a\cos(t)\rangle \to |\vec{r}\,'(t)| = \sqrt{(-a\sin(t))^2 + (a\cos(t))^2} = a \tag{7}$$

$$\to \hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\sin(t), \cos(t)\rangle \tag{8}$$

$$\to \hat{n}(t) = \hat{T}(t) \times \hat{k} = \langle \cos(t), \sin(t)\rangle \tag{9}$$

$$\text{Flux} = \int_C \vec{F}\cdot\hat{n}ds = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\hat{n}(t)|\vec{r}\,'(t)|dt \tag{10}$$

$$= \int_0^{2\pi} \langle \underbrace{\sin(t)}_{y}, \underbrace{\cos(t)}_{x}\rangle \cdot \langle \cos(t), \sin(t)\rangle a dt \tag{11}$$

$$= a\int_0^{2\pi} (\sin(t)\cos(t) + \cos(t)\sin(t))dt \tag{12}$$

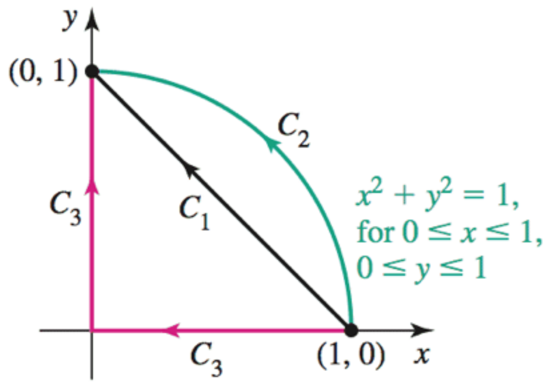
$$= a\int_0^{2\pi} \sin(2t)dt = -\frac{a}{2}\cos(2t)\Big|_0^{2\pi} = \boxed{0}. \tag{13}$$

**Remark:** We also could have used the fact that

$$\hat{n}ds = \hat{n}|\vec{r}\,'(t)|dt = |\vec{r}\,'(t)\times\hat{k}|dt = \langle a\cos(t), a\sin(t)\rangle \tag{14}$$

in order to avoid the calculation of $|\vec{r}\,'(t)|$ and $\hat{T}(t)$ and save a little effort.

**Problem 15.2.60 (4+4+4+1=13 points):** Consider the rotation field $\vec{F} = \langle -y, x \rangle$, and the three paths shown in the figure.




(1) Compute the work required in the presence of the force field $\vec{F}$ to move an object on the curve $\mathcal{C}_1$.
(2) Compute the work required in the presence of the force field $\vec{F}$ to move an object on the curve $\mathcal{C}_2$.
(3) Compute the work required in the presence of the force field $\vec{F}$ to move an object on the curve $\mathcal{C}_3$.
(4) Does it appear that the line integral $\int_{\mathcal{C}} \vec{F} \cdot \vec{T} ds$ is independent of the path, where $\mathcal{C}$ is any path from $(1,0)$ to $(0,1)$?

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**Solution to part 1:** We begin by finding a parameterization for our curve $\mathcal{C}_1$. Since $\mathcal{C}_1$ is a line segment, we can use the standard parameterization

$$\vec{r}(t) = \underbrace{\langle 1,0 \rangle}_{\text{Start Point}} + t(\underbrace{\langle 0,1 \rangle}_{\text{End Point}} - \underbrace{\langle 1,0 \rangle}_{\text{Start Point}}) = \langle 1-t, t \rangle, 0 \le t \le 1. \tag{15}$$

We now see that

$$\text{Work} = \int_{\mathcal{C}_1} \vec{F} \cdot \hat{T} ds = \int_{\mathcal{C}_1} \vec{F} \cdot \vec{r}\,' dt \tag{16}$$

$$= \int_0^1 \langle \underbrace{-t}_{-y}, \underbrace{1-t}_{x} \rangle \cdot \langle -1, 1 \rangle dt = \int_0^1 (t+1-t) dt = \int_0^1 1 dt = \boxed{1}. \tag{17}$$

**Solution to part 2:** We begin by finding a parameterization for our curve $\mathcal{C}_2$. Since $\mathcal{C}_2$ is a portion of a circle with the counter clockwise orientation, we can use the standard parameterization

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle, 0 \leq t \leq \frac{\pi}{2}. \tag{18}$$

We now see that

$$\text{Work} = \int_{\mathcal{C}_1} \vec{F} \cdot \hat{T} ds = \int_{\mathcal{C}_1} \vec{F} \cdot \vec{r}\,' dt \tag{19}$$

$$= \int_0^{\frac{\pi}{2}} \vec{F}(\vec{r}(t)) \cdot \langle -\sin(t), \cos(t) \rangle dt \tag{20}$$

$$= \int_0^{\frac{\pi}{2}} \langle \underbrace{-\sin(t)}_{-y}, \underbrace{\cos(t)}_{x} \rangle \cdot \langle -\sin(t), \cos(t) \rangle dt \tag{21}$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2(t) + \cos^2(t)) dt = \int_0^{\frac{\pi}{2}} 1 dt = \boxed{\frac{\pi}{2}}. \tag{22}$$

**Solution to part 3:** Since $\mathcal{C}_3$ is composed of 2 separate smooth curves, we will decompose $\mathcal{C}_3$ into its pieces and handle them separately. Let $\mathcal{C}_4$ denote the line segment from $(1,0)$ to $(0,0)$ and let $\mathcal{C}_5$ denote the line segment from $(0,0)$ to $(0,1)$. We see that

$$\text{Work} = \int_{\mathcal{C}_3} \vec{F} \cdot \hat{T} ds = \int_{\mathcal{C}_4} \vec{F} \cdot \hat{T} ds + \int_{\mathcal{C}_5} \vec{F} \cdot \hat{T} ds. \tag{23}$$

In light of this observation, we will first calculate the work it takes to move an object across $\mathcal{C}_4$ in the presence of the force field $\vec{F}$, then we will calculate the work it takes to move an object across $\mathcal{C}_5$ in the presence of the force field $\vec{F}$. Since $\mathcal{C}_4$ is a line segment, we see as in part 1 that

$$\vec{r}(t) = \langle -t, 0 \rangle, 0 \leq t \leq 1 \tag{24}$$

is a paramterization for $\mathcal{C}_4$. We now see that

$$\int_{\mathcal{C}_4} \vec{F} \cdot \hat{T} ds = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) dt \tag{25}$$

$$= \int_0^1 \langle \underbrace{0}_{-y}, \underbrace{-t}_{x} \rangle \cdot \langle -1, 0 \rangle dt = \int_0^1 0 dt = 0. \tag{26}$$

Since $\mathcal{C}_5$ is also a line segment, we see as before that

$$\vec{r}(t) = \langle 0, t \rangle, 0 \leq t \leq 1 \tag{27}$$

is a paramterization for $\mathcal{C}_5$. We now see that

$$\int_{\mathcal{C}_5} \vec{F} \cdot \hat{T} ds = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) dt \tag{28}$$

$$= \int_0^1 \langle \underbrace{-t}_{-y}, \underbrace{0}_{x} \rangle \cdot \langle 0, 1 \rangle dt = \int_0^1 0 dt = 0. \tag{29}$$

It follows that the total work is $0 + 0 = \boxed{0}$.

**Solution to part 4:** Since the answers to parts 1, 2, and 3 are all different, we see that the work required to move an object from $(1, 0)$ to $(0, 1)$ in the presence of the force field $\vec{F}$ depends on that path that use. In particular, the vector field $\vec{F}$ is not conservative.