Math 2173 Spring 2021 Recitation Handout 4 Solutions

Group Member 1: Sohail Farhangi

Group Member 2: ______________________________

Group Member 3: ______________________________

Group Member 4: ______________________________

Group Member 5: ______________________________

Group Member 6: ______________________________

Below is a checklist of instructions to follow when completing this assignment. Failure to follow these directions will result in penalty on your final score and/or in some problems not being graded. If multiple directions are not followed, then it is also possible that the assignment will not be accepted for any credit at all. Please contact your TA or make a post on the discussion boards for this course if you have any questions about this assignment or these directions.

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	Checklist of Instructions
	Please clearly write the names of all group members working on this assignment in the spaces allotted above.
	This assignment must be completed by a group of 3, 4, 5, or 6 members.
	This assignment is to be uploaded to gradescope as a pdf file no later than 11:59 PM EST on Sunday, February 7.
	The assignment will be uploaded by 1 group member, and that group member will be responsible for manually entering the names of all other collaborators into gradescope.
	This assignment must be completed using this template. You may either print this template to write on it and then scan it (pages ordered correctly) into a pdf file, or you may write directly on the template using programs such as notability.
	If you need more space than what is given to solve a given problem, then you will find blank pages provided at the end of this template. At the end of each problem section of this assignment you will find a space in which to indicate on what page your work is continued in case you used additional pages to complete your solution. You must provide the page number on which your work is continued in the alloted space, or write 'N/A' incase you did not use any additional pages.
	On the additional pages, you will also find space in which to indicate which problem the page is being used for, and if the page is used then that space must also be filled.
	To complete this handout, you may use your textbook, class notes, discussions with your TA and group members, and any resources that are available on Carmen. You should not receive any help from the MSLC or people outside of your group when solving these problems. You may discuss these problems on the Carmen discussion boards, but you should not provide your entire solution when answering a such question, you should only give a hint or a helpful idea.

(Ungraded Optional Problem) Problem 14.1.23: Evaluate

$$\int_0^{\ln(2)}\int_1^{\ln(3)} e^{x+2y}dydx, \tag{1}$$

$$\int_1^{\ln(3)}\int_0^{\ln(2)} e^{x+2y}dxdy, \text{ and} \tag{2}$$

$$\left(\int_0^{\ln(2)} e^{x}dx\right)\left(\int_1^{\ln(3)} e^{2y}dy\right) \tag{3}$$

Note that all 3 integrals should result in the same value once evaluated. Please show your work for the calculations of each of the 3 integrals separately.

(Ungraded Optional Problem) Problem 14.2.28: Let R be the region in the first quadrant bounded by $x = 0$, $y = x^2$, and $y = 8 - x^2$. Evaluate

$$\iint_R (x + y)dA. \tag{4}$$

Problem 14.2.22 (4 points): Evaluate

$$\int_0^{\sqrt{\frac{\pi}{2}}}\int_0^1 yx\sin(x^2)dydx, \tag{5}$$

$$\int_0^1\int_0^{\sqrt{\frac{\pi}{2}}} yx\sin(x^2)dxdy, \text{ and} \tag{6}$$

$$\left(\int_0^{\sqrt{\frac{\pi}{2}}} x\sin(x^2)dx\right)\left(\int_0^1 ydy\right) \tag{7}$$

Note that all 3 integrals should result in the same value once evaluated. Please show your work for the calculations of each of the 3 integrals separately.

Solution: We see that

$$\int_0^{\sqrt{\frac{\pi}{2}}}\int_0^1 yx\sin(x^2)dydx = \int_0^{\sqrt{\frac{\pi}{2}}}\left(\frac{y^2}{2}x\sin(x^2)\Big|_{y=0}^1\right)dx \tag{8}$$

$$= \int_0^{\sqrt{\frac{\pi}{2}}}\frac{1}{2}x\sin(x^2)dx \overset{u=x^2}{=} \int_{x=0}^{\sqrt{\frac{\pi}{2}}}\frac{1}{4}\sin(u)du = -\frac{1}{4}\cos(u)\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}} \tag{9}$$

$$= -\frac{1}{4}\cos(x^2)\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}} = \boxed{\frac{1}{4}}, \tag{10}$$

