# Math 2173 Spring 2021 Recitation Handout 1 Solutions

Group Member 1: Sohail Farhangi

Group Member 2: ______________________________

Group Member 3: ______________________________

Group Member 4: ______________________________

Group Member 5: ______________________________

Group Member 6: ______________________________

Below is a checklist of instructions to follow when completing this assignment. Failure to follow these directions will result in penalty on your final score and/or in some problems not being graded. If multiple directions are not followed, then it is also possible that the assignment will not be accepted for any credit at all. Please contact your TA or make a post on the discussion boards for this course if you have any questions about this assignment or these directions.

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| | Checklist of Instructions |
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| | Please clearly write the names of all group members working on this assignment in the spaces allotted above. |
| | This assignment must be completed by a group of 3, 4, 5, or 6 members. |
| | This assignment is to be uploaded to gradescope as a pdf file no later than 11:59 PM EST on Sunday, January 17. |
| | The assignment will be uploaded by 1 group member, and that group member will be responsible for manually entering the names of all other collaborators into gradescope. |
| | This assignment must be completed using this template. You may either print this template to write on it and then scan it (pages ordered correctly) into a pdf file, or you may write directly on the template using programs such as notability. |
| | If you need more space than what is given to solve a given problem, then you will find blank pages provided at the end of this template. At the end of each problem section of this assignment you will find a space in which to indicate on what page your work is continued in case you used additional pages to complete your solution. You must provide the page number on which your work is continued in the alloted space, or write 'N/A' incase you did not use any additional pages. |
| | On the additional pages, you will also find space in which to indicate which problem the page is being used for, and if the page is used then that space must also be filled. |
| | To complete this handout, you may use your textbook, class notes, discussions with your TA and group members, and any resources that are available on Carmen. You should not receive any help from the MSLC or people outside of your group when solving these problems. You may discuss these problems on the Carmen discussion boards, but you should not provide your entire solution when answering a such question, you should only give a hint or a helpful idea. |

**13.6.77:** The electric field due to a point charge of strength $Q$ at the origin has a potential function $V(x,y,z) = kQ/r$, where $r^2 = x^2+y^2+z^2$ is the square of the distance between a variable point $P(x,y,z)$ at the charge, and $k > 0$ is a physical constant. The electric field is given by $\mathbf{E}(x,y,z) = -\nabla V(x,y,z)$.

**a.** Show that

$$\mathbf{E}(x,y,z) = kQ\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle. \tag{1}$$

**b.** Show that $|\mathbf{E}| = kQ/r^2$. Explain why this relationship is called the inverse square law.

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**Solution to a:** We note that since $r$ represents a distance, $r$ is a nonnegative number, so

$$r = (x^2+y^2+z^2)^{\frac{1}{2}} \text{ ( and not } -(x^2+y^2+z^2)^{\frac{1}{2}}). \tag{2}$$

It follows that

$$V(x,y,z) = kQ(x^2+y^2+z^2)^{-\frac{1}{2}} \to \tag{3}$$

$$\begin{aligned} V_x(x,y,z) &= -\tfrac{1}{2}(kQ(x^2+y^2+z^2)^{-\frac{3}{2}})\tfrac{\partial}{\partial x}(x^2+y^2+z^2) \\ &= -kQx(x^2+y^2+z^2)^{-\frac{3}{2}} &&= -kQxr^{-3} \\ V_y(x,y,z) &= -\tfrac{1}{2}(kQ(x^2+y^2+z^2)^{-\frac{3}{2}})\tfrac{\partial}{\partial y}(x^2+y^2+z^2) \\ &= -kQy(x^2+y^2+z^2)^{-\frac{3}{2}} &&= -kQyr^{-3} \\ V_z(x,y,z) &= -\tfrac{1}{2}(kQ(x^2+y^2+z^2)^{-\frac{3}{2}})\tfrac{\partial}{\partial z}(x^2+y^2+z^2) \\ &= -kQz(x^2+y^2+z^2)^{-\frac{3}{2}} &&= -kQzr^{-3} \end{aligned} \tag{4}$$

It is now clear that

$$\mathbf{E}(x,y,z) = -\nabla V(x,y,z) = -\langle V_x, V_y, V_z\rangle \tag{5}$$

$$= -\langle -kQxr^{-3}, -kQyr^{-3}, -kQzr^{-3}\rangle = kQ\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle. \tag{6}$$

