Math 2173 Spring 2021 Recitation Handout 3 Solutions

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Group Member 2: ________________________________

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Below is a checklist of instructions to follow when completing this assignment. Failure to follow these directions will result in penalty on your final score and/or in some problems not being graded. If multiple directions are not followed, then it is also possible that the assignment will not be accepted for any credit at all. Please contact your TA or make a post on the discussion boards for this course if you have any questions about this assignment or these directions.

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	Checklist of Instructions
	Please clearly write the names of all group members working on this assignment in the spaces allotted above.
	This assignment must be completed by a group of 3, 4, 5, or 6 members.
	This assignment is to be uploaded to gradescope as a pdf file no later than 11:59 PM EST on Sunday, January 31.
	The assignment will be uploaded by 1 group member, and that group member will be responsible for manually entering the names of all other collaborators into gradescope.
	This assignment must be completed using this template. You may either print this template to write on it and then scan it (pages ordered correctly) into a pdf file, or you may write directly on the template using programs such as notability.
	If you need more space than what is given to solve a given problem, then you will find blank pages provided at the end of this template. At the end of each problem section of this assignment you will find a space in which to indicate on what page your work is continued in case you used additional pages to complete your solution. You must provide the page number on which your work is continued in the alloted space, or write 'N/A' incase you did not use any additional pages.
	On the additional pages, you will also find space in which to indicate which problem the page is being used for, and if the page is used then that space must also be filled.
	To complete this handout, you may use your textbook, class notes, discussions with your TA and group members, and any resources that are available on Carmen. You should not receive any help from the MSLC or people outside of your group when solving these problems. You may discuss these problems on the Carmen discussion boards, but you should not provide your entire solution when answering a such question, you should only give a hint or a helpful idea.

(Ungraded Optional Problem) Problem 13.9.53: Economists model the output of manufacturing systems using production functions that have many of the same properties as utility functions. The family of Cobb-Douglas production functions has the form $P = f(K, L) = CK^aL^{1-a}$, where K represents capital, L represents labor, and C and a are positive real numbers with $0 < a < 1$. If the cost of capital is p dollars per unit, the cost of labor is q dollars per unit, and the total available budget is B, then the constraint takes the form $pK + qL = B$. Find the values of K and L that maximize the production function

$$P = f(K, L) = K^{\frac{1}{2}}L^{\frac{1}{2}} \tag{1}$$

subject to

$$20K + 30L = 300, \tag{2}$$

assuming $K \geq 0$ and $L \geq 0$.

Problem 13.9.54 (7 points): Economists model the output of manufacturing systems using production functions that have many of the same properties as utility functions. The family of Cobb-Douglas production functions has the form $P = f(K, L) = CK^aL^{1-a}$, where K represents capital, L represents labor, and C and a are positive real numbers with $0 < a < 1$. If the cost of capital is p dollars per unit, the cost of labor is q dollars per unit, and the total available budget is B, then the constraint takes the form $pK + qL = B$. Find the values of K and L that maximize the production function