$$\int_0^1\int_0^{\sqrt{\frac{\pi}{2}}} yx\sin(x^2)dxdy \overset{u=x^2}{=} \int_0^1\int_{x=0}^{\sqrt{\frac{\pi}{2}}}\frac{1}{2}y\sin(u)dudy \tag{11}$$

$$= \int_0^1\left(-\frac{1}{2}y\cos(u)\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}}\right)dy = \int_0^1\left(-\frac{1}{2}y\cos(x^2)\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}}\right)dy = \int_0^1\frac{1}{2}ydy \tag{12}$$

$$= \frac{1}{4}y^2\Big|_0^1 = \boxed{\frac{1}{4}}\text{, and} \tag{13}$$

. . . .

$$\left(\int_0^{\sqrt{\frac{\pi}{2}}} x\sin(x^2)dx\right)\left(\int_0^1 ydy\right) \overset{u=x^2}{=} \left(\int_{x=0}^{\sqrt{\frac{\pi}{2}}} \frac{1}{2}\sin(u)du\right)\left(\frac{1}{2}y^2\Big|_0^1\right) \tag{14}$$

. . . .

$$= \left(-\frac{1}{2}\cos(u)\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}}\right)\cdot\frac{1}{2} = \left(-\frac{1}{2}\cos(x^2)\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}}\right)\cdot\frac{1}{2} = \frac{1}{2}\cdot\frac{1}{2} = \boxed{\frac{1}{4}}. \tag{15}$$

Problem 14.1.59 (10 points): Suppose that the second partial derivative of f are continuous on $R = \{(x, y) : 0 \le x \le a, 0 \le y \le b\}$. Show that

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x, y)dA = f(a, b) - f(a, 0) - f(0, b) + f(0, 0). \tag{16}$$

Hint: Think about the fundamental theorem of calculus.

Solution: We see that

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x, y)dA = \int_0^b \int_0^a \frac{\partial^2 f}{\partial x \partial y}(x, y)dxdy = \int_0^b \frac{\partial f}{\partial y}(x, y)\Big|_{x=0}^{a} dy \tag{17}$$

$$= \int_0^b \left(\frac{\partial f}{\partial y}(a, y) - \frac{\partial f}{\partial y}(0, y)\right) dy = (f(a, y) - f(0, y))\Big|_0^b. \tag{18}$$

$$= f(a, b) - f(a, 0) - f(0, b) + f(0, 0). \tag{19}$$

Alternatively, since the second partial derivatives of f are continuous on R, we can use Clairaut's Theorem to perform the calculations in the following fashion.

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x, y)dA = \int_0^a \int_0^b \frac{\partial^2 f}{\partial y \partial x}(x, y)dydx = \int_0^a \frac{\partial f}{\partial x}(x, y)\Big|_{y=0}^{b} dx \tag{20}$$

$$= \int_0^a \left(\frac{\partial f}{\partial x}(x, b) - \frac{\partial f}{\partial x}(x, 0)\right) dx = (f(x, b) - f(x, 0))\Big|_0^a. \tag{21}$$