**Solution to b:** We see that

$$|\mathbf{E}| = |kQ\langle\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle| = kQ|\langle\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle| = kQ\left((\frac{x}{r^3})^2 + (\frac{y}{r^3})^2 + (\frac{z}{r^3})^2\right)^{\frac{1}{2}} \tag{7}$$

$$= kQ\left(\frac{x^2+y^2+z^2}{r^6}\right)^{\frac{1}{2}} = kQ\left(\frac{r^2}{r^6}\right)^{\frac{1}{2}} = kQ\left(\frac{1}{r^4}\right)^{\frac{1}{2}} = \frac{kQ}{r^2}. \tag{8}$$

The fact that $|\mathbf{E}| = \frac{kQ}{r^2}$ is known as the inverse square law because the magnitude of the electric field $\mathbf{E}$ is proportional to the inverse of the square (or the square of the inverse) of the distance $r$.

**Problem 13.6.64:** Consider the function $F(x,y,z) = e^{xyz}$.

**a.** Write $F$ as a composite function $f \circ g$, where $f$ is a function of one variable and $g$ is a function of three variables.

**b.** Calculate $\nabla F(x,y,z)$ as well as $\nabla g(x,y,z)$. Find a relationship between $\nabla F(x,y,z)$ and $\nabla g(x,y,z)$.

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**Solution to a:** Letting $f(t) = e^t$ and $g(x,y,z) = xyz$, we see that $F(x,y,z) = f(g(x,y,z)$, so $F = f \circ g$.

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**Solution to b:** We see that

$$\begin{aligned} F_x(x,y,z) &= e^{xyz}\frac{\partial}{\partial x}(xyz) = yze^{xyz} \\ F_y(x,y,z) &= e^{xyz}\frac{\partial}{\partial y}(xyz) = xze^{xyz} \\ F_z(x,y,z) &= e^{xyz}\frac{\partial}{\partial z}(xyz) = xye^{xyz} \end{aligned} \tag{9}$$

$$\to \nabla F(x,y,z) = \langle F_x(x,y,z), F_y(x,y,z), F_z(x,y,a)\rangle \tag{10}$$

$$= yze^{xyz}, xze^{xyz}, xye^{xyz}\rangle = e^{xyz}\langle yz, xz, xy\rangle, \text{ and} \tag{11}$$

$$\begin{aligned} g_x(x,y,z) &= yz \\ g_y(x,y,z) &= xz \\ g_z(x,y,z) &= xy \end{aligned} \tag{12}$$

$$\to \nabla g(x,y,z) = \langle g_x(x,y,z), g_y(x,y,z), g_z(x,y,z)\rangle = \langle yz, xz, xy\rangle. \tag{13}$$

We now see that

$$\nabla F(x,y,z) = f(g(x,y,z))\nabla g(x,y,z) = F(x,y,z)\nabla g(x,y,z). \tag{14}$$

However, this is purely a coincidence. We will see later on that if if $F$, $f$, and $g$ are functions for which $F = f \circ g$, then

$$\boxed{\nabla F(x,y,z) = f'(g(x,y,z))\nabla g(x,y,z)}. \tag{15}$$

**Problem 13.6.33 (NOT GRADED):** Consider the function $f(x, y) = 10 - 2x^2 - 3y^2$ and the point $P = (3, 2)$.

**a.** Find the gradient field $\nabla f(x, y)$ of $f(x, y)$ and then evaluate it at $P$.

**b.** Find the angles $\theta$ (with respect to the x-axis) associated with the directions of maximum increase, maximum decrease, and zero change.

**c.** Write the directional derivative at $P$ as a function of $\theta$; call this function $g(\theta)$.

**d.** Find the value of $\theta$ that maximizes $g(\theta)$ and find the maximum value.

**e.** Verify that the value of $\theta$ that maximizes $g$ corresponds to the direction of the gradient vector at $P$. Verify that the maximum value of $g$ equals the magnitude of the gradient vector at $P$.

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**Problem 13.6.38:** Consider the function $f(x,y) = \ln(1+2x^2+3y^2)$ and the point $P = (\frac{3}{4}, -\sqrt{3})$.

**a.** Find the gradient field $\nabla f(x,y)$ of $f(x,y)$ and then evaluate it at $P$.

**b.** Find the angles $\theta$ (with respect to the x-axis) associated with the directions of maximum increase, maximum decrease, and zero change.