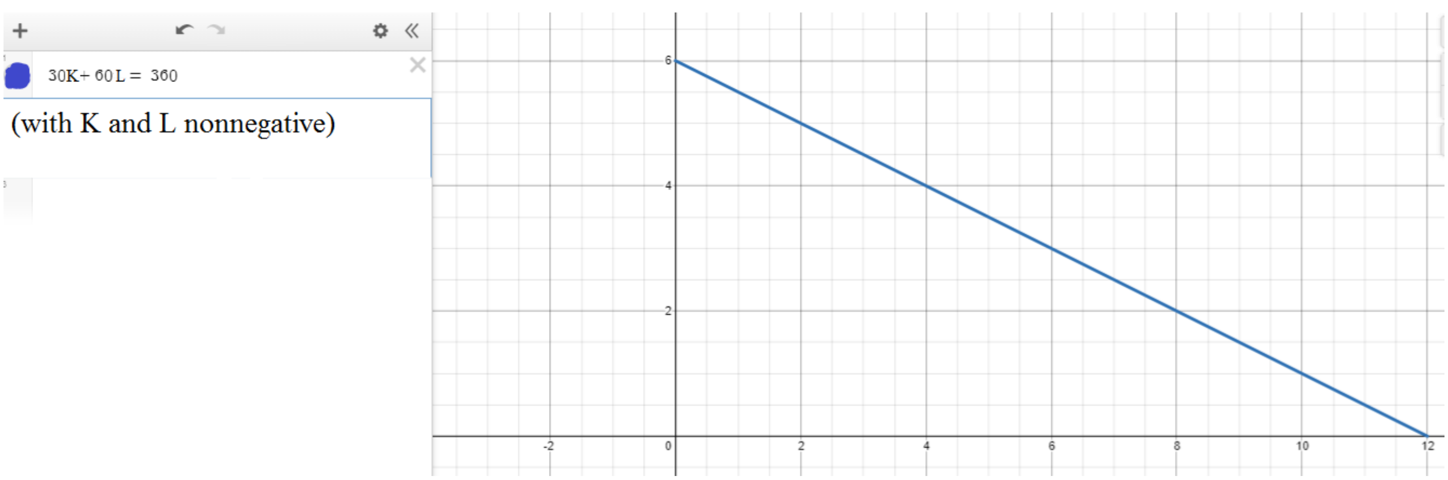
$$P = f(K, L) = 10K^{\frac{1}{3}}L^{\frac{2}{3}} \tag{3}$$

subject to

$$30K + 60L = 360, \tag{4}$$

assuming $K \geq 0$ and $L \geq 0$.

Solution: We see that the region defined by the constraint is the line segment from $(K, L) = (0, 6)$ to $(K, L) = (12, 0)$, which is a closed and bounded region with boundary.



The method of Lagrange multipliers will give us all of the critical points in the interior of the line segment, and we will then compare the values of f at the critical points with the values of f at the boundary (the 2 end points of the line segment) in order to find the absolute maximum and absolute minimum values. We begin by identifying our constraint function $g(K, L)$, its gradient field $\nabla g(K, L)$, and the gradient field $\nabla f(K, L)$ of our optimization function as

(5) $$g(K,L) = 30K + 60L - 360, \nabla g(K,L) = \langle 30, 60\rangle, \text{ and}$$

(6) $$\nabla f(K,L) = \langle \frac{10}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}}, \frac{20}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}}\rangle.$$

The method of Lagrange multipliers gives us the system of equations

(7) $$\begin{aligned} g(K,L) &= 0 \\ \nabla f(K,L) &= \lambda \nabla g(K,L) \end{aligned}$$

. . . .

(8) $$\Leftrightarrow \begin{aligned} 30K + 60L - 360 &= 0 \\ \langle \tfrac{10}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}}, \tfrac{20}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}}\rangle &= \lambda\langle 30, 60\rangle \end{aligned}$$

. . . .

(9) $$\Leftrightarrow \begin{aligned} 30K + 60L - 360 &= 0 \\ \tfrac{10}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}} &= 30\lambda \\ \tfrac{20}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}} &= 60\lambda \end{aligned}$$

. . . .

(10) $$\rightarrow \frac{20}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}} = 60\lambda = \frac{20}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}} \rightarrow K = L$$

. . . .

(11) $$\rightarrow 0 = 30K + 60L - 360 = 90L - 360 \rightarrow L = 4 \rightarrow \boxed{(K,L) = (4,4)}.$$

Since $(4,4)$ is the only critical point given to use by the method of Lagrange multipliers and

(12) $$f(4,4) = 10 \cdot 4^{\frac{1}{3}}4^{\frac{2}{3}} = 10 \cdot 4 = 40 > 0 = f(12,0) = f(0,6),$$

we see that the production function attains its absolute maximum value (subject to the given constraint) of 40 at $(4,4)$.

Problem 13.9.55 (10 points): Given the production function $P = f(K, L) = K^a L^{1-a}$ and the budget constraint $pK + qL = B$, where a, p, q, and B are given, show that P is maximized when $K = aB/p$ and $L = (1-a)B/q$. (Recall that $K \geq 0$ and $L \geq 0$ in order for the model to make sense in the real world and in order for the production function f to be well defined.)

