Math 2173 Spring 2021 Recitation Handout 5 Solutions

Group Member 1: Sohail Farhangi

Group Member 2: ______________________________

Group Member 3: ______________________________

Group Member 4: ______________________________

Group Member 5: ______________________________

Group Member 6: ______________________________

Below is a checklist of instructions to follow when completing this assignment. Failure to follow these directions will result in penalty on your final score and/or in some problems not being graded. If multiple directions are not followed, then it is also possible that the assignment will not be accepted for any credit at all. Please contact your TA or make a post on the discussion boards for this course if you have any questions about this assignment or these directions.

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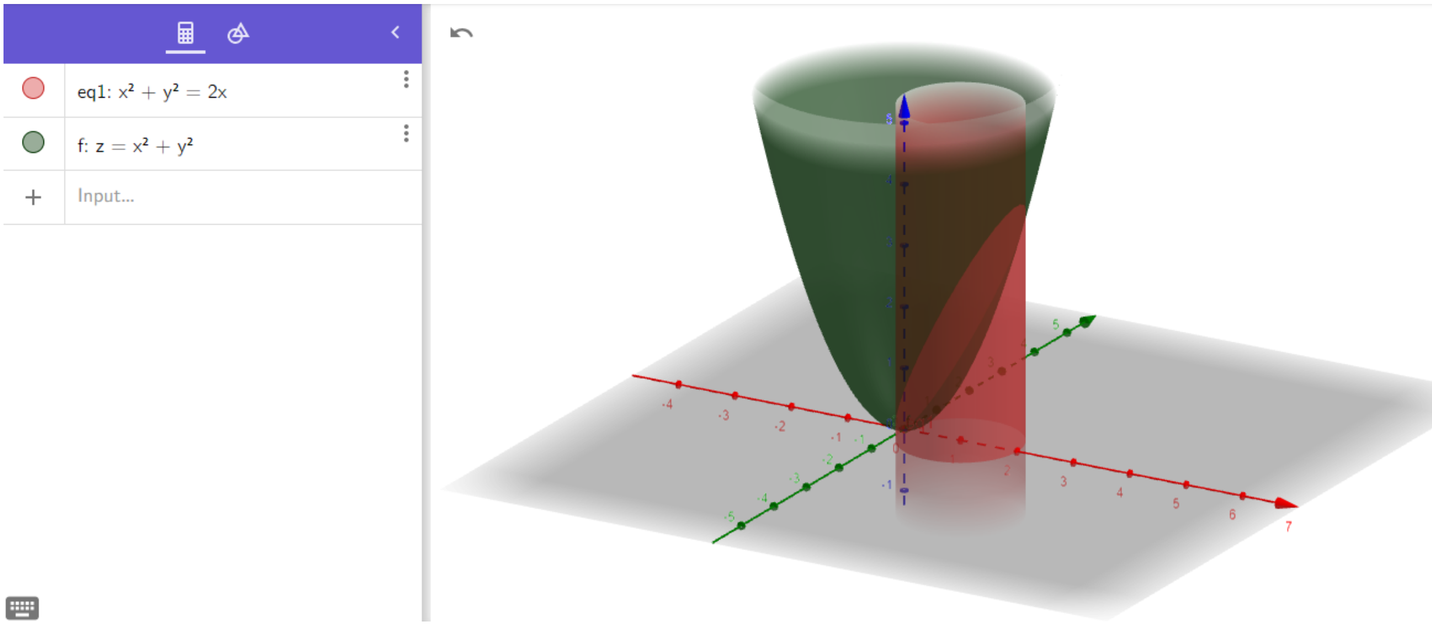
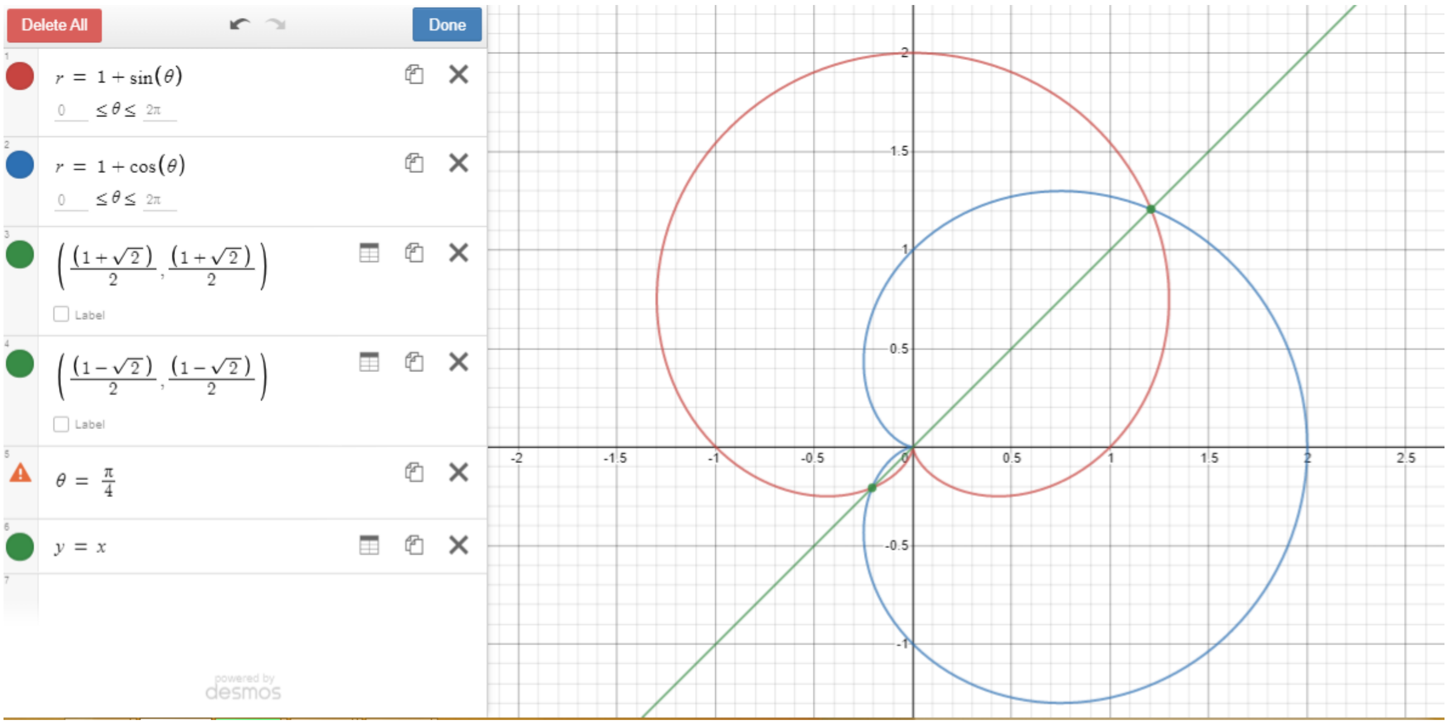
	Checklist of Instructions
	Please clearly write the names of all group members working on this assignment in the spaces allotted above.
	This assignment must be completed by a group of 3, 4, 5, or 6 members.
	This assignment is to be uploaded to gradescope as a pdf file no later than 11:59 PM EST on Sunday, February 14.
	The assignment will be uploaded by 1 group member, and that group member will be responsible for manually entering the names of all other collaborators into gradescope.
	This assignment must be completed using this template. You may either print this template to write on it and then scan it (pages ordered correctly) into a pdf file, or you may write directly on the template using programs such as notability.
	If you need more space than what is given to solve a given problem, then you will find blank pages provided at the end of this template. At the end of each problem section of this assignment you will find a space in which to indicate on what page your work is continued in case you used additional pages to complete your solution. You must provide the page number on which your work is continued in the alloted space, or write 'N/A' incase you did not use any additional pages.
	On the additional pages, you will also find space in which to indicate which problem the page is being used for, and if the page is used then that space must also be filled.
	To complete this handout, you may use your textbook, class notes, discussions with your TA and group members, and any resources that are available on Carmen. You should not receive any help from the MSLC or people outside of your group when solving these problems. You may discuss these problems on the Carmen discussion boards, but you should not provide your entire solution when answering a such question, you should only give a hint or a helpful idea.