$$= f(a, b) - f(a, 0) - f(0, b) + f(0, 0). \tag{22}$$

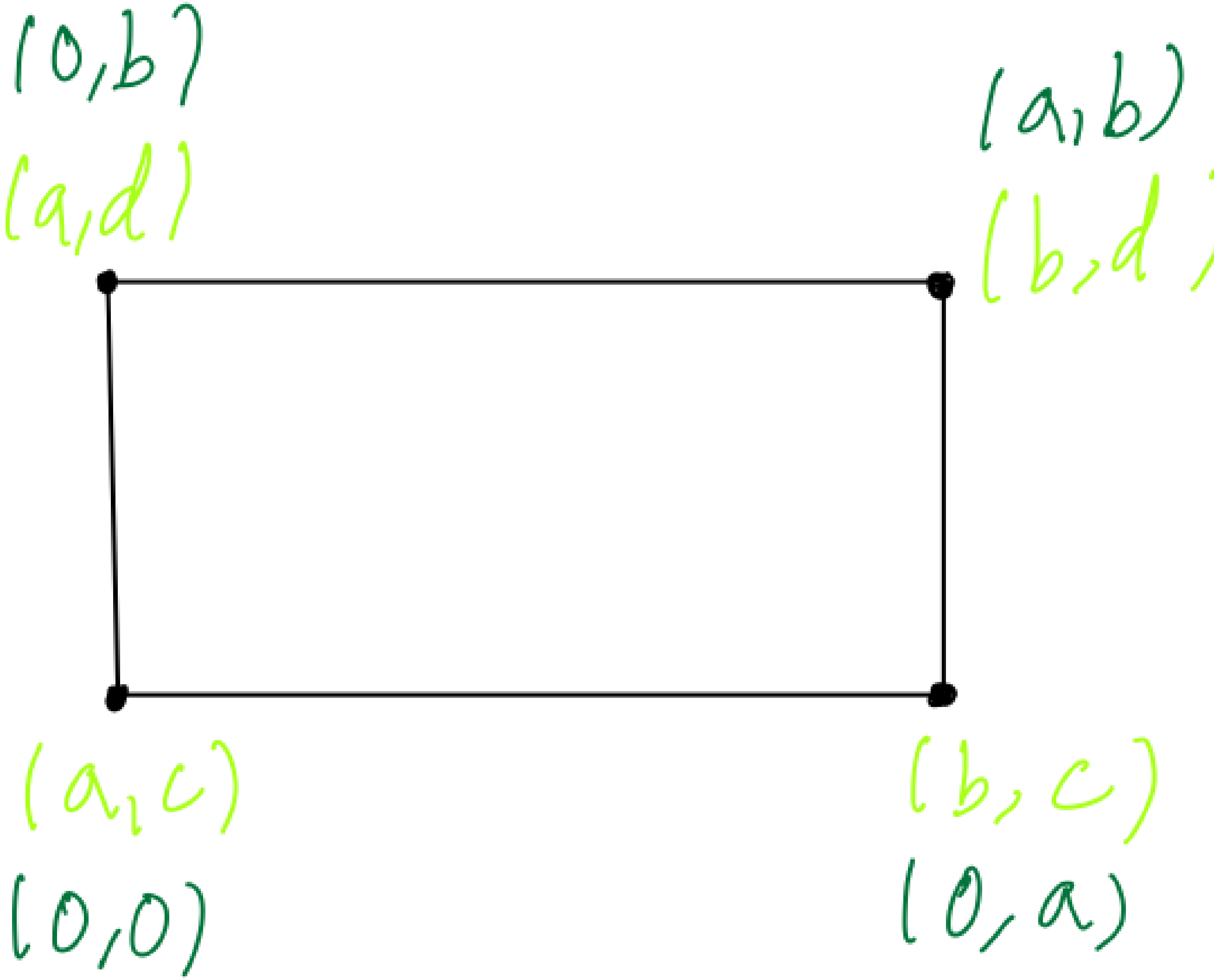
Remark: A similar method can show that if $R = \{(x, y) : a \le x \le b, c \le y \le d\}$, then

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x, y)dA = f(b, d) - f(a, d) - f(b, c) + f(a, c). \tag{23}$$

The Fundamental Theorem of Calculus told us that

$$\int_a^b \frac{df}{dx}(x)dx = f(b) - f(a). \tag{24}$$

Comparing equations (24) and (23), we see that instead taking the difference at the 2 endpoints of a line segment, we are adding 2 opposite corners of the rectangular region R ($f(b,d)$ and $f(a,c)$, or $f(a,b)$ and $f(0,0)$ from the original problem) and subtracting from that the sum of the other 2 opposite corners ($f(a,d)$ and $f(b,c)$, or $f(a,0)$ and $f(0,b)$ from the original problem).



Problem 14.1.60 (8 points): Let $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

a. Evaluate $\iint_R \cos(x\sqrt{y})dA$.

b. Evaluate $\iint_R x^3 y \cos(x^2y^2)dA$.

Hint: Choose a convenient order of integration.