Solution: We see that the region defined by the constraint is the line segment from $(K, L) = (0, \frac{B}{q})$ to $(K, L) = (\frac{B}{p}, 0)$, which is a closed and bounded region with boundary. The method of Lagrange multipliers will give us all of the critical points in the interior of the line segment, and we will then compare the values of f at the critical points with the values of f at the boundary (the 2 end points of the line segment) in order to find the absolute maximum and absolute minimum values. We begin by identifying our constraint function $g(K, L)$, its gradient field $\nabla g(K, L)$, and the gradient field $\nabla f(K, L)$ of our optimization function as

$$g(K, L) = pK + qL - B, \nabla g(K, L) = \langle p, q \rangle, \text{ and} \tag{13}$$

$$\nabla f(K, L) = \langle aK^{a-1}L1 - a, (1-a)K^a L^{-a} \rangle. \tag{14}$$

The method of Langrange multipliers gives us the system of equations

$$\begin{aligned} g(K, L) &= 0 \\ \nabla f(K, L) &= \lambda \nabla g(K, L) \end{aligned} \tag{15}$$

$$\Leftrightarrow \begin{aligned} pK + qL - B &= 0 \\ \langle aK^{a-1}L1 - a, (1-a)K^a L^{-a} \rangle &= \lambda \langle p, q \rangle \end{aligned} \tag{16}$$

$$\Leftrightarrow \begin{aligned} pK + qL - B &= 0 \\ aK^{a-1}L1 - a &= p\lambda \\ (1-a)K^a L^{-a} &= q\lambda \end{aligned} \tag{17}$$

$$\rightarrow qaK^{a-1}L^{1-a} = pq\lambda = p(1-a)K^a L^{-a} \tag{18}$$

$$\to qaL = p(1-a)K \to L = \frac{p(1-a)}{qa}K \tag{19}$$

. . . .

$$\to 0 = pK + qL - B = pK + \frac{p(1-a)}{a}K - B \to K = \frac{Ba}{p} \tag{20}$$

. . . .

$$\to L \overset{\text{(By (19))}}{=} \frac{B(1-a)}{q}, \text{ so} \tag{21}$$

. . . .

$$(K, L) = (\frac{Ba}{p}, \frac{B(1-a)}{q}) \tag{22}$$

is the only critical point obtained by the method of Lagrange multipliers. We see that $K, L > 0$ at this critical point, so

$$f(K, L) > 0 = f(0, \frac{B}{q}) = f(\frac{B}{p}, 0). \tag{23}$$

Since the value of f at the (only) critical point is larger than the values of f on the boundary (the end points) we see that f attains its absolute maximum value at the critical point as desired.

Problem 13.9.30 (10 points): Find the point on the plane $2x + 3y + 6z - 10 = 0$ closest to the point $(-2, 5, 1)$ by using the method of Lagrange Multipliers.

Solution: We see that our constraint function is $g(x, y, z) = 2x+3y+6z-10$, and the function that we are trying to optimize is the distaince from a point (x, y, z) on the plane to the point $(-2, 5, 1)$, which is given by

$$h(x, y, z) = \sqrt{(x - (-2))^2 + (y - 5)^2 + (z - 1)^2} = \sqrt{x^2 + 4x + 4 + y^2 - 10y + 25 + z^2 - 2z + 1}. \tag{24}$$

Since $h(x, y, z)$ and $f(x, y, z) = (h(x, y, z))^2$ have their absolute minimum(s) occurring at the same location(s), we will optimize $f(x, y, z)$ subject to $g(x, y, z) = 0$ instead since the resulting calculations will be easier. Since our constraint function defines an open region (a plane) the method of Lagrange multipliers will give us all of the critical points in the open region, and we will compare the values of $f(x, y, z)$ at the critical points to the values of $f(x, y, z)$ as (x, y, z) approaches the boundary. Noting that

$$\nabla g(x, y, z) = \langle 2, 3, 6\rangle \text{ and} \tag{25}$$

$$\nabla f(x, y, z) = \langle 2x + 4, 2y - 10, 2z - 2\rangle, \tag{26}$$

the method of Lagrange multipliers gives us the system of equations

$$\begin{aligned} g(x, y, z) &= 0 \\ \vec{\nabla} f(x, y, z) &= \lambda \vec{\nabla} g(x, y, z) \end{aligned} \tag{27}$$

. . . .