(Ungraded Optional) Problem: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.



(Ungraded Optional Problem) Problem 14.3.44: Let R be the region inside both the cardiod $r = 1 + \sin(\theta)$ and the cardiod $r = 1 + \cos(\theta)$. Sketch a picture of the region R, or create an image of the region R using a graphing program, then use double integration to find the area of R.

Solution: We begin by drawing a picture of the region R.



We see that the 2 cardiods intersect when $\sin(\theta) = \cos(\theta)$, which occurs when $\theta = \frac{\pi}{4}, -\frac{3\pi}{4}$. We now see that when $-\frac{3\pi}{4} \le \theta \le \frac{\pi}{4}$ we have $1 + \sin(\theta) \le 1 + \cos(\theta)$ and when $\frac{\pi}{4} \le \theta \le \frac{5\pi}{4}$ we have $1 + \cos(\theta) \le 1 + \sin(\theta)$. It follows that

$$\text{(1)} \qquad \text{Area}(R) = \iint_R 1dA = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \int_0^{1+\sin(\theta)} rdrd\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_0^{1+\cos(\theta)} rdrd\theta$$

$$\text{(2)} \qquad = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}r^2\Big|_{r=0}^{1+\sin(\theta)} d\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2}r^2\Big|_{r=0}^{1+\cos(\theta)} d\theta$$

$$\text{(3)} \qquad = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(1 + 2\sin(\theta) + \sin^2(\theta))d\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2}(1 + 2\cos(\theta) + \cos^2(\theta))d\theta$$

$$= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(1 + 2\sin(\theta) + \frac{1-\cos(2\theta)}{2})d\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2}(1 + 2\cos(\theta) + \frac{1+\cos(2\theta)}{2})d\theta \tag{4}$$

$$= \left(\frac{3}{4}\theta - \cos(\theta) + \frac{-\sin(2\theta)}{4}\Big|_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\right) + \left(\frac{3}{4}\theta + \sin(\theta) + \frac{\sin(2\theta)}{4}\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\right) \tag{5}$$

$$= \boxed{\frac{3\pi}{2} - 2\sqrt{2}}. \tag{6}$$

Problem 14.2.66 (10 points): Evaluate

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1} dydx \tag{7}$$

by changing the order of integration.