**c.** Write the directional derivative at $P$ as a function of $\theta$; call this function $g(\theta)$.

**d.** Find the value of $\theta$ that maximizes $g(\theta)$ and find the maximum value.

**e.** Verify that the value of $\theta$ that maximizes $g$ corresponds to the direction of the gradient vector at $P$. Verify that the maximum value of $g$ equals the magnitude of the gradient vector at $P$.

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**Solution to a:** We see that

$$\begin{aligned} f_x(x,y) &= \frac{1}{1+2x^2+3y^2}\frac{\partial}{\partial x}(1+2x^2+3y^2) = \frac{4x}{1+2x^2+3y^2} \\ f_y(x,y) &= \frac{1}{1+2x^2+3y^2}\frac{\partial}{\partial y}(1+2x^2+3y^2) = \frac{6y}{1+2x^2+3y^2} \end{aligned} \tag{16}$$

$$\to \nabla f(x,y) = \langle \frac{4x}{1+2x^2+3y^2}, \frac{6y}{1+2x^2+3y^2} \rangle. \tag{17}$$

$$\nabla f(\frac{3}{4}, -\sqrt{3}) = \langle \frac{3}{1+\frac{9}{8}+9}, \frac{-6\sqrt{3}}{1+\frac{9}{8}+9} \rangle = \langle \frac{24}{89}, \frac{-48\sqrt{3}}{89} \rangle. \tag{18}$$

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**Solution to b:** We recall that $\nabla f(P)$ points in the direction of maximum increase from $P$. Since $\nabla f(P)$ is in the fourth quadrant, we see that

$$\theta_{\max} = \tan^{-1}\left(\frac{\frac{-48\sqrt{3}}{89}}{\frac{24}{89}}\right) = \tan^{-1}(-2\sqrt{3}) \tag{19}$$

| This quantity does not simplify to a more pleasant expression. |
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is the angle associated with the direction of maximum increase. Since $-\nabla f(P)$ points in the direction of maximum decrease from $P$, we see that $\theta_{\min} = \theta_{\max} + \pi$ is the angle associated with the direction of maximum decrease. Since the directions of no change are orthogonal to $\nabla f(P)$ (and to $-\nabla f(P)$), we see that $\theta_1 = \theta_{\max} + \frac{\pi}{2}$ and $\theta_2 = \theta_{\max} - \frac{\pi}{2}$ are the angles associated to the directions of zero change.

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**Solution to c:** We recall that $\vec{u}(\theta) = \langle \cos(\theta), \sin(\theta)\rangle$ is the unit vector associated with the angle $\theta$. We also recall that for any unit vector $\vec{u}$, we have that

$$d_{\vec{u}} f(a,b) = \nabla f(a,b)\cdot\vec{u}, \text{ so} \tag{20}$$

$$g(\theta) = d_{\vec{u}(\theta)} f(P) = \nabla f(P)\cdot\vec{u}(\theta) = \langle \frac{24}{89}, \frac{-48\sqrt{3}}{89}\rangle \cdot \langle \cos(\theta), \sin(\theta)\rangle \tag{21}$$

$$= \frac{24}{89}\cos(\theta) - \frac{48\sqrt{3}}{89}\sin(\theta). \tag{22}$$

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**Solution to d:** We see that

$$g'(\theta) = -\frac{24}{89}\sin(\theta) - \frac{48\sqrt{3}}{89}\cos(\theta) \to \tag{23}$$

$$g'(\theta) = 0 \Leftrightarrow -\frac{24}{89}\sin(\theta) = \frac{48\sqrt{3}}{89}\cos(\theta) \Leftrightarrow \tan(\theta) = -2\sqrt{3} \Leftrightarrow \tag{24}$$

$$\theta = \pm\tan^{-1}(-2\sqrt{3}). \tag{25}$$

We see that

$$g''(\theta) = -\frac{24}{89}\cos(\theta) + \frac{48\sqrt{3}}{89}\sin(\theta) = \frac{24}{89}\cos(\theta)\left(-1 + 2\sqrt{3}\tan(\theta)\right) \tag{26}$$

$$\to g''(\tan^{-1}(-2\sqrt{3})) \tag{27}$$

$$= \frac{24}{89}\underbrace{\cos(\tan^{-1}(-2\sqrt{3}))}_{\text{positive}}\left(-1 + 2\sqrt{3}\tan(\tan^{-1}(-2\sqrt{3}))\right) \tag{28}$$

$$= \frac{24}{89}\underbrace{\cos\big(\tan^{-1}(-2\sqrt{3})\big)}_{\text{positive}}(-1-12) < 0. \tag{29}$$