$$\Leftrightarrow \begin{aligned} 2x + 3y + 6z - 10 &= 0 \\ \langle 2x + 4, 2y - 10, 2z - 2\rangle &= \lambda\langle 2, 3, 6\rangle \end{aligned} \tag{28}$$

. . . .

$$\Leftrightarrow \begin{array}{rcl} 2x+3y+6z-10 &=& 0 \\ 2x+4 &=& 2\lambda \\ 2y-10 &=& 3\lambda \\ 2z-2 &=& 6\lambda \end{array} \Leftrightarrow \begin{array}{rcl} 2x+3y+6z-10 &=& 0 \\ x &=& \lambda-2 \\ y &=& \frac{3}{2}\lambda+5 \\ z &=& 3\lambda+1 \end{array} \tag{29}$$

. . . .

$$\rightarrow 0 = 2(\lambda-2)+3(\frac{3}{2}\lambda+5)+6(3\lambda+1)-10 = \frac{49}{2}\lambda+7 \rightarrow \lambda = -\frac{2}{7} \tag{30}$$

. . . .

$$\rightarrow \boxed{(x,y,z) = (-\frac{16}{7}, \frac{32}{7}, \frac{1}{7})}. \tag{31}$$

We see that a point (x, y, z) in the plane $2x + 3y + 6z - 10 = 0$ approaches the boundary of the plane (the 'outer edges' of the plane) if at least one of x, y, or z appraoches infinity. It follows that the square of the distance function $(f(x, y, z))$ approaches positive infinity as (x, y, z) approaches the boundary, so the absolute minimum exists and occurs at the critical point that we found.

Problem 13.9.46 (10 points): Find the absolute minimum and absolute maximum values of the function

$$f(x,y) = x^2 + 4y^2 + 1 \tag{32}$$

over the region

$$R = \{(x,y) : x^2 + 4y^2 \leq 1\}. \tag{33}$$

You should know how to solve this type of problem using lagrange multipliers, but you can also avoid using lagrange multipliers and parameterization of the boundary in this particular problem (and still receive full credit) if you think about it carefully. You will not receive full credit for this problem if your solution involves parameterization of the boundary.