Hint: Start by drawing a picture of the region of integration.

Solution: We change the order of integration as shown in the pictures below.

FIGURE 1. Traversing the region of integration when dA = dydx.

. . . .

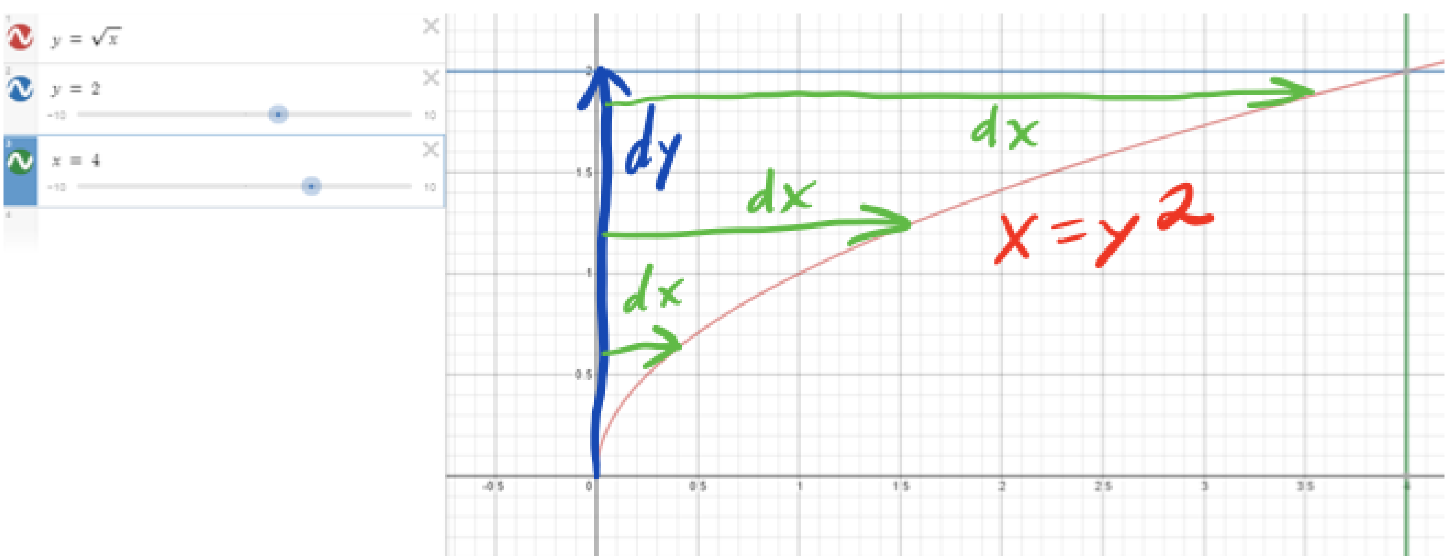


FIGURE 2. Traversing the region of integration when dA = dxdy.

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1} dy dx = \int_0^2 \int_0^{y^2} \frac{x}{y^5+1} dx dy = \int_0^2 \left(\frac{x^2}{2(y^5+1)} \Big|_{x=0}^{y^2} \right) dy \tag{8}$$

$$= \int_0^2 \frac{y^4}{2y^5+2} dy \overset{u=y^5}{=} \int_{y=0}^2 \frac{1}{2u+2} \frac{du}{5} = \frac{1}{10} \ln(u+1) \Big|_{y=0}^2 \tag{9}$$

$$= \frac{1}{10} \ln(y^5+1) \Big|_{y=0}^2 = \boxed{\frac{\ln(33)}{10}}. \tag{10}$$

Problem 14.2.92 (8+4 points): Let R be the region inside of the ellipse $\frac{x^2}{18}+\frac{y^2}{36}=1$ for which we also have $y\le\frac{4}{3}x$.