Solution to a: Noting that $\int \cos(cx)dx$ is easily computable, but $\int \cos(c\sqrt{y})dy$ is not easily computable, we decide to use the order of integration given by $dA = dxdy$. It follows that

$$\iint_R \cos(x\sqrt{y})dA = \int_0^1 \int_0^1 \cos(x\sqrt{y})dxdy \tag{25}$$

$$\overset{u=x\sqrt{y}}{=} \int_0^1 \int_0^1 \frac{\cos(x\sqrt{y})}{\sqrt{y}}\sqrt{y}dxdy \overset{u=x\sqrt{y}}{=} \int_0^1 \int_{x=0}^1 \frac{\cos(u)}{\sqrt{y}}dudy \tag{26}$$

$$= \int_0^1 \left(\frac{\sin(u)}{\sqrt{y}}\Big|_{x=0}^1\right) dy = \int_0^1 \left(\frac{\sin(x\sqrt{y})}{\sqrt{y}}\Big|_{x=0}^1\right) dy = \int_0^1 \frac{\sin(\sqrt{y})}{\sqrt{y}}dy \tag{27}$$

$$\overset{u=\sqrt{y}}{=} \int_0^1 2\sin(\sqrt{y})\frac{dy}{2\sqrt{y}} \overset{u=\sqrt{y}}{=} \int_{y=0}^1 2\sin(u)du = -2\cos(u)\Big|_{y=0}^1 \tag{28}$$

$$= -2\cos(\sqrt{y})\Big|_{y=0}^1 = \boxed{2 - 2\cos(1)}. \tag{29}$$

Solution to b: Noting that $\int c_1x^3\cos(c_2x^2)dx$ is not easily computable, but $\int c_1y\cos(c_2y^2)dy$ is easily computable, we decide to use the order of integration given by $dA = dydx$. It follows that

$$\iint_R x^3y\cos(x^2y^2)dA = \int_0^1 \int_0^1 x^3y\cos(x^2y^2)dydx \tag{30}$$

$$\overset{u=y^2}{=} \int_0^1 \int_0^1 \frac{x^3}{2} \cos(x^2y^2)2ydydx \overset{u=y^2}{=} \int_0^1 \int_{y=0}^1 \frac{x^3}{2} \cos(x^2u)dudx \tag{31}$$

$$\overset{v=x^2u}{=} \int_0^1 \int_{y=0}^1 \frac{x}{2} \cos(x^2u)x^2dudx \overset{v=x^2u}{=} \int_0^1 \int_{y=0}^1 \frac{x}{2} \cos(v)dvdx \tag{32}$$

$$= \int_0^1 \left(\frac{x}{2} \sin(v)\Big|_{y=0}^1 \right) dx = \int_0^1 \left(\frac{x}{2} \sin(x^2u)\Big|_{y=0}^1 \right) dx \tag{33}$$

$$= \int_0^1 \left(\frac{x}{2} \sin(x^2y^2)\Big|_{y=0}^1 \right) dx = \int_0^1 \frac{x}{2} \sin(x^2)dx \overset{u=x^2}{=} \int_0^1 \frac{1}{4} \sin(x^2)2xdx \tag{34}$$

$$\overset{u=x^2}{=} \int_{x=0}^1 \frac{1}{4} \sin(u)du = -\frac{1}{4} \cos(u)\Big|_{x=0}^1 \tag{35}$$

$$= -\frac{1}{4} \cos(x^2)\Big|_{x=0}^1 = \boxed{\frac{1}{4} - \frac{1}{4}\cos(1)}. \tag{36}$$

Problem 14.1.61 (10 points): Let $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$. Let F be an antiderivative of f satisfying $F(0) = 0$, and let G be an antiderivative of F. Show that if f and F are integrable, and $r, s \ge 1$ are real numbers, then

$$\iint_R x^{2r-1}y^{s-1}f(x^r y^s)dA = \frac{G(1)-G(0)}{rs}. \tag{37}$$

Hint: Pick a convenient order of integration, then apply u-substition. It also helps if you do problem 14.1.60 before doing this problem.

Solution: We note that Problem 14.1.60b was a special instance of this problem in which $r = s = 2$ and $f(t) = \cos(t)$. Therefore we will proceed in a similar fashion, but we will slightly simplify our solution by merging the first 2 u-substitutions that were performed in the solution to Problem 14.1.60b into a single u-substitution. We now see that

$$\iint_R x^{2r-1}y^{s-1}f(x^r y^s)dA = \int_0^1\int_0^1 x^{2r-1}y^{s-1}f(x^r y^s)dydx \tag{38}$$