In fact, we can calculate $\cos\big(\tan^{-1}(-2\sqrt{3})\big)$ as follows. For $\theta$ in quadrants 1 and 4 we have

$$\cos(\theta) = \left(\frac{1}{\sec^2(\theta)}\right)^{\frac{1}{2}} = \left(\frac{1}{1+\tan^2(\theta)}\right)^{\frac{1}{2}} \to \tag{30}$$

$$\cos\big(\tan^{-1}(-2\sqrt{3})\big) = \left(\frac{1}{1+\tan^2\big(\tan^{-1}(-2\sqrt{3})\big)}\right)^{\frac{1}{2}} \tag{31}$$

$$= \left(\frac{1}{1+(-2\sqrt{3})^2}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{13}}. \tag{32}$$

You could also arrive at the above conclusion by drawing a right triangle in the cartesian plane corresponding to the angle $\theta = \tan^{-1}(-2\sqrt{3})$.

Equation (29) lets us apply the second derivative test to see that $g(\theta)$ has a local maximum at $\theta = \tan^{-1}(-2\sqrt{3})$.

$$g\big(\tan^{-1}(-2\sqrt{3})\big) = \frac{24}{89}\cos\big(\tan^{-1}(-2\sqrt{3})\big)\Big(1-2\sqrt{3}\tan^{-1}(-2\sqrt{3})\Big) \tag{33}$$

$$= \frac{24}{89}\frac{1}{\sqrt{13}}(1+12) = \frac{24\sqrt{13}}{89} > \frac{24}{89} = g(0) = g(2\pi), \tag{34}$$

we see that $g$ attains its maximum value of $\frac{24\sqrt{13}}{89}$ on $[0, 2\pi]$ at $\theta = \tan^{-1}(-2\sqrt{3})$.

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**Solution to e:** From parts $b$ and $d$ we have already seen that the value of $\theta$ that maximizes $g(\theta)$ is the same as the angle $\theta$ associated with the direction of maximum increase. To finish, we just note that

$$|\nabla f(\frac{3}{4}, -\sqrt{3})| = |\langle\frac{24}{89}, \frac{-48\sqrt{3}}{89}\rangle| = \frac{24}{89}|\langle 1, -2\sqrt{3}\rangle| \tag{35}$$

$$= \frac{24}{89}\sqrt{1^2 + (-2\sqrt{3})^2} = \frac{24\sqrt{13}}{89}. \tag{36}$$

**Simplified Problem 13.6.38:** Consider the function $f(x,y) = \ln(1+4x^2+3y^2)$ and the point $P = (\frac{3}{4}, -\sqrt{3})$.

**a.** Find the gradient field $\nabla f(x,y)$ of $f(x,y)$ and then evaluate it at $P$.

**b.** Find the angles $\theta$ (with respect to the x-axis) associated with the directions of maximum increase, maximum decrease, and zero change.

**c.** Write the directional derivative at $P$ as a function of $\theta$; call this function $g(\theta)$.

**d.** Find the value of $\theta$ that maximizes $g(\theta)$ and find the maximum value.

**e.** Verify that the value of $\theta$ that maximizes $g$ corresponds to the direction of the gradient vector at $P$. Verify that the maximum value of $g$ equals the magnitude of the gradient vector at $P$.

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**Solution to a:** We see that

$$\begin{aligned} f_x(x,y) &= \tfrac{1}{1+4x^2+3y^2}\tfrac{\partial}{\partial x}(1+4x^2+3y^2) = \tfrac{8x}{1+4x^2+3y^2} \\ f_y(x,y) &= \tfrac{1}{1+4x^2+3y^2}\tfrac{\partial}{\partial y}(1+4x^2+3y^2) = \tfrac{6y}{1+4x^2+3y^2} \end{aligned} \tag{37}$$

$$\to \nabla f(x,y) = \langle \frac{8x}{1+4x^2+3y^2}, \frac{6y}{1+4x^2+3y^2} \rangle. \tag{38}$$

$$\nabla f(\frac{3}{4}, -\sqrt{3}) = \langle \frac{6}{1+\frac{9}{4}+9}, \frac{-6\sqrt{3}}{1+\frac{9}{4}+9} \rangle = \langle \frac{24}{49}, \frac{-24\sqrt{3}}{49} \rangle. \tag{39}$$