(a) Find the area of R by evaluating

$$\iint_R 1dydx. \tag{11}$$

(b) Evaluate

$$\iint_R xydydx. \tag{12}$$

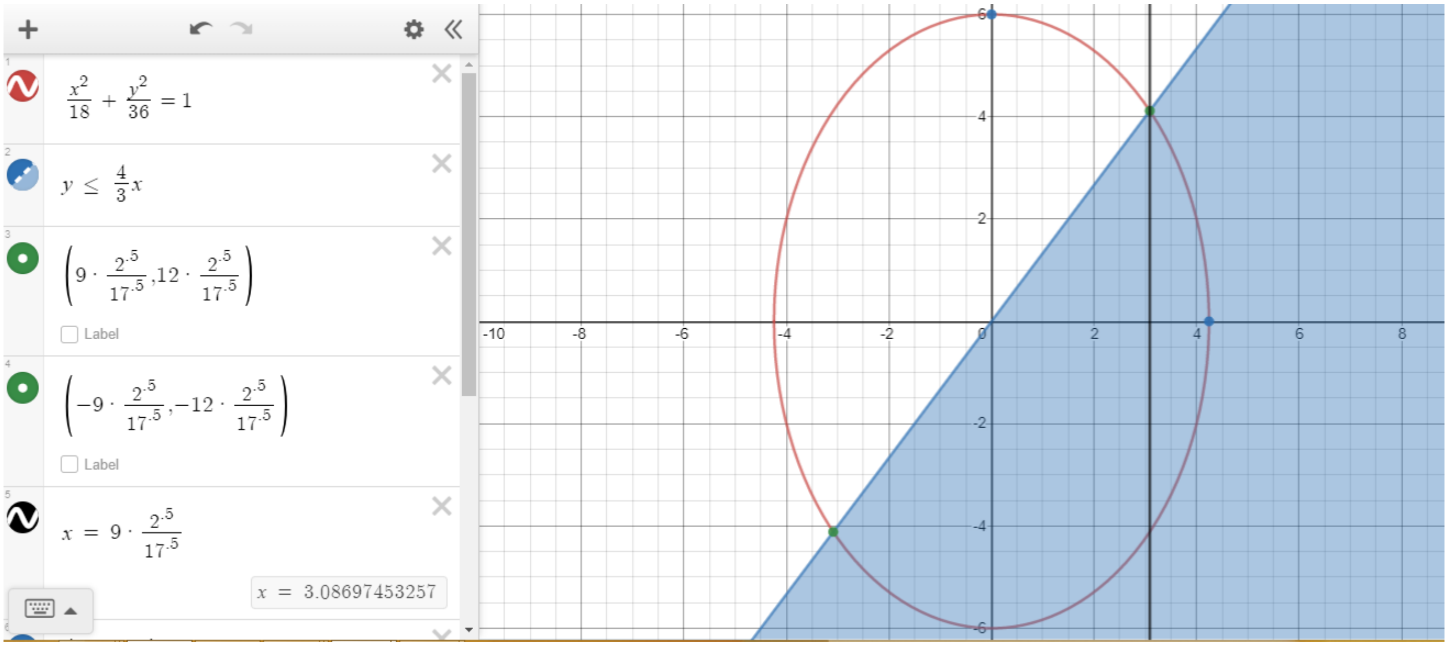
You will recieve 8/8 points for this problem if you just set up but do not evaluate the integrals for parts (a) and (b). You will receive 4 bonus points for the evaluation of the integrals in parts (a) and (b) for the opportunity to earn 12/8 points on this problem.

Hint: Draw a picture of the region R on a graphing program. To help you solve this problem, you may also use the fact that

$$\int\sqrt{1-x^2}=\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}(x)+C \tag{13}$$

It is also possible to solve either part of this problem using symmetry. If you figure out how to do this, it will greatly reduce the amount of calculations that you have to do.

Solution to (a): We first sketch a picture of the region R.



We now solve for the intersection points of the curves $\frac{x^2}{18} + \frac{y^2}{36} = 1$ and $y = \frac{4}{3}x$. We see that

(14) $$\begin{matrix} \frac{x^2}{18} + \frac{y^2}{36} = 1 \\ y = \frac{4}{3}x \end{matrix} \rightarrow \frac{x^2}{18} + \frac{\frac{16}{9}x^2}{36} = 1$$

(15) $$\rightarrow x = \pm\frac{9\sqrt{2}}{\sqrt{17}} \rightarrow (x, y) = (-\frac{9\sqrt{2}}{\sqrt{17}}, -\frac{12\sqrt{2}}{\sqrt{17}}), (\frac{9\sqrt{2}}{\sqrt{17}}, \frac{12\sqrt{2}}{\sqrt{17}}).$$

We now see that the area of R is

(16) $$\iint_R 1dA = \iint_R 1dydx$$

(17) $$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x} 1dydx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} 1dydx$$

(18) $$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} y\Big|_{y=-\sqrt{36-2x^2}}^{\frac{4}{3}x} dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} y\Big|_{y=-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} dx$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{4}{3}x + \sqrt{36-2x^2}\right) dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} 2\sqrt{36-2x^2}dx \tag{19}$$

Since

$$\int \sqrt{1-x^2} = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x), \quad \text{(substitute } x = \sin(\theta)\text{)} \tag{20}$$

we see that

$$\int \sqrt{36-2x^2}dx = \int 6\sqrt{1-(\frac{x}{3\sqrt{2}})^2}dx \overset{y=\frac{x}{3\sqrt{2}}}{=} \int 18\sqrt{2}\sqrt{1-y^2}dy \tag{21}$$

. . . .

$$= 9\sqrt{2}y\sqrt{1-y^2} + 9\sqrt{2}\sin^{-1}(y) = \frac{1}{2}x\sqrt{36-2x^2} + 9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}}). \tag{22}$$

Applying this result to equation (19), we see that

$$\int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{4}{3}x + \sqrt{36-2x^2}\right) dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} 2\sqrt{36-2x^2}dx \tag{23}$$

. . . .