$$\overset{u=x^r y^s}{=} \int_0^1\int_0^1 x^{r-1}f(x^r y^s)x^r y^{s-1}dydx \overset{u=x^r y^s}{=} \int_0^1\int_{y=0}^1 \frac{x^{r-1}}{s}f(u)dudx \tag{39}$$

$$= \int_0^1 \left(\frac{x^{r-1}}{s}F(u)\Big|_{y=0}^1\right)dx = \int_0^1 \left(\frac{x^{r-1}}{s}F(x^r y^s)\Big|_{y=0}^1\right)dx \tag{40}$$

$$= \int_0^1 \left(\frac{x^{r-1}}{s}F(x^r) - \frac{x^{r-1}}{s}\underbrace{F(0)}_{=0}\right)dx = \int_0^1 \frac{x^{r-1}}{s}F(x^r)dx \tag{41}$$

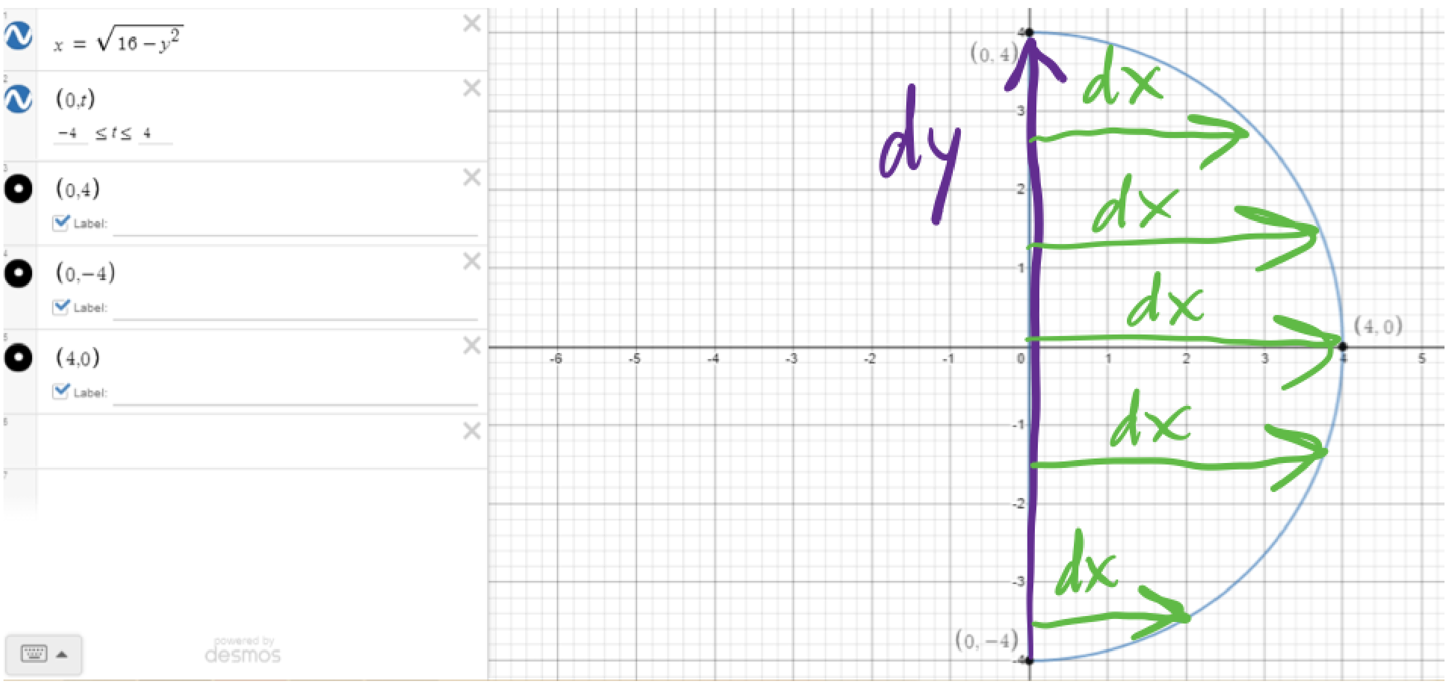
$$\overset{u=x^r}{=} \int_0^1 \frac{1}{rs}F(x^r)rx^{r-1}dx \overset{u=x^r}{=} \int_{x=0}^1 \frac{1}{rs}F(u)du \tag{42}$$

$$= \frac{1}{rs} G(u) \Big|_{x=0}^{1} = \frac{1}{rs} G(x^r) \Big|_{x=0}^{1} = \frac{G(1) - G(0)}{rs}. \tag{43}$$