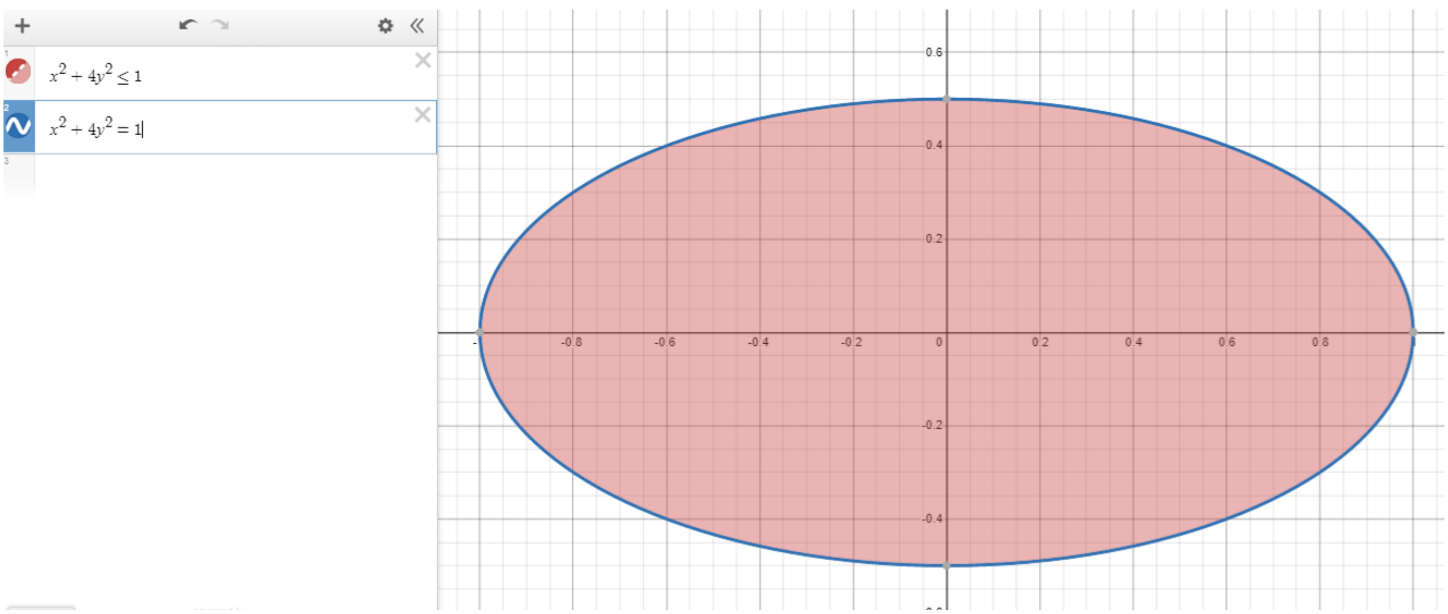


FIGURE 1. The interior of the R is shaded in red and the boundary of R is blue.

Solution: Since the region R is a closed and bounded region, and the function f is continuous, the extreme value theorem tells us that the absolute minimum and absolute maximum values of f must be achieved on the boundary of R or at a critical point in the interior of R. We first find all of the critical points of f. We see that

$$\text{(34)}\qquad \begin{array}{rcl} f_x(x,y) &=& 0 \\ f_y(x,y) &=& 0 \end{array} \Leftrightarrow \begin{array}{rcl} 2x &=& 0 \\ 8y &=& 0 \end{array} \Leftrightarrow (x,y) = (0,0).$$

We see that $(0,0) \in R$ and that $f(0,0) = 1$. Next we will determine the absolute minimum and absolute maximum values of f on the boundary of R. Since the boundary of R is given by $x^2 + 4y^2 = 1$, we see that $f(x,y) = 2$ for every (x,y) on the boundary of R, so we immediately see that f achieves its absolute minimum value of 1 at $(0,0)$ and its absolute maximum value of 2 at any (x,y) on the boundary of R.

If we were not lucky enough to instatly notice that $f(x,y) = 2$ for every (x,y) on the boundary of R, then we would try to handle the boundary by using the method of Lagrange multipliers. More specifically, we would try to optimize the function $f(x,y) = 1 + x^2 + 4y^2$ subject to the constraint $g(x,y) = x^2 + 4y^2 - 1 = 0$. Noting that

$$\text{(35)}\qquad \nabla g(x,y) = \langle 2x, 8y\rangle \text{ and } \nabla f(x,y) = \langle 2x, 8y\rangle$$

the method of Lagrange multipliers gives us the system of equations

$$\text{(36)}\qquad \begin{array}{rcl} g(x,y) &=& 0 \\ \nabla f(x,y) &=& \lambda\nabla g(x,y) \end{array} \Leftrightarrow \begin{array}{rcl} g(x,y) &=& 0 \\ \langle 2x, 8y\rangle &=& \lambda\langle 2x, 8y\rangle \end{array}$$

$$\text{(37)}\qquad \Leftrightarrow \begin{array}{rcl} g(x,y) &=& 0 \\ 2x &=& 2\lambda x \\ 8y &=& 8\lambda y \end{array}.$$

Letting $\lambda = 1$, we see that every point (x,y) on the boundary of R (which is the same as every point (x,y) satisfying the constraint $g(x,y) = 0$ also satisfies the system of equations given to us by the method of Lagrange multipliers. This seems bad at first since the boundary has infinitely many points, so it looks like the method of Lagrange multipliers did not help us in our search for the absolute minimum and absolute maximum values that occur on the boundary. However, it turns out that the only time every point on the boundary of our region R (assuming that R has a piecewise smooth boundary, which it always will in this class) is a critical point is when $f(x,y)$ is constant on the region R

(as it was in this problem), so the problem turns out to be easier in these cases since you can determine the value of $f(x, y)$ on the boundary of R by checking the value at any random point (x, y) on the boundary of R.