$$\begin{aligned} = &\left(\frac{2}{3}x^2 + \frac{1}{2}x\sqrt{36-2x^2} + 9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\right)\Big|_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \\ &+ \left(x\sqrt{36-2x^2} + 18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \end{aligned} \tag{24}$$

. . . .

$$\begin{aligned} 2&\left(\frac{1}{2}x\sqrt{36-2x^2} + 9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} \\ &+ x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} + 18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \end{aligned} \tag{25}$$

. . . .

$$x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + x\sqrt{36-2x^2}\Big|_{3\sqrt{2}} - x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{3\sqrt{2}} - 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} \tag{26}$$

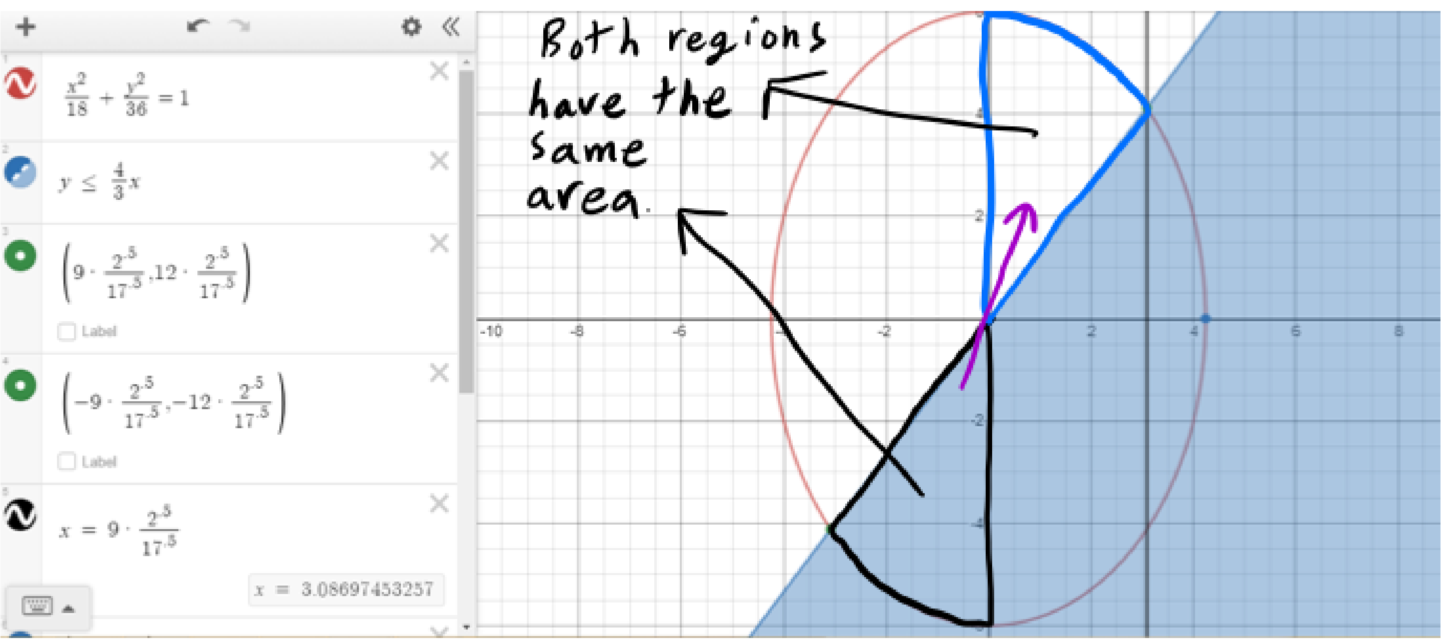
. . . .

$$= x\sqrt{36-2x^2}\Big|_{3\sqrt{2}} + 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{3\sqrt{2}} \tag{27}$$

. . . .

$$= 0 + 18\sqrt{2}\sin^{-1}(1) = \boxed{9\sqrt{2}\pi}. \tag{28}$$

Remark: For the ellipse $\frac{y^2}{36} + \frac{x^2}{18} = 1$ we see that the major radius is 6 and the minor radius is $3\sqrt{2}$, so the area of the ellipse is $6 \cdot 3\sqrt{2} \cdot \pi = 18\sqrt{2}\pi$. We now see that our region R has half the area of the ellipse containing it. In fact, we can prove this directly with symmetry and no calculus at all! We just have to remember that when we reflect the point (x, y) across the origin we get the point $(-x, -y)$, and that reflection across the origin (or reflection across any other point) preserves area as shown in the picture below.