Problem 14.2.30 (8 points): Let R be the region in quadrants 1 and 4 bounded by the semicircle of radius 4 centered at $(0,0)$. Sketch a picture of R, then evaluate

$$\iint_R x^2 y dA. \tag{44}$$

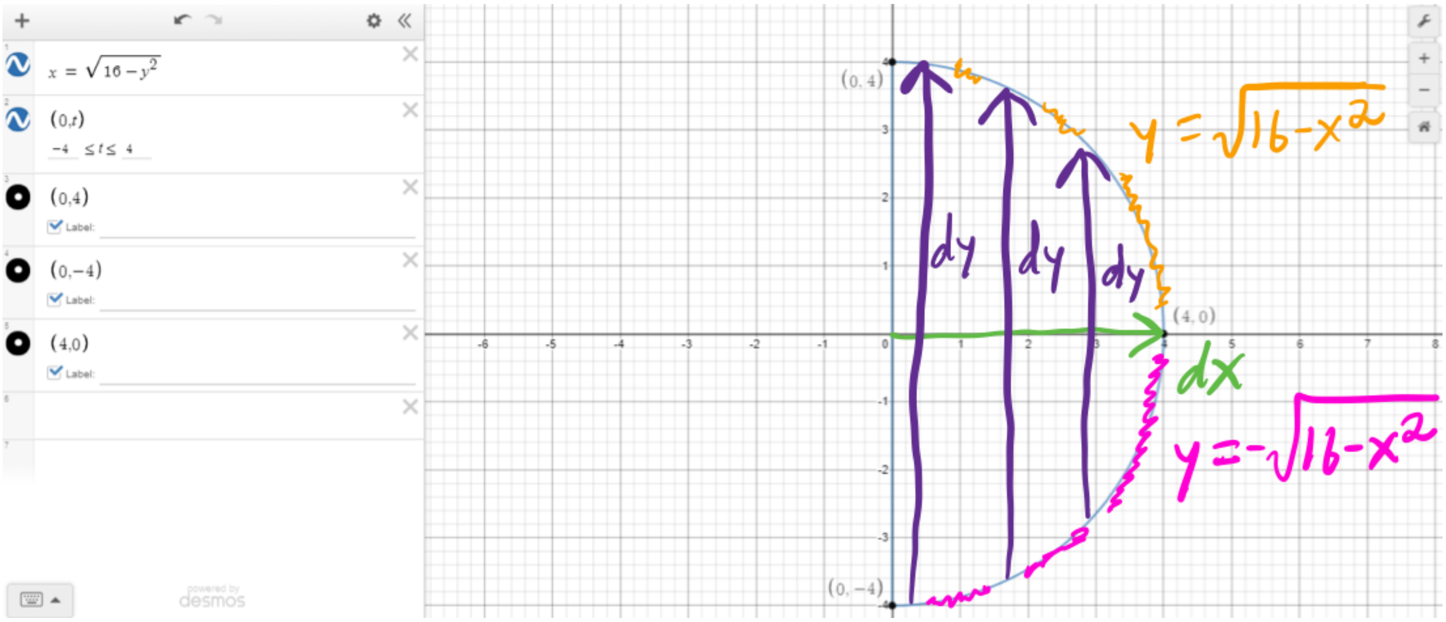
Solution 1: If we decide to integrate using the order $dA = dxdy$, then we obtain the picture and calculations show below.



$$\iint_R x^2 y dA = \int_{-4}^{4} \int_{0}^{\sqrt{16-y^2}} x^2 y dx dy = \int_{-4}^{4} \left(\frac{x^3}{3} y \Big|_{x=0}^{\sqrt{16-y^2}} \right) dy \tag{45}$$

$$= \int_{-4}^{4} \frac{1}{3} y (16 - y^2)^{\frac{3}{2}} dy = -\frac{1}{15}(16 - y^2)^{\frac{5}{2}} \Big|_{-4}^{4} = \boxed{0}. \tag{46}$$