Problem 13.9.16 (13 points): Use the method of Lagrange multipliers to find the absolute maximum and minimum of the function

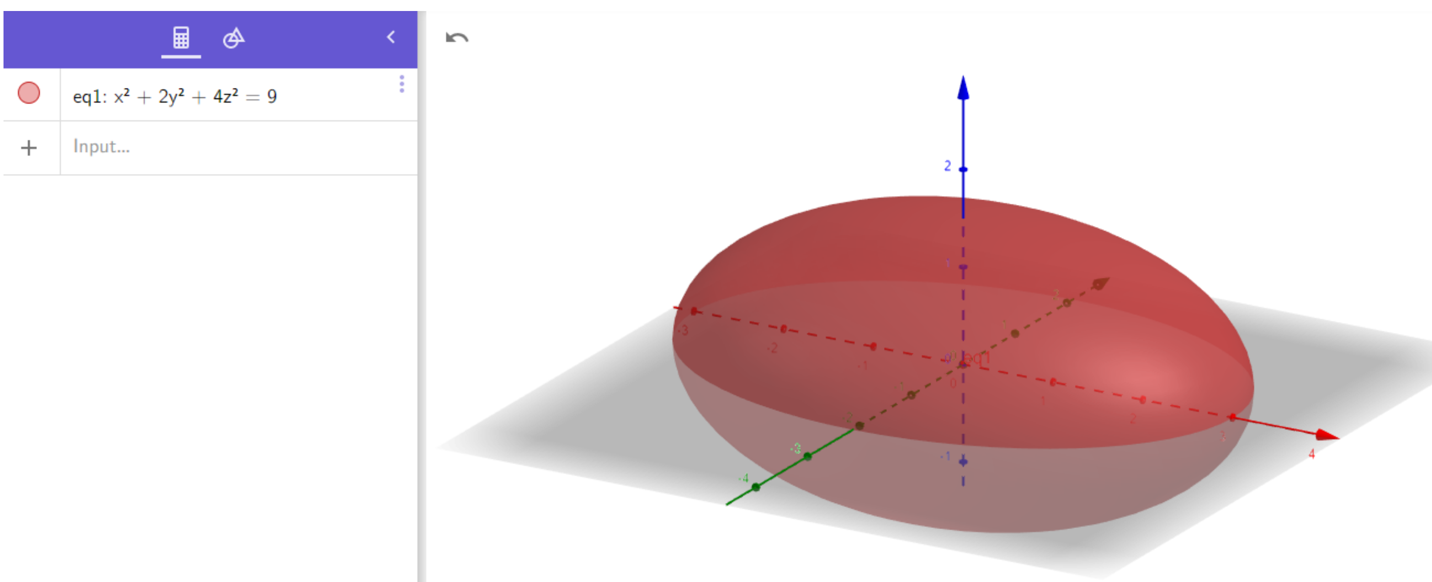
$$f(x, y, z) = xyz \tag{38}$$

subject to the constraint

$$x^2 + 2y^2 + 4z^2 = 9. \tag{39}$$

Please remember to use the 0 product property when solving this problem and solving any systems of equations that arise. You should have 14 critical points that satisfy the system of equations given by the method of Lagrange Multipliers.

Solution: We see that the region defined by the constraint is a closed and bounded region with no boundary, so the method of Lagrange multipliers will give us the complete list of critical points that we need to check in order to determine the absolute minimum and absolute maximum values of f subject to the constraint.



We see that

$$x^2 + 2y^2 + 4z^2 = 9 \Leftrightarrow x^2 + 2y^2 + 4z^2 - 9 = 0, \tag{40}$$

so we may take our constraint function to be $g(x, y, z) = x^2 + 2y^2 + 4z^2 - 9$. We see that

$$\vec{\nabla} f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle = \langle yz, xz, xy\rangle, \text{ and} \tag{41}$$

$$\vec{\nabla} g(x, y, z) = \langle g_x(x, y, z), g_y(x, y, z), g_z(x, y, z) = \langle 2x, 4y, 8z\rangle. \tag{42}$$

We now want to find all (x, y, z, λ) (although we don't really care about the value of λ) such that

$$\begin{matrix} g(x, y, z) = 0 \\ \vec{\nabla} f(x, y, z) = \lambda \vec{\nabla} g(x, y, z) \end{matrix} \Leftrightarrow \begin{matrix} x^2 + 2y^2 + 4z^2 - 9 = 0 \\ \langle yz, xz, xy\rangle = \lambda\langle 2x, 4y, 8z\rangle \end{matrix} \tag{43}$$

$$\begin{matrix} x^2 + 2y^2 + 4z^2 - 9 = 0 \\ yz = 2\lambda x \\ xz = 4\lambda y \\ xy = 8\lambda z \end{matrix} \tag{44}$$

By cross-multiplying the second and third equations in (44) we see that

$$4\lambda y^2 z = 2\lambda x^2 z \to 0 = 4\lambda y^2 z - 2\lambda x^2 z = 2\lambda z(2y^2 - x^2), \tag{45}$$

so by the zero product property we see that either $\lambda = 0$, $z = 0$, or $2y^2 - x^2 = 0$. We will handle each case separately.