Solution to (b): Using our diagram from part (a) we see that

$$\iint_R xy\,dA = \iint_R xy\,dy\,dx \tag{29}$$

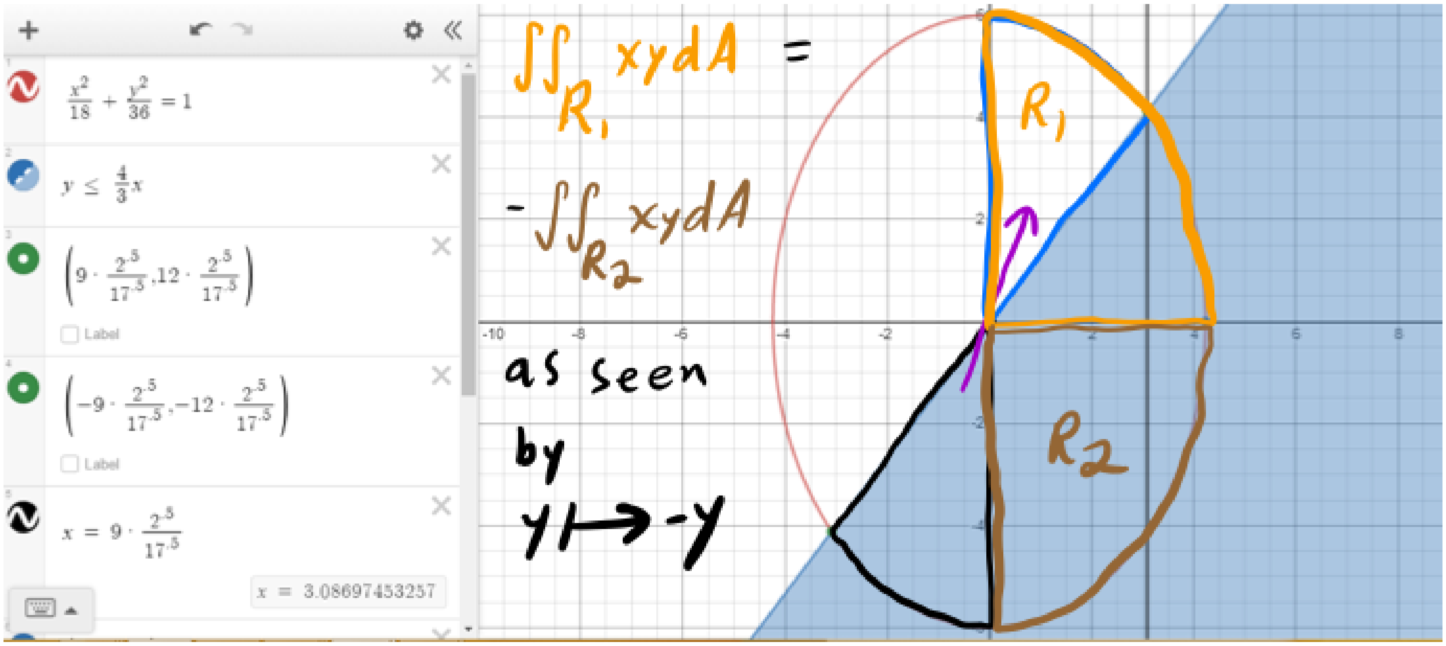
$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x} xy\,dy\,dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} xy\,dy\,dx \tag{30}$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{1}{2}xy^2\right)\Big|_{y=-\sqrt{36-2x^2}}^{\frac{4}{3}x} dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \left(\frac{1}{2}xy^2\right)\Big|_{y=-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} dx \tag{31}$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{1}{2}x(\frac{4}{3}x)^2 - \frac{1}{2}x(-\sqrt{36-2x^2})^2\right) dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \left(\frac{1}{2}x(\sqrt{36-2x^2})^2 - \frac{1}{2}x(-\sqrt{36-2x^2})^2\right) dx \tag{32}$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{16}{9}x^3 - 18x + x^3\right) dx = \boxed{0}. \tag{33}$$

Remark: We see that both integrals appearing in equation (30) are 0. It turns out that this can also be shown directly with symmetry instead of evaluating the integrals! Firstly, we recall that (x, y) turns into $(-x, -y)$ when reflected across the origin and that reflection across the origin preserves area. We also note that $xy = (-x)(-y)$, so we can rewrite our double integral as a double integral that takes place over the right (or left) half of the ellipse instead of the region R. We then notice that $x(-y) = -(xy)$, so the integrals over the top right and lower right quarters of the ellipse cancel each other out to yield 0 as shown in the picture below.



You may see the 3rd solution to problem 14.2.30 from last week to see how you would rigorously show that $\iint_{R_2} xydA = -\iint_{R_1} xydA$.

Problem 14.3.22 (8 points): Find the volume of the solid S bounded by the paraboloid $z = 8 - x^2 - 3y^2$ and the hyperbolic paraboloid $z = x^2 - y^2$.