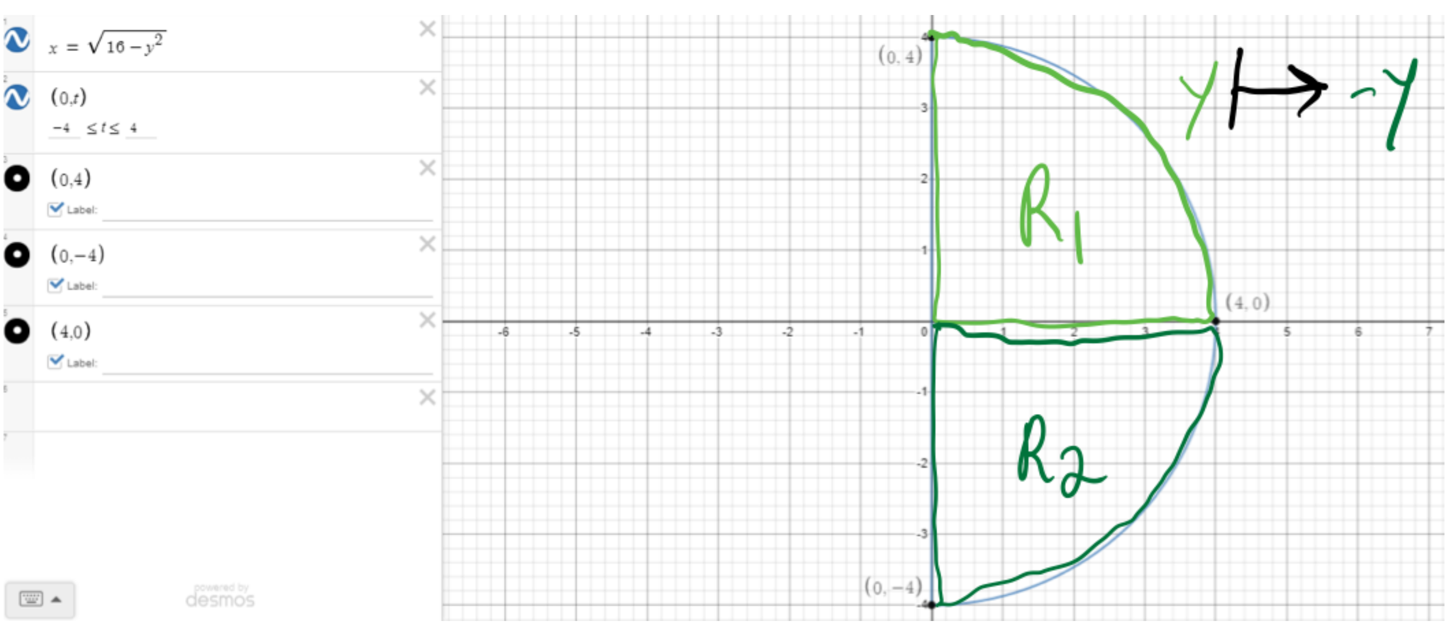
Solution 2: If we decide to integrate using the order $dA = dydx$, then we obtain the picture and calculations show below.



$$\iint_R x^2y\,dA = \int_0^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} x^2y\,dy\,dx = \int_0^4 \frac{1}{2}\left(x^2y^2\Big|_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\right)dx \tag{47}$$

$$= \int_0^4 0\,dx = \boxed{0}. \tag{48}$$

Solution 3: Using the symmetry shown below,



we see that

$$\iint_R x^2ydA = \iint_{R_1} x^2ydA + \iint_{R_2} x^2ydA \tag{49}$$

$$= -\iint_{R_2} x^2ydA + \iint_{R_2} x^2ydA = \boxed{0}. \tag{50}$$

To see the details of the above calculation worked out in more detail, we proceed as we did in solution 2.

$$\iint_{R_1} x^2ydA = \int_0^4 \int_0^{\sqrt{16-x^2}} x^2ydydx = \int_0^4 \int_0^{\sqrt{16-x^2}} x^2(-y)(-dy)dx \tag{51}$$

$$\overset{y=-y}{=} \int_0^4 \int_0^{-\sqrt{16-x^2}} x^2ydydx = -\int_0^4 \int_{-\sqrt{16-x^2}}^0 x^2ydydx \tag{52}$$

$$= -\iint_{R_2} x^2ydA. \tag{53}$$