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**Solution to b:** We recall that $\nabla f(P)$ points in the direction of maximum increase from $P$. Since $\nabla f(P)$ is in the fourth quadrant, we see that

$$\theta_{\max} = \tan^{-1}\left(\frac{\frac{-24\sqrt{3}}{49}}{\frac{24}{49}}\right) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}. \tag{40}$$

is the angle associated with the direction of maximum increase. Since $-\nabla f(P)$ points in the direction of maximum decrease from $P$, we see that $\theta_{\min} = \theta_{\max} + \pi = \frac{2\pi}{3}$ is the angle associated with the direction of maximum decrease. Since the directions of no change are orthogonal to $\nabla f(P)$ (and to $-\nabla f(P)$), we see that $\theta_1 = \theta_{\max} + \frac{\pi}{2} = \frac{5\pi}{6}$ and $\theta_2 = \theta_{\max} - \frac{\pi}{2} = -\frac{\pi}{6}$ are the angles associated to the directions of zero change.

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**Solution to c:** We recall that $\vec{u}(\theta) = \langle\cos(\theta), \sin(\theta)\rangle$ is the unit vector associated with the angle $\theta$. We also recall that for any unit vector $\vec{u}$, we have that

$$d_{\vec{u}}f(a,b) = \nabla f(a,b)\cdot\vec{u}, \text{ so} \tag{41}$$

$$g(\theta) = d_{\vec{u}(\theta)}f(P) = \nabla f(P)\cdot\vec{u}(\theta) = \langle\frac{24}{49}, \frac{-24\sqrt{3}}{49}\rangle\cdot\langle\cos(\theta), \sin(\theta)\rangle \tag{42}$$

$$= \frac{24}{49}\cos(\theta) - \frac{24\sqrt{3}}{49}\sin(\theta). \tag{43}$$

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**Solution to d:** We see that

$$g'(\theta) = -\frac{24}{49}\sin(\theta) - \frac{24\sqrt{3}}{49}\cos(\theta) \to \tag{44}$$

$$g'(\theta) = 0 \Leftrightarrow -\frac{24}{49}\sin(\theta) = \frac{24\sqrt{3}}{49}\cos(\theta) \Leftrightarrow \tan(\theta) = -\sqrt{3} \Leftrightarrow \tag{45}$$

$$\theta = -\frac{\pi}{3}, \frac{2\pi}{3} \tag{46}$$

We see that

$$g''(\theta) = -\frac{24}{49}\cos(\theta) + \frac{24\sqrt{3}}{49}\sin(\theta) \tag{47}$$

$$\to g''(-\frac{\pi}{3}) = -\frac{24}{49}\cos(-\frac{\pi}{3}) + \frac{24\sqrt{3}}{49}\sin(-\frac{\pi}{3}) = -\frac{48}{89} < 0. \tag{48}$$

The second derivative test shows us that $g(\theta)$ has a local maximum at $\theta = -\frac{\pi}{3}$.

$$g(-\frac{\pi}{3}) = \frac{24}{49}\cos(-\frac{\pi}{3}) - \frac{24\sqrt{3}}{49}\sin(-\frac{\pi}{3}) = \frac{48}{49}. \tag{49}$$

we see that $g$ attains its maximum value of $\frac{48}{89}$ on $[0, 2\pi]$ at $\theta = -\frac{\pi}{3}$.

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**Solution to e:** From parts $b$ and $d$ we have already seen that the value of $\theta$ that maximizes $g(\theta)$ is the same as the angle $\theta$ associated with the direction of maximum increase. To finish, we just note that

$$|\nabla f(\frac{3}{4}, -\sqrt{3})| = |\langle\frac{24}{49}, \frac{-24\sqrt{3}}{49}\rangle| = \frac{24}{49}|\langle 1, -\sqrt{3}\rangle| \tag{50}$$

$$= \frac{24}{49}\sqrt{1^2 + (-\sqrt{3})^2} = \frac{48}{49}. \tag{51}$$