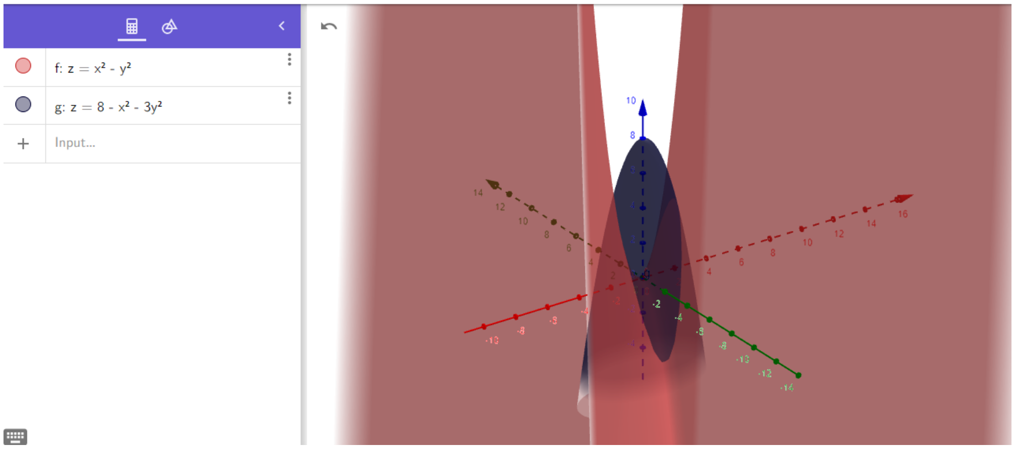


FIGURE 3. A view of the solid S whose volume we are calculating.

Solution: We begin by finding the (x, y)-coordinates of the curves of intersection of the 2 given surfaces. We see that

$$8 - x^2 - 3y^2 = z = x^2 - y^2 \to 2x^2 + 2y^2 = 8 \to x^2 + y^2 = 4, \tag{34}$$

so the (x, y)-coordinates of the curve of intersection is simply the circle of radius 2 centered at the origin.

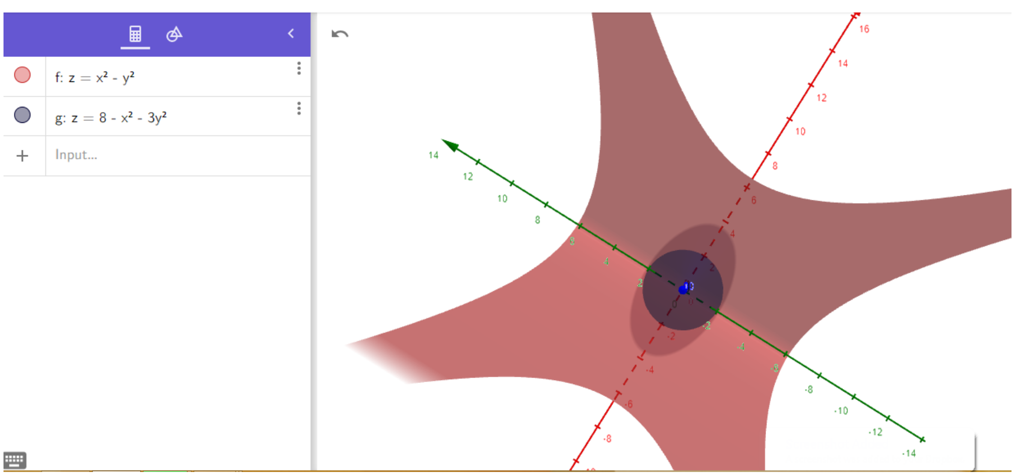


FIGURE 4. A bird's eye view of the solid S that is used to find the region of integration R.

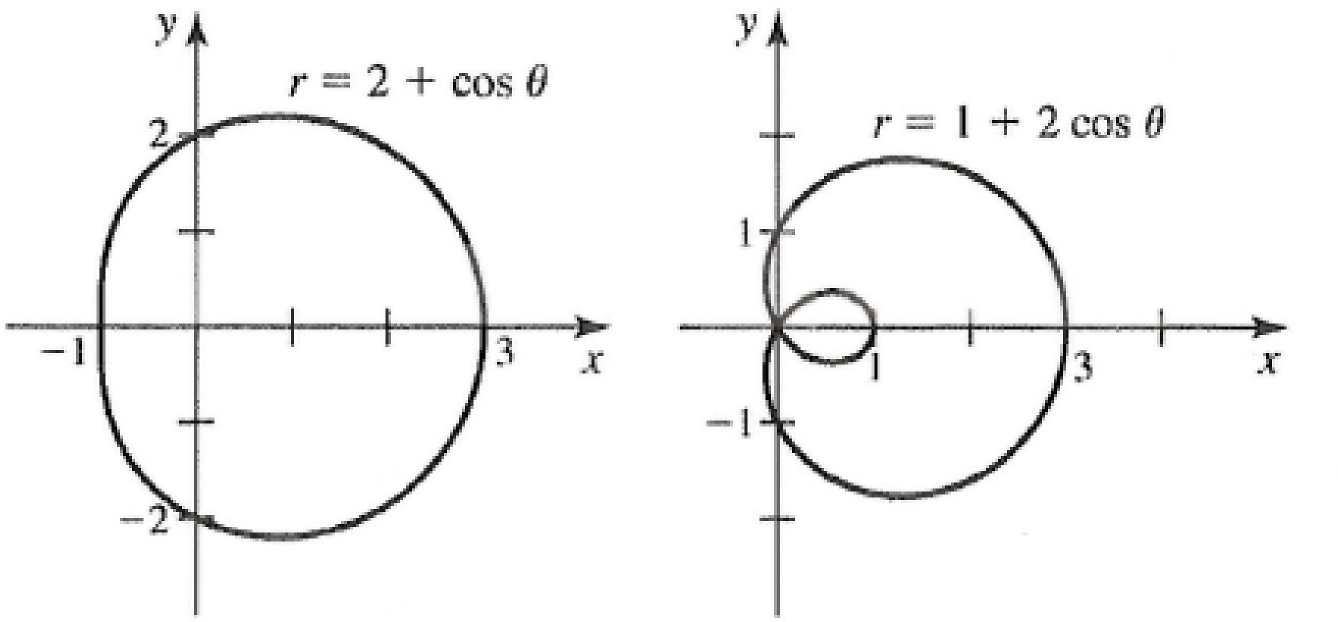
Noting that $8-0^2-3\cdot0^2 = 8 > 0 = 0^2-0^2$, we see that the curve $z = 8-x^2-3y^2$ lies above the curve $z = x^2 - y^2$ for all (x, y) inside of R, the disc of radius 2 centered at the origin. We now see that