Case 1 ($\lambda = 0$): By plugging $\lambda = 0$ back into (44) we see that

$$\begin{matrix} x^2 + 2y^2 + 4z^2 - 9 = 0 \\ yz = 0 \\ xz = 0 \\ xy = 0 \end{matrix} . \tag{46}$$

Using the zero product property once again on the second, third, and fourth equations of (46), we see that 2 of x, y, and z must be 0. In conjunction with the first equation of (44) (the constraint equation) we see that $(x, y, z, \lambda) \in$

$\{(0,0,\pm\frac{3}{2},0),(0,\pm\frac{3}{\sqrt{2}},0,0),(\pm 3,0,0,0)\}$ are the solutions that we obtain from this case.

Case 2 ($\mathbf{z=0}$): By plugging $z=0$ back into (44) we see that

$$\begin{gathered} x^2+2y^2-9=0 \\ 0=2\lambda x \\ 0=4\lambda y \\ xy=0 \end{gathered}. \tag{47}$$

Since we are done with case 1, we may also assume that $\lambda \neq 0$. It now follows from the second and third equations in (47) that $x=y=0$, but this contradicts the first equation in (47), so we obtain no additional solutions in this case.

Case 3 ($2y^2-x^2=0$): In this case we see that $x^2=2y^2$ so $x=\pm\sqrt{2}y$. Plugging $x=\sqrt{2}y$ back into (44) gives us

$$\begin{gathered} 2y^2+2y^2+4z^2-9=0 \\ yz=2\sqrt{2}\lambda y \\ \sqrt{2}yz=4\lambda y \\ \sqrt{2}y^2=8\lambda z \end{gathered}. \tag{48}$$

By cross-multiplying the third and fourth equations in (48) we see that

$$8\sqrt{2}\lambda yz^2=4\sqrt{2}\lambda y^3 \to 0=8\sqrt{2}\lambda yz^2-4\sqrt{2}\lambda y^3=4\sqrt{2}\lambda y(2z^2-y^2). \tag{49}$$

Since we are no longer in case 1, we may assume that $\lambda \neq 0$, so either $y=0$ or $2z^2-y^2=0$. If $y=0$, then $x=\sqrt{2}y=0$, and we reobtain the solution $(x,y,z)=(0,0,\frac{3}{2})$. If $2z^2-y^2=0$, then $y^2=2z^2$. Plugging this back into the first equation of (48) yields

$$12z^2=9 \to z=\pm\frac{\sqrt{3}}{2}, \tag{50}$$

so we obtain the solutions

$$(x,y,z) \in \{(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}), (\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})\}. \quad (51)$$

If $x = -\sqrt{2}y$ then a similar calculation yields the additional solutions

$$(x,y,z) \in \{(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}), (-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})\}. \quad (52)$$

Now that we have found all solutions to the system of equations in (44), we see that

(x,y,z)	f(x,y,z)		(x,y,z)	f(x,y,z)
$(0,0,\frac{3}{2})$	0		$(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
$(0,\frac{3}{\sqrt{2}},0)$	0		$(\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
(3,0,0)	0		$(\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(0,0,-\frac{3}{2})$	0		$(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
$(0,-\frac{3}{\sqrt{2}},0)$	0		$(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
(-3,0,0)	0		$(-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$		$(-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$

In conclusion, we see that the absolute minimum value of $f(x, y, z)$ subject to $g(x, y, z) = 0$ is $-\frac{3\sqrt{3}}{2\sqrt{2}}$ and the absolute maximum value of $f(x, y, z)$ subject to $g(x, y, z) = 0$ is $\frac{3\sqrt{3}}{2\sqrt{2}}$.