$$\text{Volume}(S) = \iint_R (z_{\text{top}} - z_{\text{bot.}})dA = \iint_R ((8 - x^2 - 3y^2) - (x^2 - y^2))dA \tag{35}$$

$$= \iint_R (8 - 2x^2 - 2y^2)dA = \int_0^{2\pi} \int_0^2 (8 - 2r^2) r\,dr\,d\theta \tag{36}$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 (8r - 2r^3)dr\right) = (2\pi)\left(4r^2 - \frac{1}{2}r^4\Big|_0^2\right) = \boxed{16\pi}. \tag{37}$$

Problem 14.3.67 (10 points): The limaçon $r = b + a\cos(\theta)$ has an inner loop if $b < a$ and no inner loop if $b > a$.



(a) Find the area of the region bounded by the limaçon $r = 2 + \cos(\theta)$.
(b) Find the area of the region outside the inner loop and inside the outer loop of the limaçon $r = 1 + 2\cos(\theta)$.
(c) Find the area of the region inside the inner loop of the limaçon $r = 1 + 2\cos(\theta)$.

Note: Be careful not to double count any portion of area when solving parts (b) and (c) of this problem.

Solution to (a): Letting R denote the interior of the limaçon $r = 2+\cos(\theta)$, we see that

$$\text{(38)} \qquad \text{Area}(R) = \iint_R 1dA = \iint_R rdrd\theta = \int_0^{2\pi}\int_0^{2+\cos(\theta)} rdrd\theta$$

$$\text{(39)} \qquad = \int_0^{2\pi} \frac{1}{2}r^2\Big|_{r=0}^{2+\cos(\theta)} d\theta = \int_0^{2\pi}\frac{1}{2}(2+\cos(\theta))^2 d\theta$$

$$(40) \quad = \int_0^{2\pi} (2+2\cos(\theta)+\frac{1}{2}\cos^2(\theta))d\theta = \int_0^{2\pi} (2+2\cos(\theta)+\frac{1}{4}\cos(2\theta)+\frac{1}{4})d\theta$$

. . . .

$$(41) \quad (\frac{9}{4}\theta + 2\sin(\theta) + \frac{1}{8}\sin(2\theta))\Big|_0^{2\pi} = \boxed{\frac{9}{2}\pi}.$$

Solution to (c): Let R denote the region inside of the inner loop of the limaçon $r = 1 + 2\cos(\theta)$. We see that the inner loop of the limaçon begins and ends when $r = 0$, which occurs when $\cos(\theta) = -\frac{1}{2}$, which occurs when $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. It follows that

$$(42) \quad \text{Area}(R) = \iint_R 1dA = \iint_R rdrd\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \int_0^{1+2\cos(\theta)} rdrd\theta$$

. . . .

$$(43) \quad = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}r^2\Big|_{r=0}^{1+2\cos(\theta)} d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(1+2\cos(\theta))^2 d\theta$$

. . . .

$$(44) \quad = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\frac{1}{2}+2\cos(\theta)+2\cos^2(\theta))d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\frac{1}{2}+2\cos(\theta)+\cos(2\theta)+1)d\theta$$

. . . .

$$(45) \quad = (\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{2}\sin(2\theta))\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = \boxed{\pi - \frac{3}{2}\sqrt{3}}.$$

Remark: We see that for $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$ we have that $r < 0$. Since the $r = 1 + 2\cos(\theta)$ in the upper bound of the inner integral and the r in the rdr from the integrand are both negative, the net result is still a positive area.

Solution to (b): Letting R' denote the region inside of the outer loop and outside of the inner loop of the limaçon $r = 1 + 2\cos(\theta)$, we see that

$$\text{Area}(R') + 2\text{Area}(R) = \int_0^{2\pi}\int_0^{1+2\cos(\theta)} r\,dr\,d\theta \tag{46}$$

. . . .

$$= (\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{2}\sin(2\theta))\Big|_0^{2\pi} = 3\pi. \tag{47}$$

Using our answer from part (c), we see that

$$\text{Area}(R') = 3\pi - 2\text{Area}(R) = 3\pi - 2(\pi - \frac{3}{2}\sqrt{3}) = \boxed{\pi + 3\sqrt{3}}. \tag{48}$$